Adjoint method and runaway electron dynamics in momentum space

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Apr 6 2016

Outline

- Introduction to runaway electron dynamics in momentum space
- Adjoint Method
  - Runaway Probability function
  - Expected Loss time
- Large angle scattering in runaway electron energy loss.
- Summary

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- Due to the decrease of the Coulomb collision force with p, electrons with momentum larger than p_{crit} can be continuously accelerated by the toroidal electric field to very high energy.
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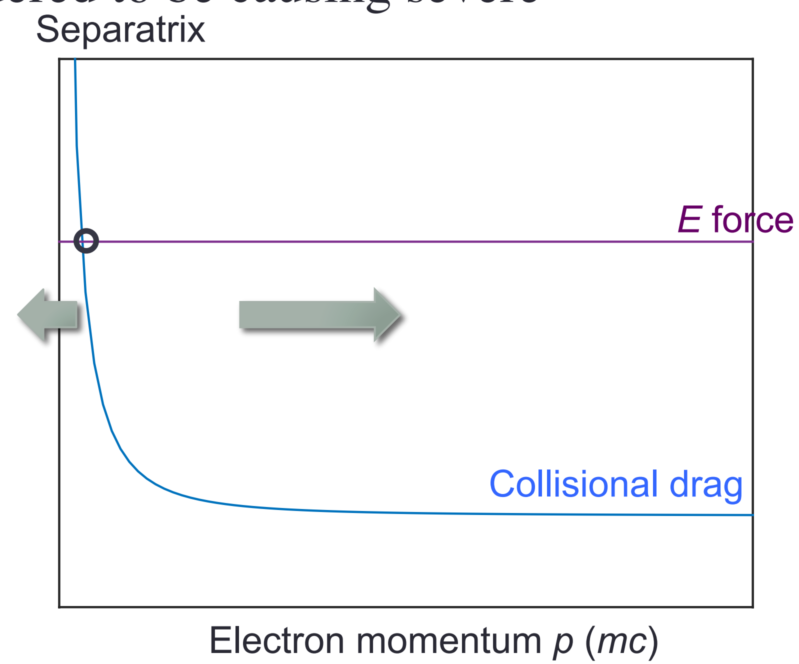

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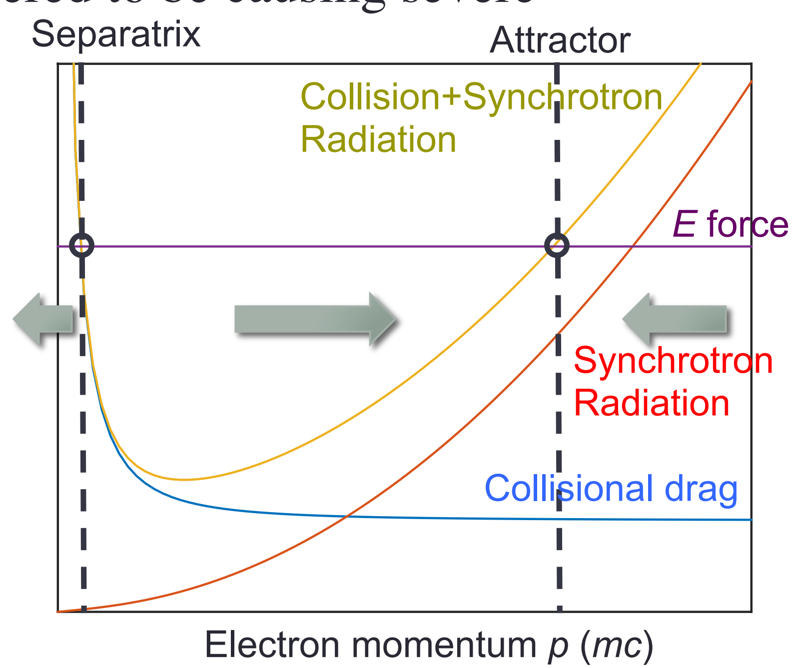

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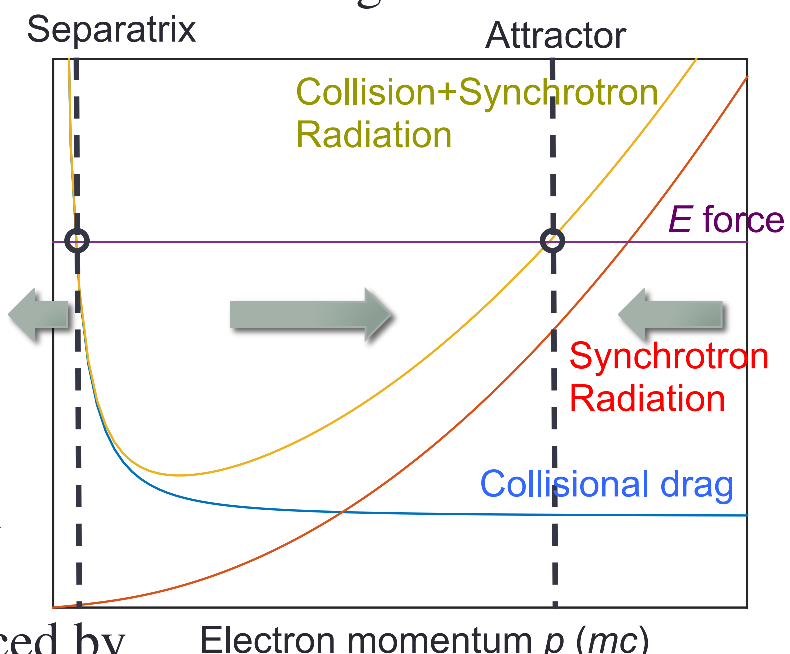

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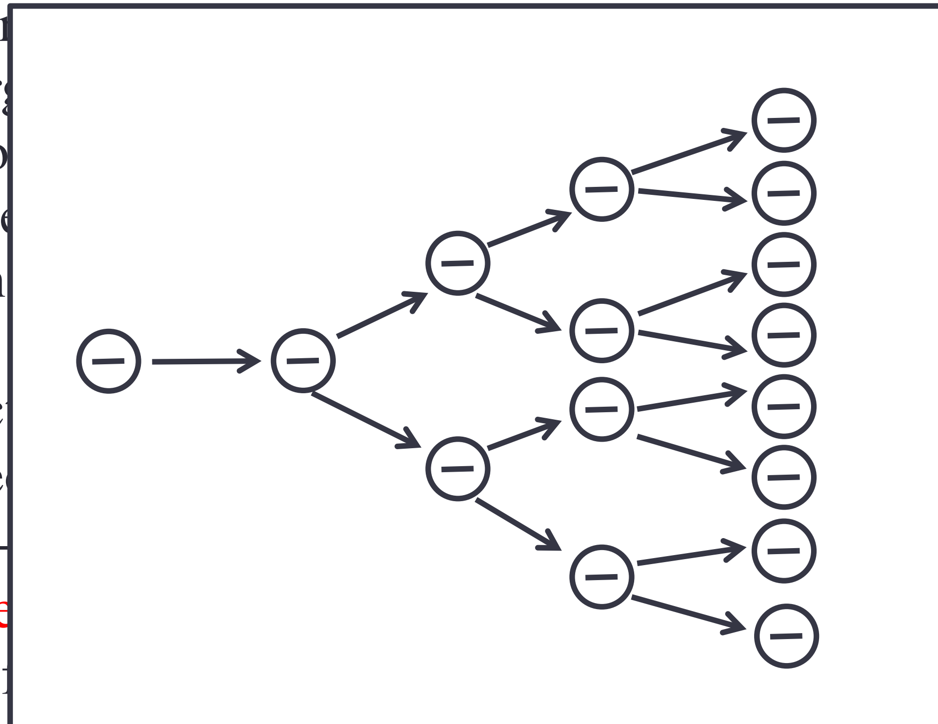

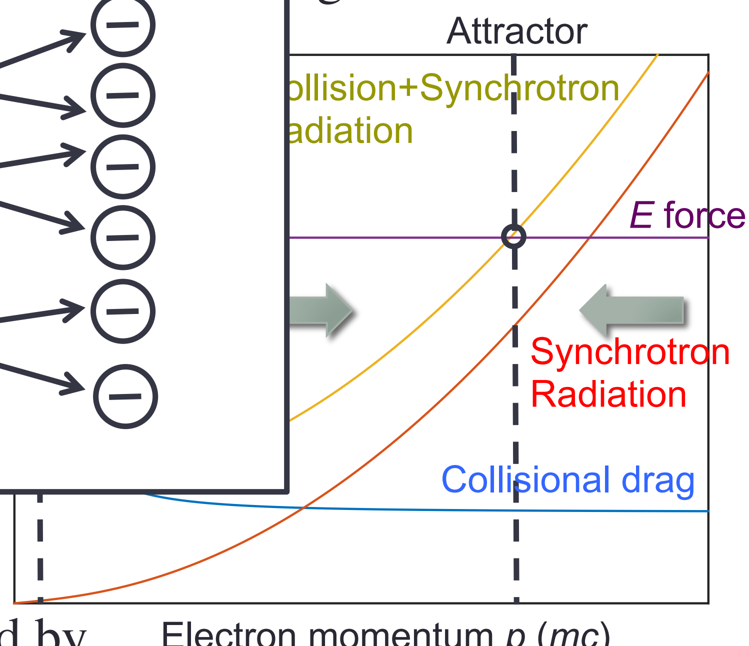

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RE Kinetic Equation in Momentum Space

$$\boxed{\frac{\partial f}{\partial t} + E\{f\} + C\{f\} + R_S\{f\} + R_B\{f\} = S\{f\}}$$

E: Parallel electric field acceleration

C: Relativistic collision operator (slowing-down and pitch angle scattering)

R_S: Synchrotron radiation reaction force (SRRF)

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- The kinetic equation is a 2-D Fokker-Planck equation (ignoring the source term).
  - Diffusion term mainly comes from pitch angle scattering term.

$$\frac{\partial f}{\partial t} = \hat{L}[f] = -\nabla\cdot\left[a\, f\right] + \nabla\nabla\cdot\left[D\, f\right]$$

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Adjoint method I: Runaway Probability Function

P is solution of adjoint Fokker-Planck equation.

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$$P(p_1) = 0, P(p_2) = 1$$

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C. Liu, D.P. Brennan, A. Bhattacharjee, and A.H. Boozer, Phys. Plasmas **23**, 010702 (2016).

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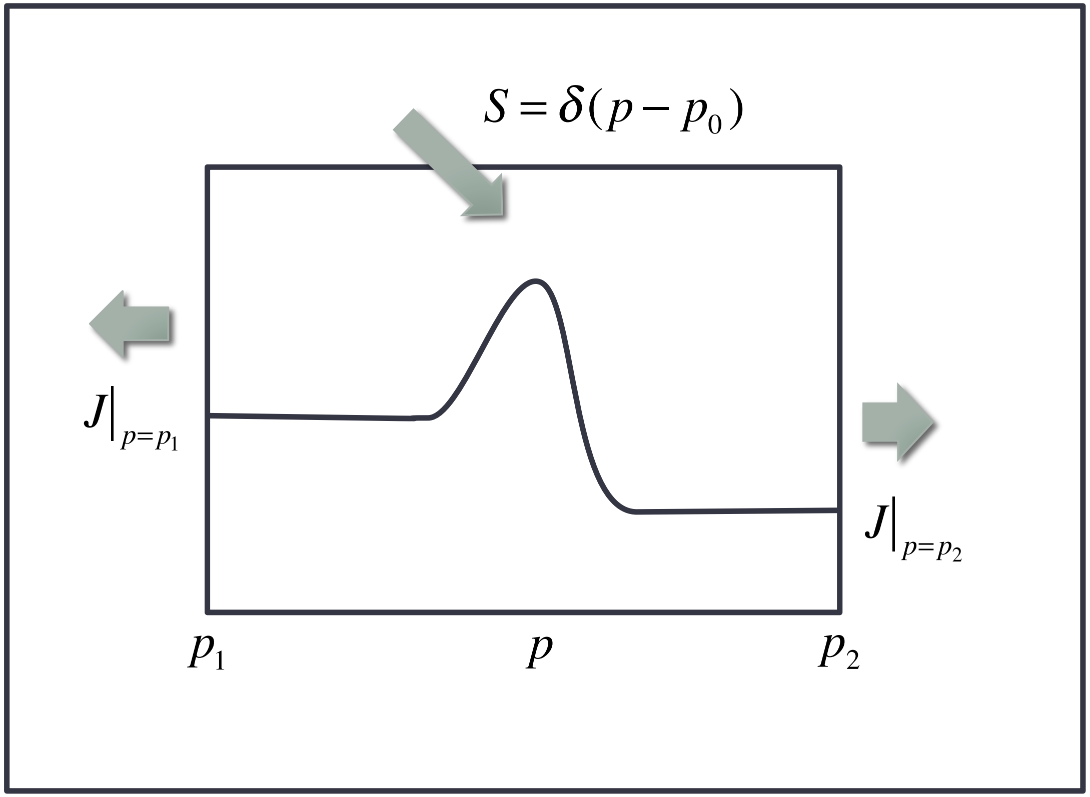

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$$\int_{p_1}^{p_2}\hat{L}[F]P\,dp=\left[PU+D\frac{\partial P}{\partial p}F\right]_{p1}^{p2}+\int_{p_1}^{p_2}F\hat{L}^{\dagger}[P]dp$$

$$P(p=p_0)=J\big|_{p=p_2}$$

P characterize the probability for electron to eventually reach boundary *p*=p_2.

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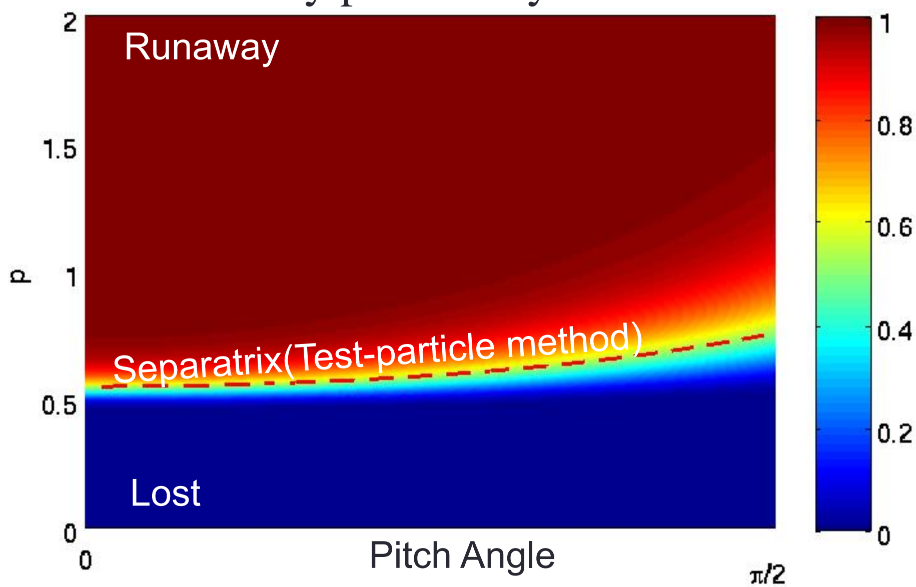
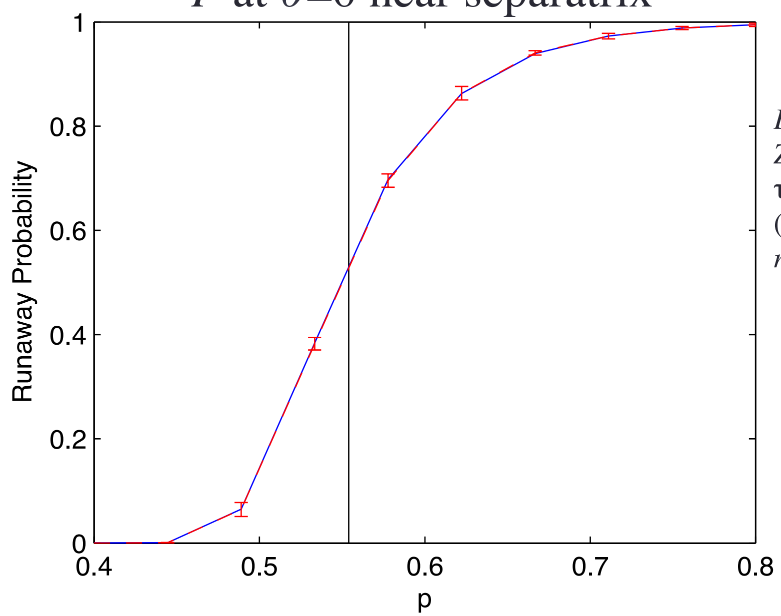
Runaway Probability Function for Z=1

$$a(p)\frac{dP(p)}{dx}+D(p)\frac{d^2P(p)}{dp^2}=0 \quad P|_{p_1}=0 \quad P|_{p_2}=1$$

Runaway probability function



P at θ=0 near separatrix



E/E_{CH}=6
Z=1
τ_r/τ=100
(B=3T,
n_e=10^{21}m^3

- P gives probability for electron to reach high momentum boundary
- Result of P shows smooth transition near separatrix
  - The test-particle method (relying on truncation of pitch angel scattering) only gives a line of separatrix, equivalent to a Heaviside P function.

$$\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial f}{\partial\xi}=\frac{\partial}{\partial\xi}(\xi f)+\boxed{\frac{\partial^2}{\partial\xi^2}\left(\frac{1-\xi^2}{2}f\right)}$$

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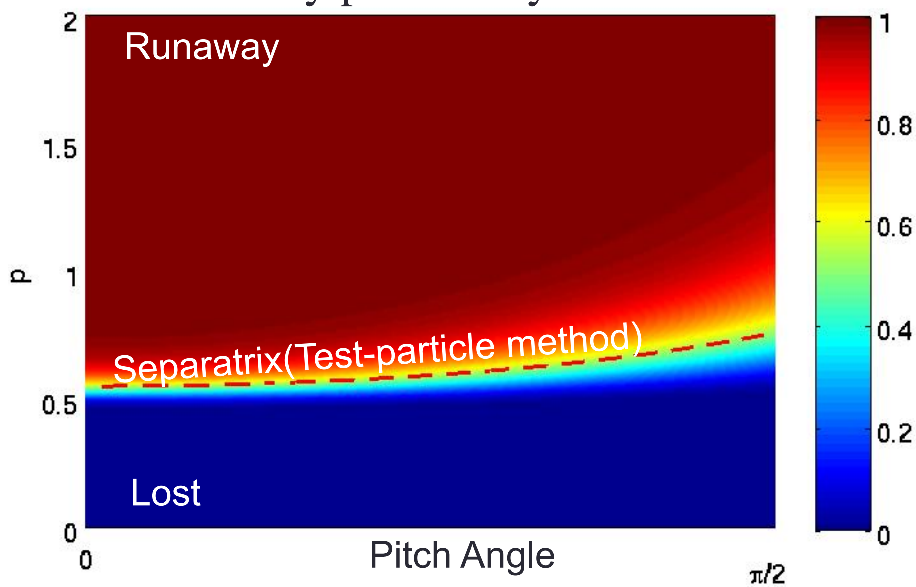

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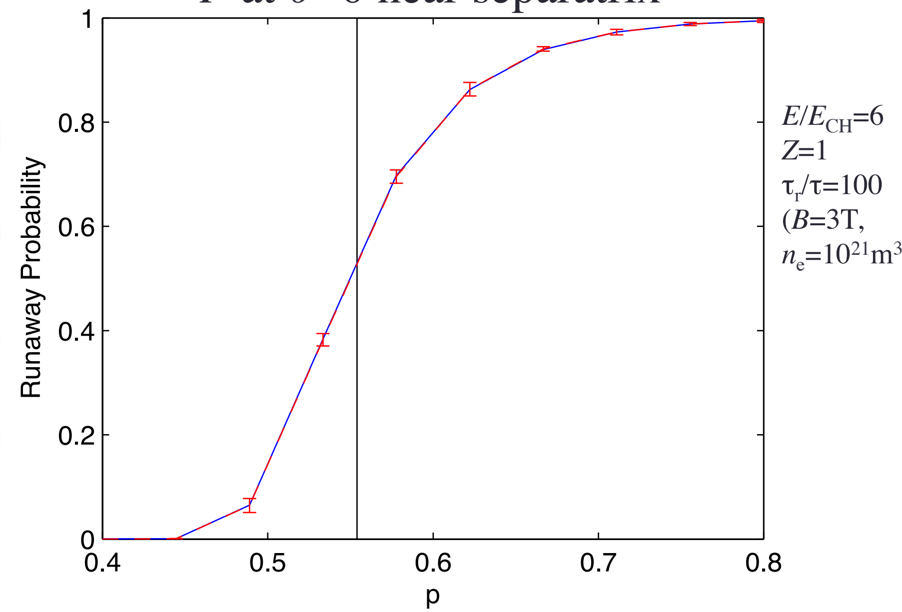

- P gives probability for electron to reach high momentum boundary
- Result of P shows smooth transition near separatrix
  - The test-particle method (relying on truncation of pitch angel scattering) only gives a line of separatrix, equivalent to a Heaviside P function.
- Results agree well with Monte-Carlo Simulation

Runaway Probability Function for *Z*=7

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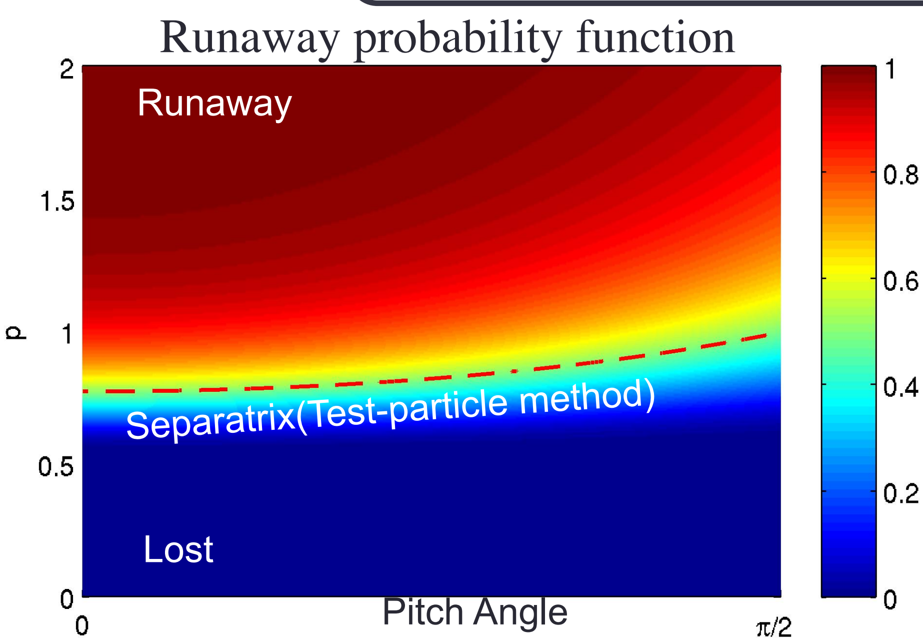
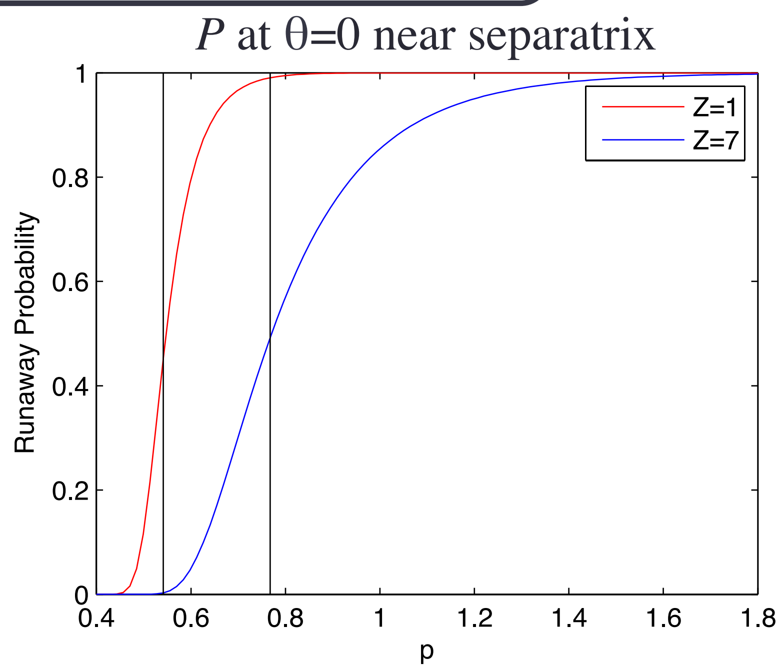


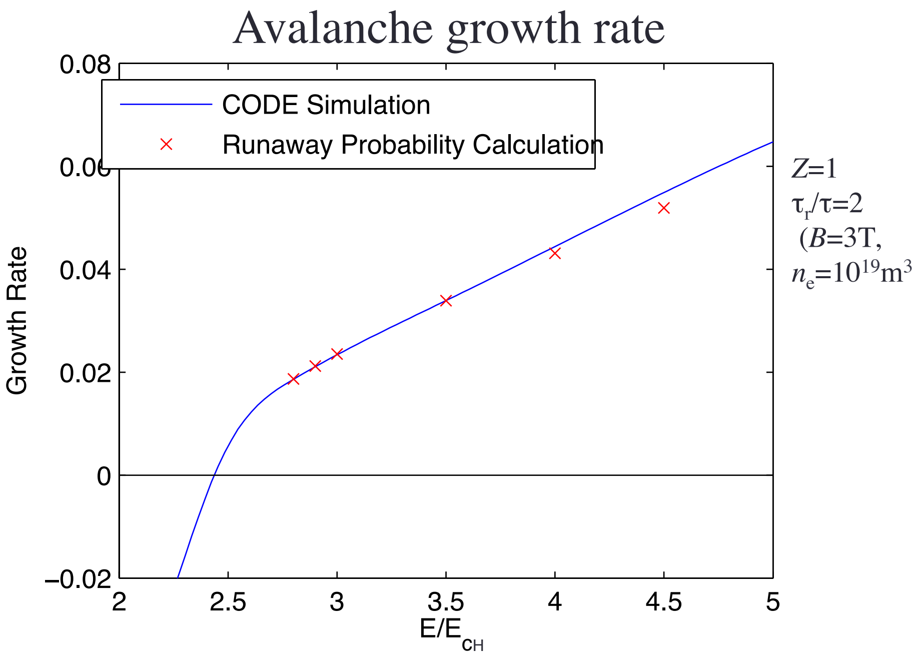


- Separatrix location and width of transition region both increase with pitch angle scattering (*Z*).
  - Transition region is asymmetric at two sides of separatrix.

Use Runaway Probability to Calculate the Avalanche Growth Rate

$$\frac{\partial f}{\partial t} + E\{f\} + C\{f\} + R_S\{f\} + R_B\{f\} = S\{f\}$$

$$\gamma_A = \frac{\int dp\, S\{f\} \cdot P}{n_{RE}}$$



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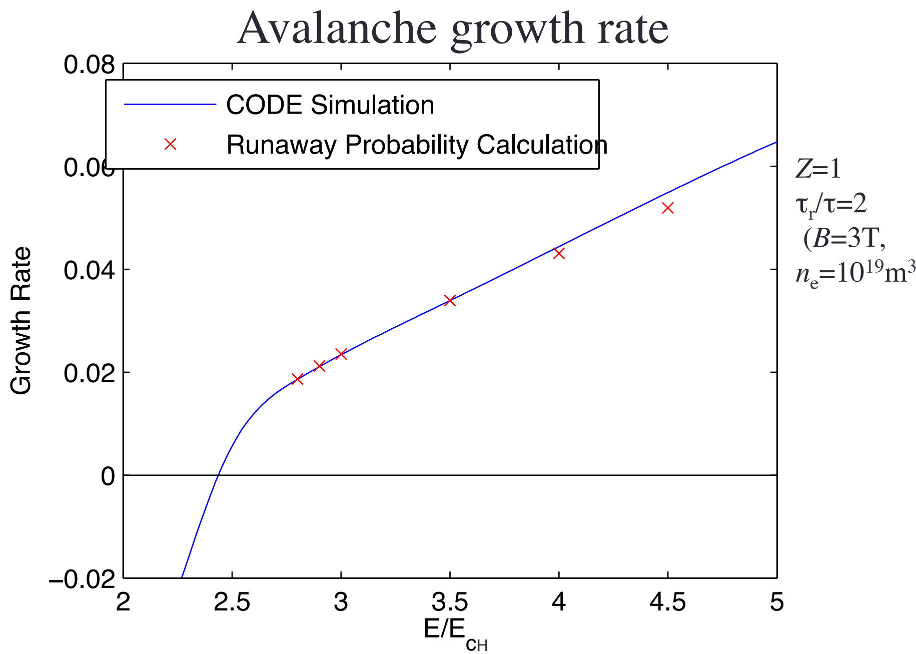

- Calculated γ_A agrees well with CODE simulation result.
- For tokamak disruption, P can be used to estimate the number of seed RE in thermal quench.

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Adjoint Method II: Expected Loss Time (ELT)

T is solution of nonhomogeneous adjoint Fokker-Planck equation.

$$\hat{L}^{\dagger}[T] = a(p)\frac{\partial T}{\partial p} + D(p)\frac{\partial^2 T}{\partial p^2} = -1$$

$$T(p_1) = 0, T(p_2) = 0$$

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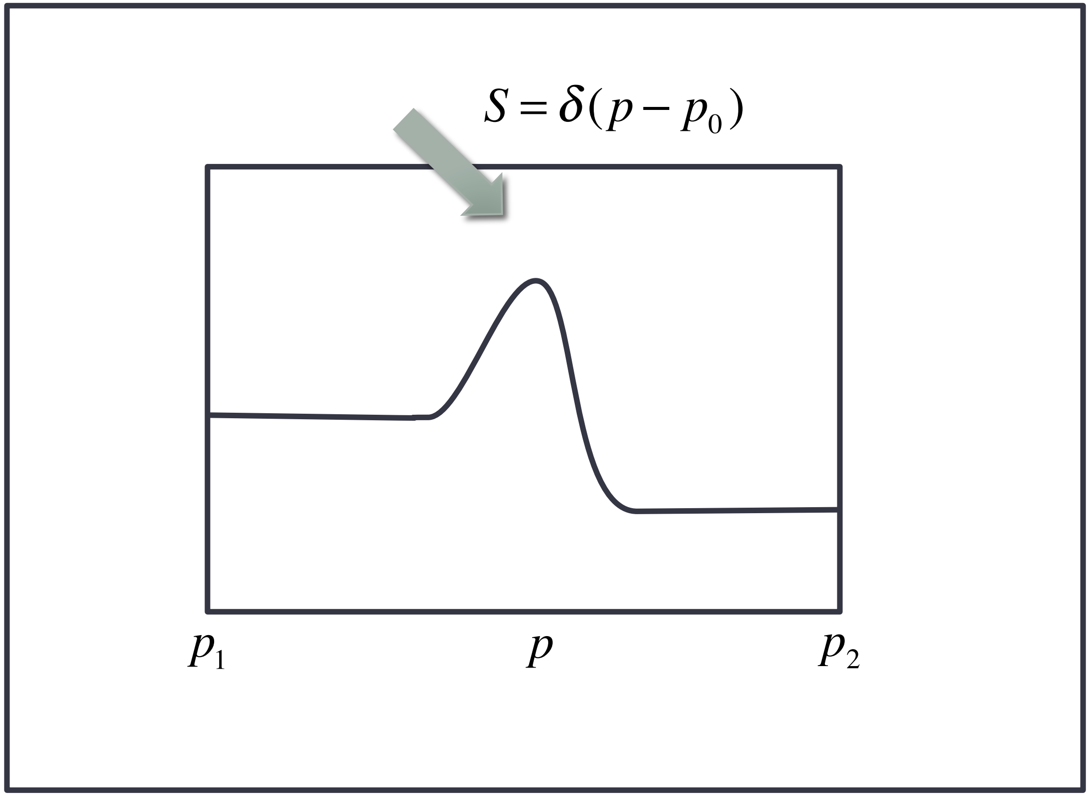

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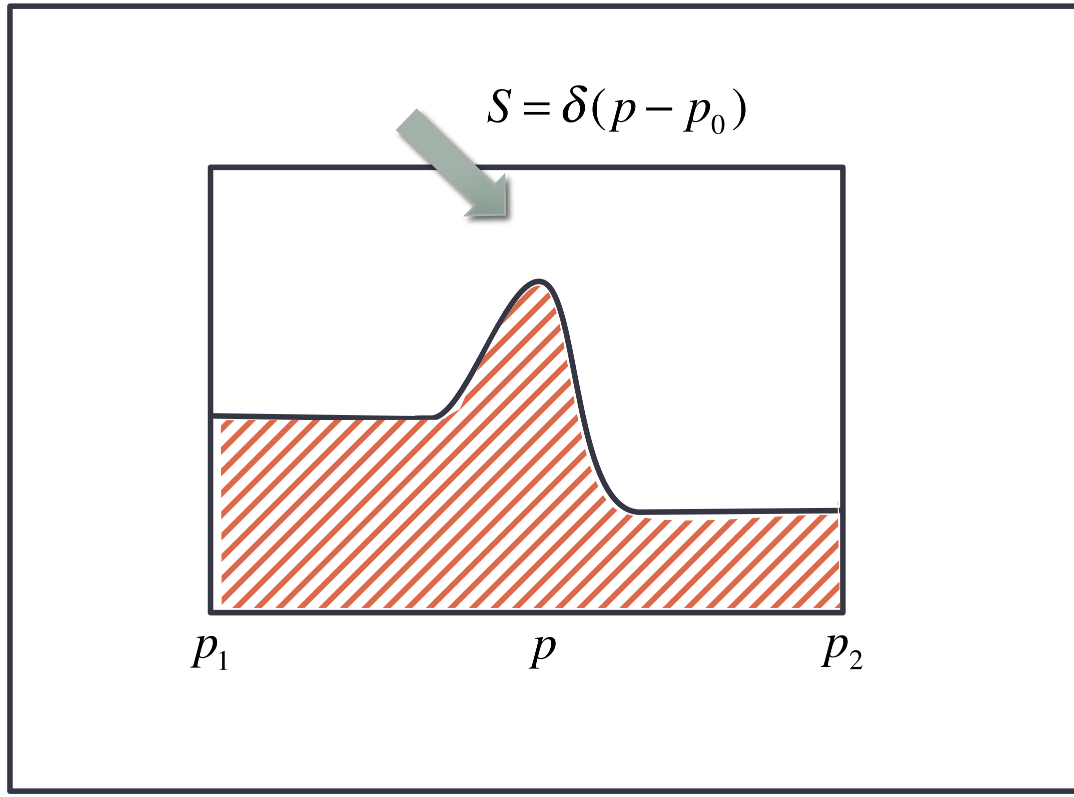

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$$T(p = p_0) = \int_{p1}^{p2} Fdp$$

T characterize the expected loss time, which is the expected time for an electron to reach the boundary.

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Expected Loss Time for Runaway Electron Decay

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- $1/T=1/T_S+1/T_R$, T_S (slowing-down time) and T_R (runaway time) are expected time to reach low/high energy boundary.
- For $E<E_0$, all electrons will end up in low energy boundary. T_S represents the timescale for runaway electron energy decay.

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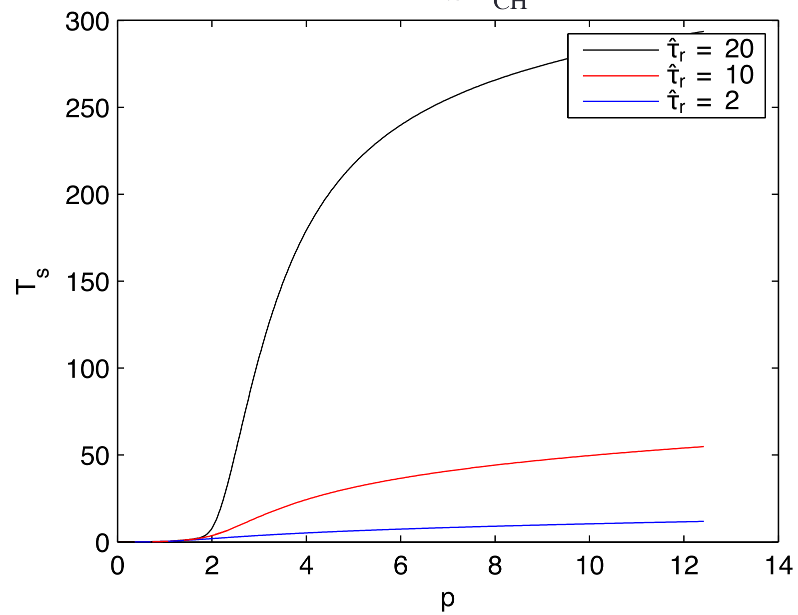
- For $E<E_0$, all electrons will end up in low energy boundary. T_S represents the timescale for runaway electron energy decay.
- In the marginal case (E is close to E_0), T has a big jump near the separatrix.
  - E field force can form a potential barrier near the separatrix that hinder particle losing energy.

$E=1.5E_{CH}$ Z=1



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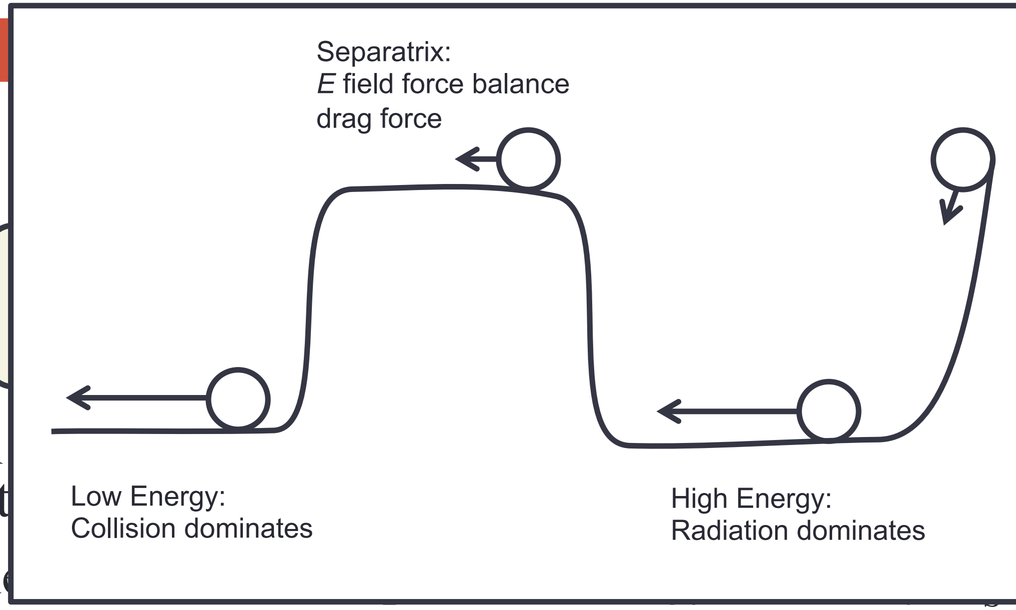

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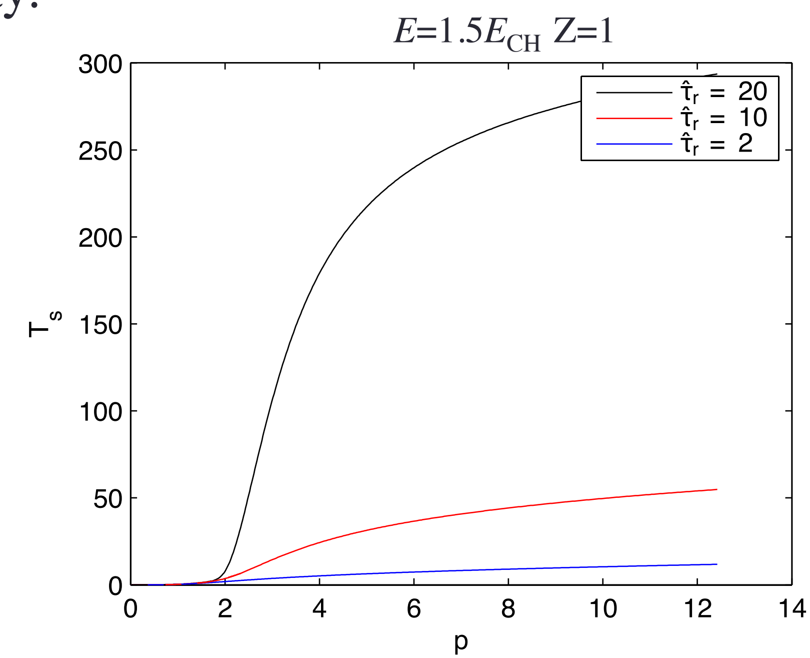

Expected Loss Time including Secondary RE Generation

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} - \hat{L}[f] = \delta(p - p_0) + S\{F\}$$

$$S\{F\} = \int dq \sigma(p,q) F(q)$$

$$\hat{L}^{\dagger}[T] = -1 + \int dq \sigma(q,p) T(q)$$

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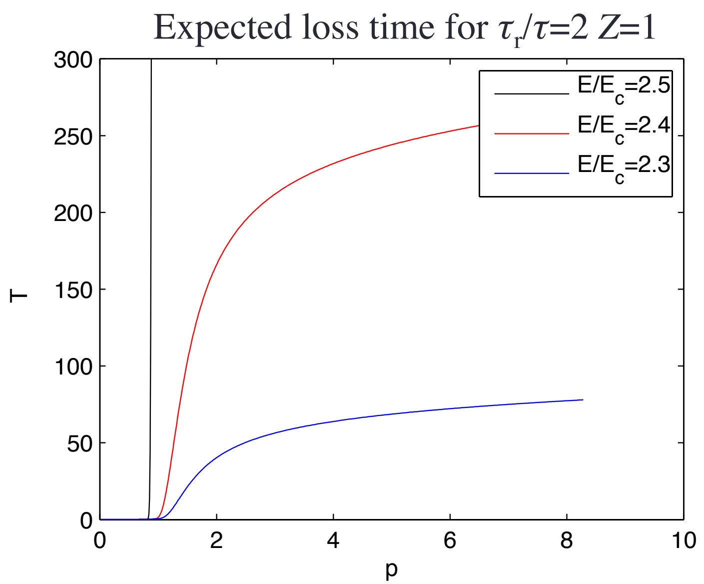
- $T\rightarrow\infty$ when RE growth rate (with avalanche) is positive.



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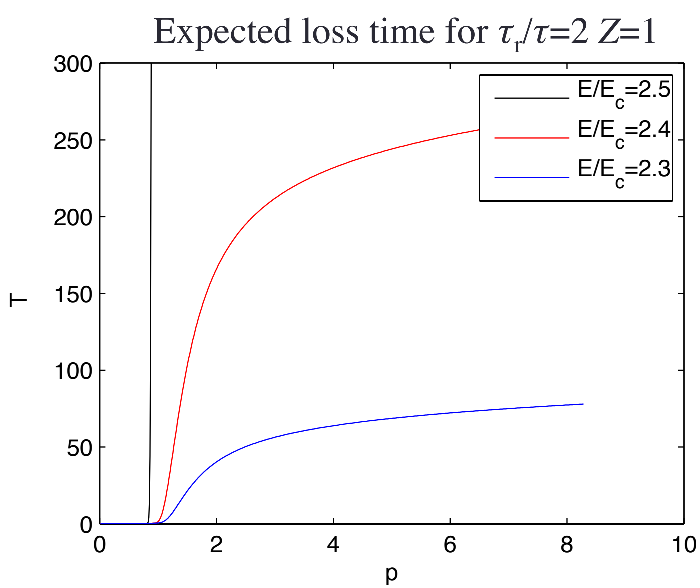

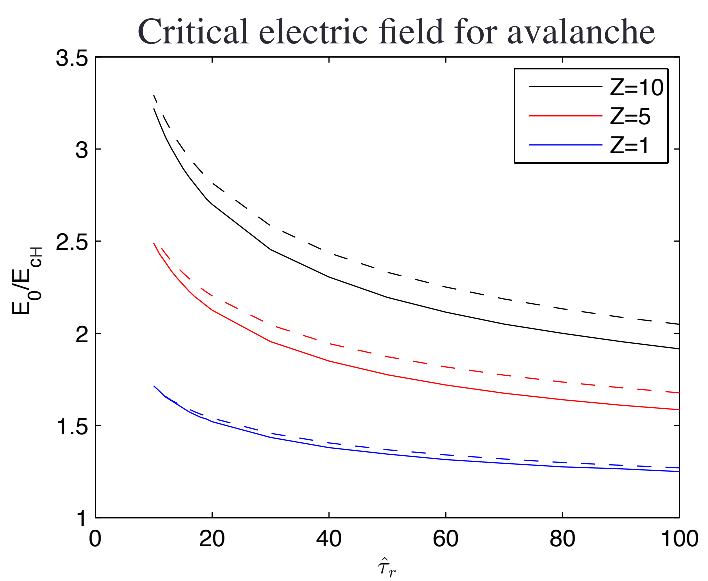

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No electric field

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Large angle scattering

1 $\quad$ $1/\ln\Lambda + v_{th}^2/v_{test}^2$ $\quad$ $1/\ln\Lambda$

Large Angle Scattering in Nonhomogeneous Momentum Space

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  - For collisional energy loss, the contribution of LAS is $1/\ln\Lambda$ of the accumulation of small angle scattering.
  - In nonhomogeneous momentum space, *E* field balances collisional drag near the separatrix, thus LAS can be more important.

No electric field

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p}\left[\left\langle \frac{\Delta p}{\Delta t}\right\rangle_c f\right] + \frac{\partial^2}{\partial p^2}\left[\left\langle \frac{\Delta p \Delta p}{\Delta t}\right\rangle_c f\right] + C_L[f]$$

Large angle scattering

1 $\quad 1/\ln\Lambda + v_{th}^2/v_{test}^2 \quad 1/\ln\Lambda$

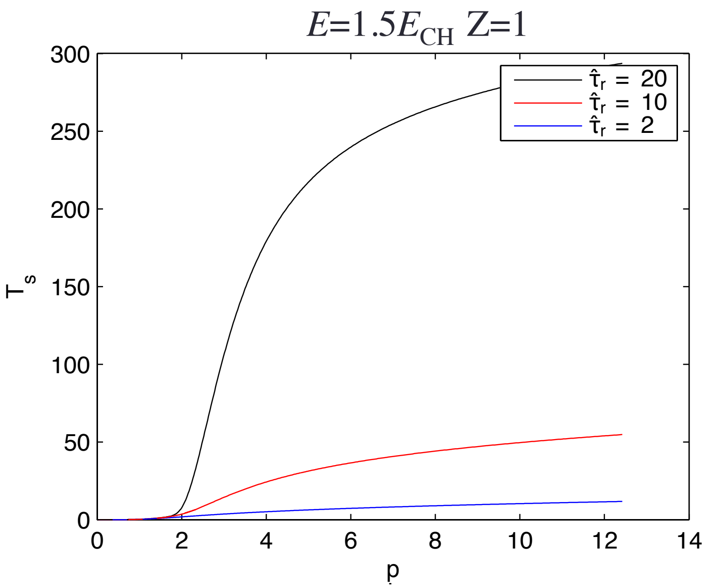
E field balance collisional drag

$$\frac{\partial f}{\partial t} = \cancel{-\frac{\partial}{\partial p}\left[eE + \left\langle \frac{\Delta p}{\Delta t}\right\rangle_c f\right]} + \frac{\partial^2}{\partial p^2}\left[\left\langle \frac{\Delta p \Delta p}{\Delta t}\right\rangle_c f\right] + C_L[f]$$

1-1 $\quad 1/\ln\Lambda + v_{th}^2/v_{test}^2 \quad 1/\ln\Lambda$

Expected Loss Time in Nonhomogeneous Momentum Space

- Large angle collision is important for electron energy loss when E is close to E_0 (marginal case).



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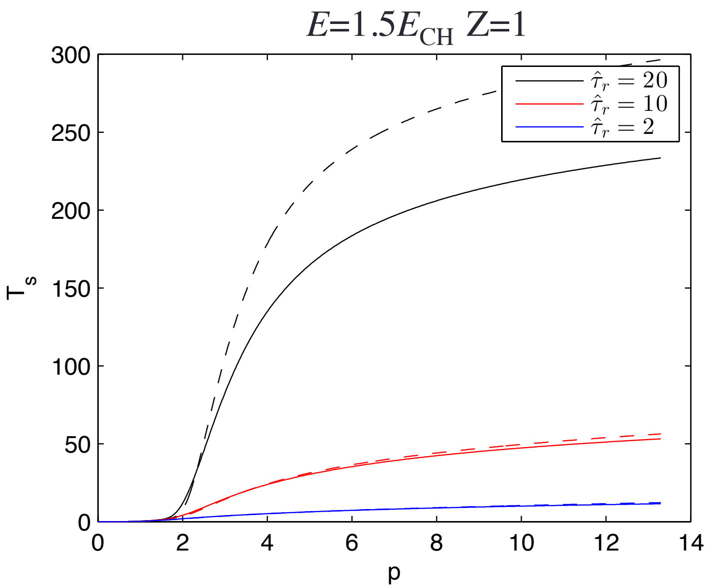
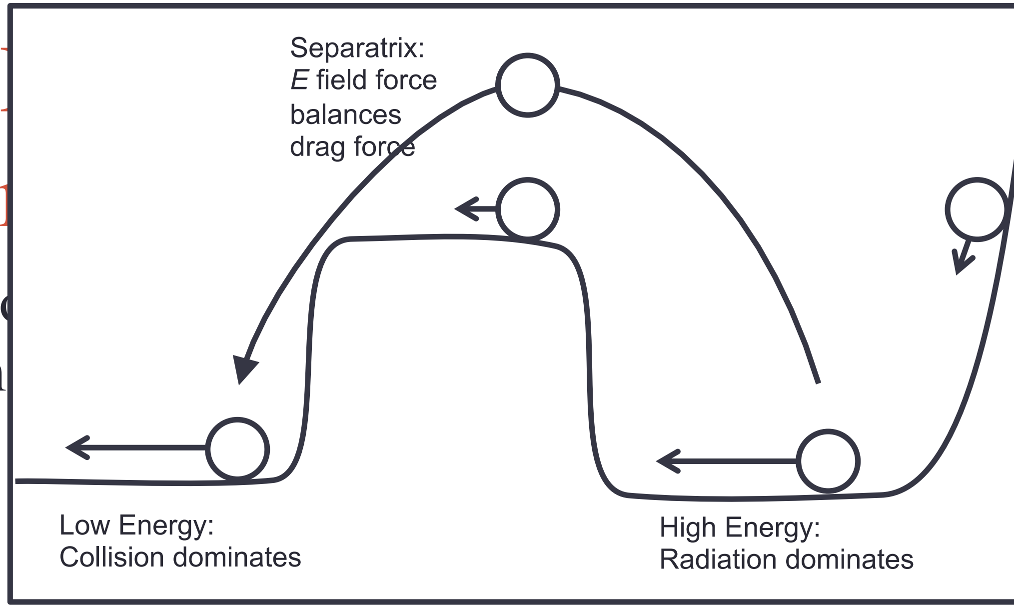
  - Results of expected loss time shows that large angle collisions help electrons overpass the potential barrier, therefore significantly reduce the jump of T at marginal case.



Expected ... ous Momentu...

- Large angle c... when E is close to E_0 (m...



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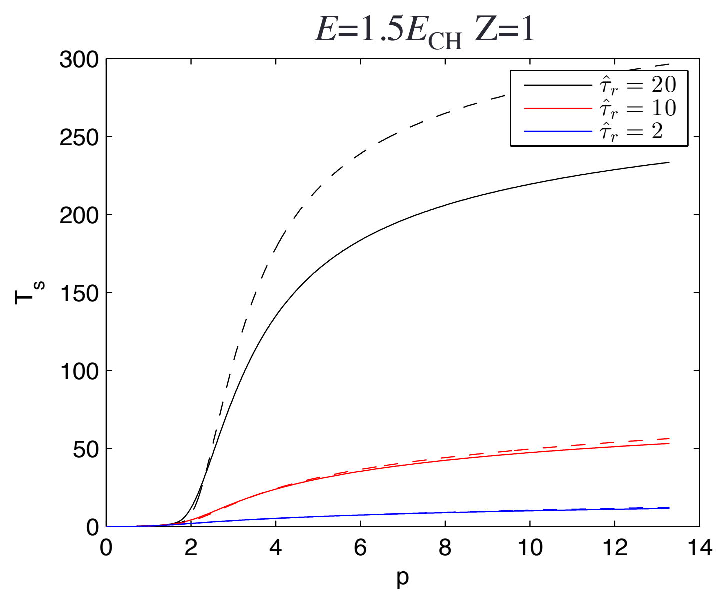

Outline

- Introduction to runaway electron dynamics in momentum space
- Adjoint Method
  - Runaway Probability function
  - Expected Loss time
- Large angle scattering in runaway electron energy loss.
- Summary

Summary

- Adjoint method gives a new angle to study the nonhomogeneous momentum space of runaway electrons.
- Both runaway probability (P) and expected loss time (T) are derived from the adjoint method.
- For marginal case (E close to E_0), large angle scattering (LAS) plays an important role in energy decaying of existing RE population.
- The adjoint method can also be applied to other dynamical systems.

Thanks!