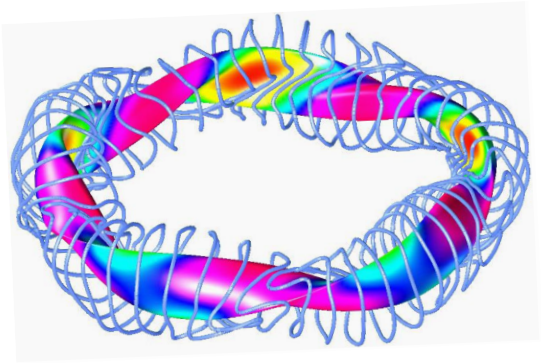
3D MHD equilibria with current sheets, magnetic islands, and chaos in stellarators and tokamaks

Joaquim Loizu

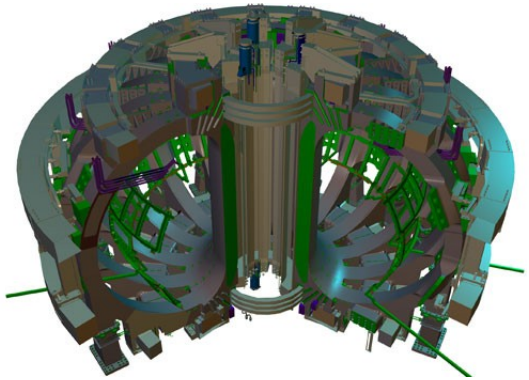
joaquim.loizu@ipp.mpg.de

S. Hudson, P. Helander, S. Lazerson, C. Nührenberg, A. Bhattacharjee

3D MHD brings together tokamaks and stellarators

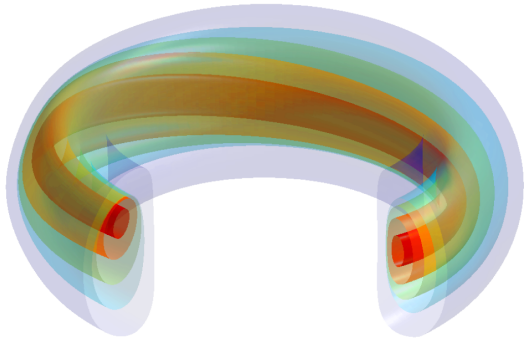


Stellarator three-dimensional topology



Tokamak non-axisymmetric designs

(magnetic ripple, resonant magnetic perturbations,…)



Tokamak MHD helical modes and bifurcations

(saturated internal kink, sawteeth)

Computational 3D MHD is a numerical challenge

- Singular current densities (current sheets) are predicted to form at rational surfaces in 3D ideal-MHD equlibria with nested surfaces.

- Non-ideal effects allow plasma relaxation towards lower-energy states: an intricate combination of flux-surfaces, magnetic islands, and chaos.

General, outstanding question:

*How to compute the **equilibrium** magnetic field that is consistent with the established **equilibrium** pressure profile?*

On the menu today

On the importance of 3D ideal-MHD

What is the origin and physical meaning of singular currents?

Can these be computed numerically?

Are there equilibria with nested resonant surfaces and smooth pressure?

What are the implications for external resonant fields?

What are the implications for internal kink theory?

On partially relaxed 3D MHD equilibria

How to compute general stellarator equilbria?

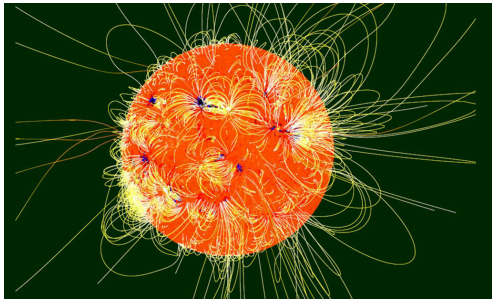
Can we predict the beta limit?

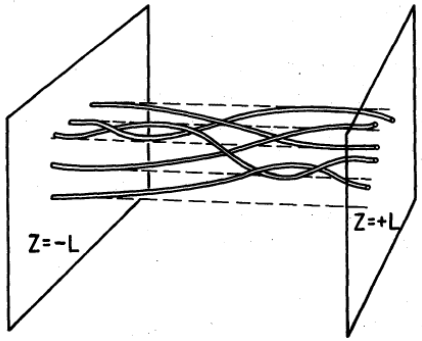
Current sheets in fusion and astrophysical plasmas

Current sheets are predicted to form in 3D ideal-MHD equlibria…

…in the solar corona, where ideal plasma convection on the surface produces field entanglement.

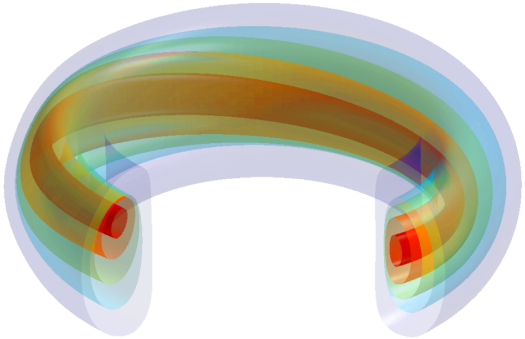
[Parker, 1972]

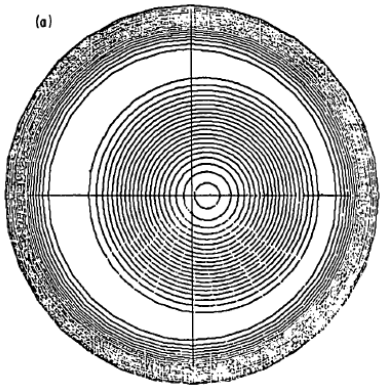




…in toroidally confined plasmas, where ideal kink instabilities bring the plasma to resonant 3D states.

[Rosenbluth, 1973]





Singular current densities come in two flavours

$$\nabla \cdot \mathbf{j} = 0 \ , \ \nabla \cdot \mathbf{B} = 0$$

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$

$$\mathbf{j} \times \mathbf{B} = \nabla p$$

$\Longrightarrow$

magnetic differential equation

$$\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$$

$$\mathbf{j}_\perp = (\mathbf{B} \times \nabla p)/B^2$$

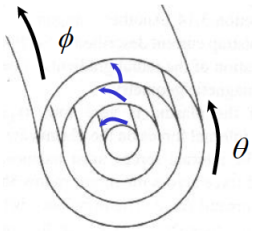
nested surfaces

Magnetic coordinates

$$(\psi, \theta, \phi)$$

$\Longrightarrow$

$$\sqrt{g}\ \mathbf{B} \cdot \nabla \equiv \iota \partial_\theta + \partial_\phi$$



Fourier decomposition

$$u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$$

$\Longrightarrow$

$$(\iota m - n) u_{mn} = i(\sqrt{g}\ \nabla \cdot \mathbf{j}_\perp)_{mn}$$

Equation type

$$x f(x) = h(x)$$

$$x \equiv \iota m - n, \ h(x) \sim p'$$

$\Longrightarrow$

$$\boxed{u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)}$$

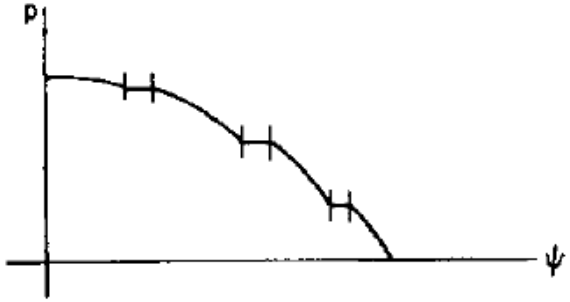
Pfirsch-Schlüter current **Dirac δ-current**

Existence of 3D ideal-MHD equilibria?

- $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$ is not the current, but the current density [A/m^2] .
- Physically-valid equilibrium if the current $J = \int_\Sigma \mathbf{j} \cdot \mathbf{d}\sigma$ across any surface is finite (weak formulation of the problem).
- Problem: Pfirsch-Schlüter current diverges across certain surfaces.
- Historical conclusion: pressure gradients cannot be supported at resonant rationals and thus pressure is either fractal or stepped.

The function p is continuous but its derivative is pathological. We have obtained an equilibrium solution without infinite currents, but at the price of a very pathological pressure distribution.

[H. Grad, 1967]



Existence of 3D ideal-MHD equilibria?

- $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$ is not the current, but the current density [A/m^2] .
- Physically-valid equilibrium if the current $J = \int_\Sigma \mathbf{j} \cdot \mathbf{d}\sigma$ across any surface is finite (weak formulation of the problem).
- Problem: Pfirsch-Schlüter current diverges across certain surfaces.
- Historical conclusion: pressure gradients cannot be supported at resonant rationals and thus pressure is either fractal or stepped.

[...] More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps. © 1996 John Wiley & Sons, Inc.

[Bruno and Laurence, 1996]

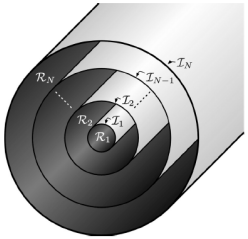
Existence of 3D ideal-MHD equilibria?

- $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$ is not the current, but the current density [A/m^2] .
- Physically-valid equilibrium if the current $J = \int_\Sigma \mathbf{j} \cdot \mathbf{d}\sigma$ across any surface is finite (weak formulation of the problem).
- Problem: Pfirsch-Schlüter current diverges across certain surfaces.
- Historical conclusion: pressure gradients cannot be supported at resonant rationals and thus pressure is either fractal or stepped.

How to compute 3D ideal equilibria with current sheets?

Are there 3D MHD equilibria with nested surfaces & smooth pressure?

Multiregion Relaxed MHD



N = 1 | N –> ∞

Taylor's theory → **MRxMHD** → **Ideal MHD**

[Dennis, 2013]

Fewer constraints → More constraints

Helicity is conserved globally | **Helicity is conserved discretely** | **Helicity is conserved locally**

$$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\, dV}_{H} - H_0\Big)$$

$$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\Big(H_l - H_{l0}\Big)\Big]$$

$$W = \int_V \Big(\frac{p}{\gamma-1} + \frac{B^2}{2}\Big)dV$$

$$\delta F = 0 \Longrightarrow \boxed{\nabla\times\mathbf{B} = \mu\mathbf{B}}$$

$$\delta F = 0 \Longrightarrow \boxed{\begin{array}{c}\nabla\times\mathbf{B} = \mu_l\mathbf{B}\\ [[p + B^2/2]] = 0\end{array}}$$

$$\delta W = 0 \Longrightarrow \boxed{\mathbf{j}\times\mathbf{B} = \nabla p}$$

[Taylor, 1974] | [Dewar, Hole, Hudson, 2006] | [Kruskal, Kulsrud, 1958]

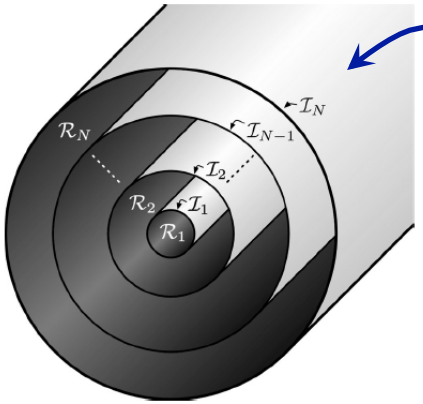
Stepped-Pressure Equilibrium Code (SPEC)

An implementation of MRxMHD

$$\mathcal{R}_l : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$$

$$\mathcal{I}_l : \quad [[p + B^2/2]] = 0$$

$l = 1, 2, \ldots N$

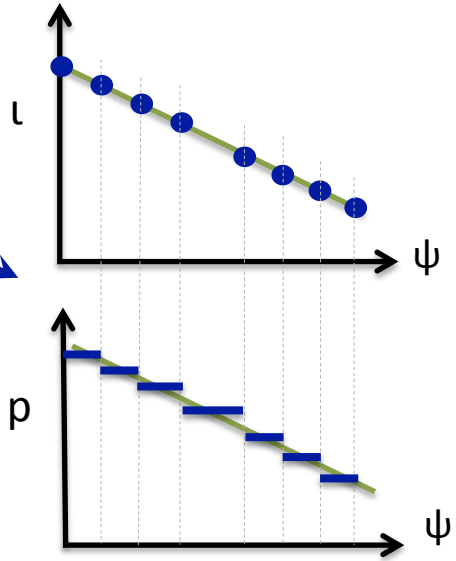


Input

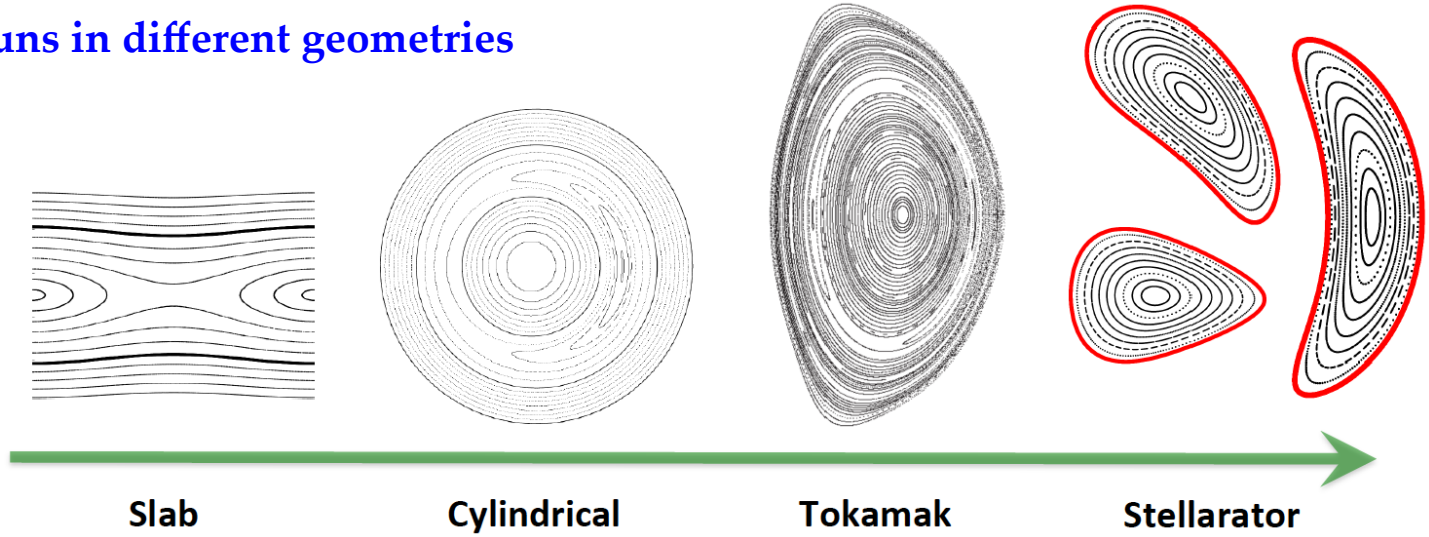
Boundary geometry
+
Two profiles

Output

B-field in each volume
+
Shape of KAM surfaces



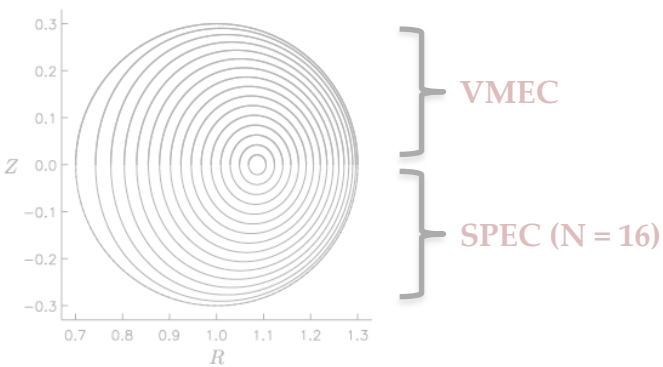
SPEC runs in different geometries



Stepped-**P**ressure **E**quilibrium **C**ode (SPEC)

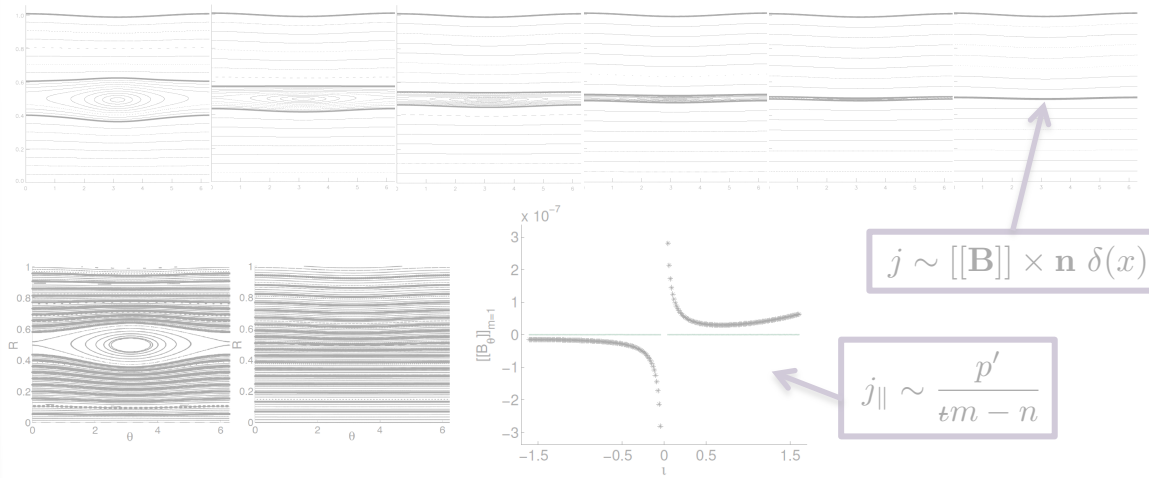
AXISYMMETRIC IDEAL MHD

[Hudson et al, PoP, 2012]



NON-AXISYMMETRIC IDEAL MHD

[Loizu et al, PoP, 2015]

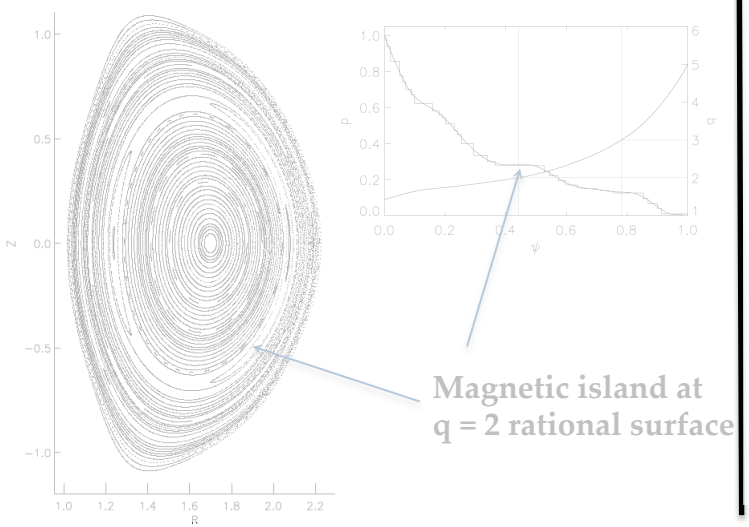


$$j \sim [[\mathbf{B}]] \times \mathbf{n}\, \delta(x)$$

$$j_{\parallel} \sim \frac{p'}{\iota m - n}$$

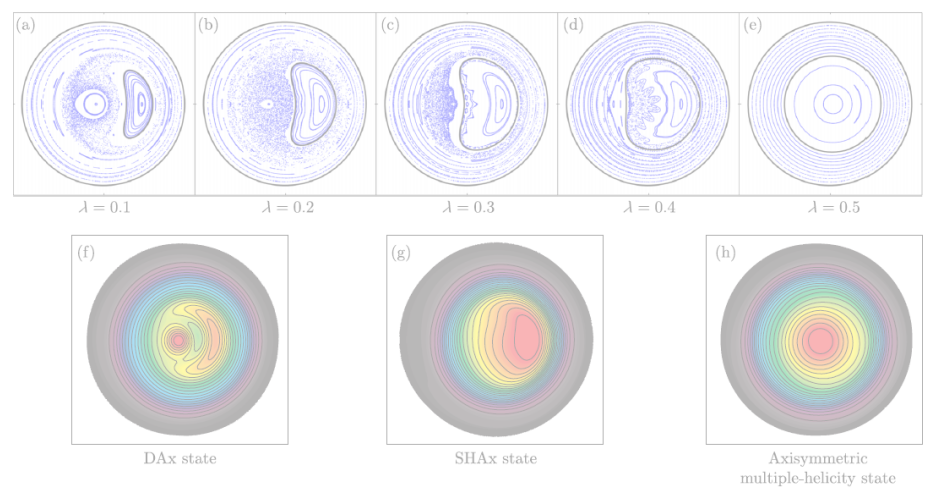
RMP IN DIII-D

[Hudson et al, PoP, 2012]

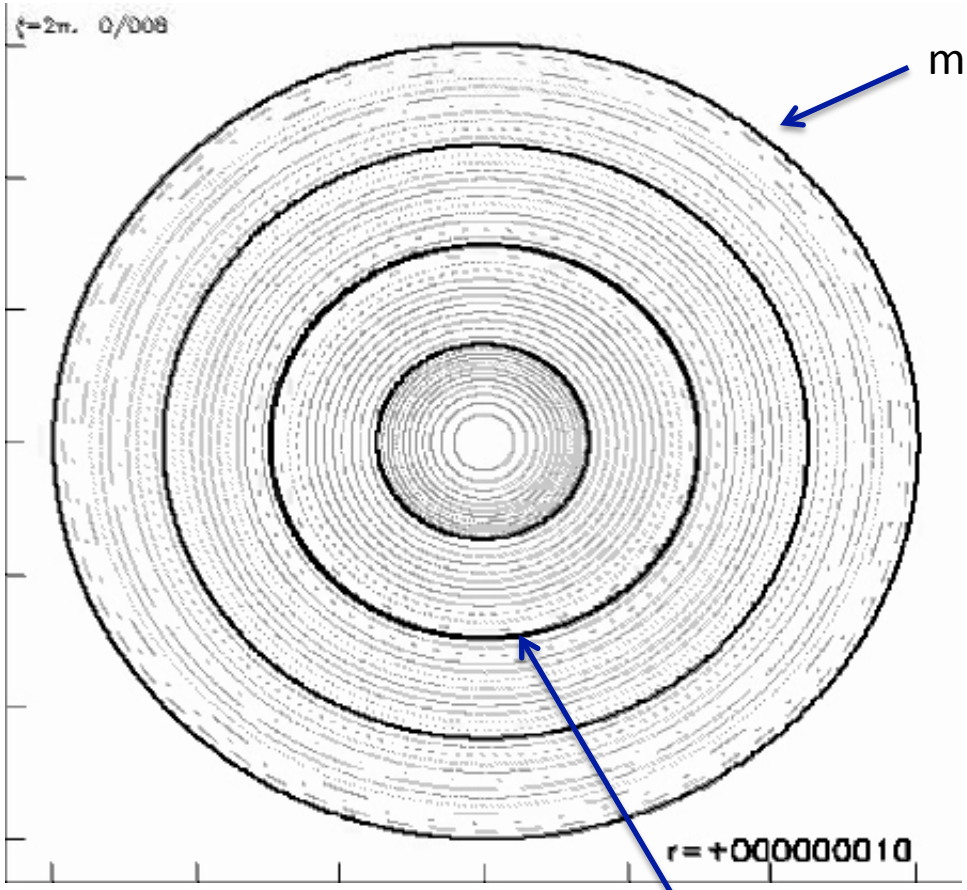


HELICAL STATES IN RFP

[Dennis et al, PRL, 2013]



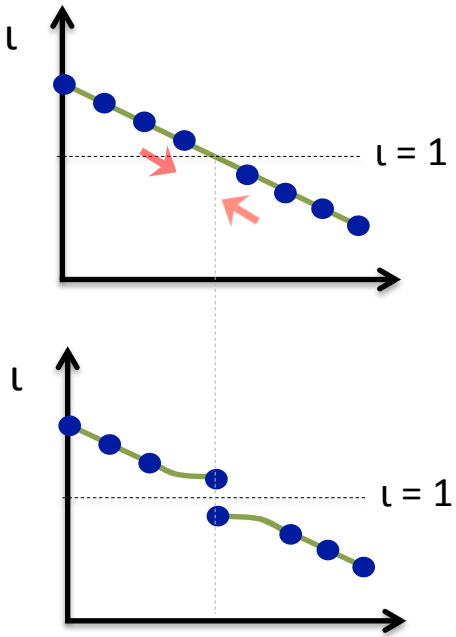
Complete shielding requires discontinuous transform



Z

R

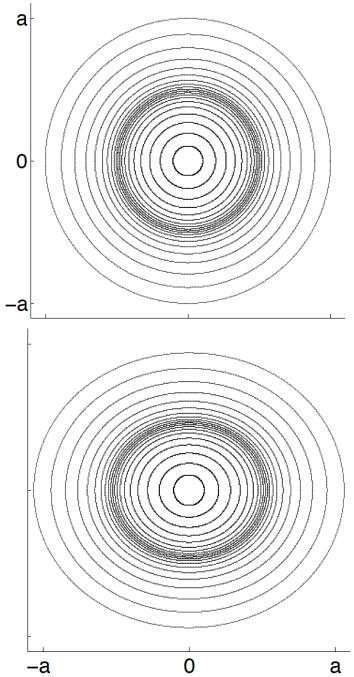
m=1, n=1 perturbation: $\xi_{11}/a = 10^{-2}$



$[[B_\theta]]_{n=1,m=1} \neq 0$

A new class of 3D MHD equilibria

- Consider equilibria with discontinuous transform across resonances.



- This class of equilbria allows for
  - Nested surfaces
  - Arbitrary 3D geometry
  - Arbitrary continuous and smooth pressure
  - Integrable current sheets

[Loizu et al, Phys Plasmas 22 090704, 2015]

- This class of ideal-MHD states may be accessed when island-healing mechanisms are at play. [Bhattacharjee PoP 1995, Hegna PoP 2012]

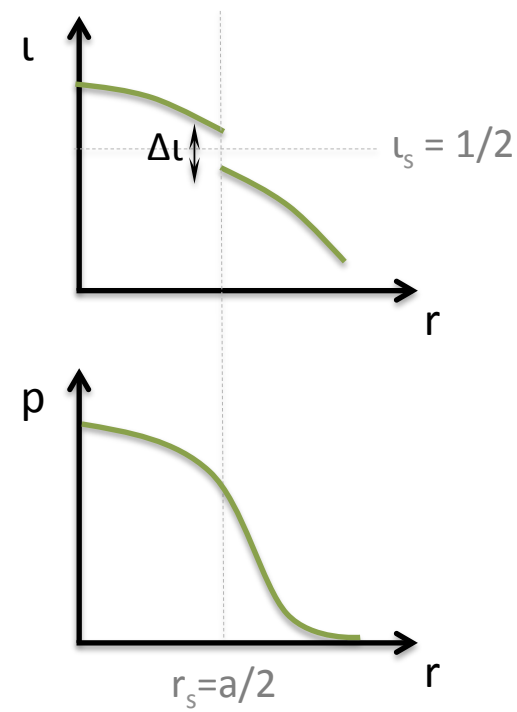
Application: resonant magnetic perturbations

- Consider a screw-pinch axisymmetric equilibrium:

$$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z^2(1+\iota^2\frac{r^2}{R^2})\right] + \frac{r\iota^2 B_z^2}{R^2} = 0$$

- Choose equilibrium profiles:

$$\iota(r) = \iota_0 - \iota_1 (r/a)^2 \pm \Delta\iota/2 \ ,$$
$$p(r) = p_0[1 - 2(r/a)^2 + (r/a)^4]$$



- **Outstanding question**: what is the ideal response to a resonant boundary perturbation ?

[Turnbull et al, PoP 2013; Reiman et al, NF 2015]

Ideal linear response to an RMP at β = 0

- Perturbed equilibrium satisfies:

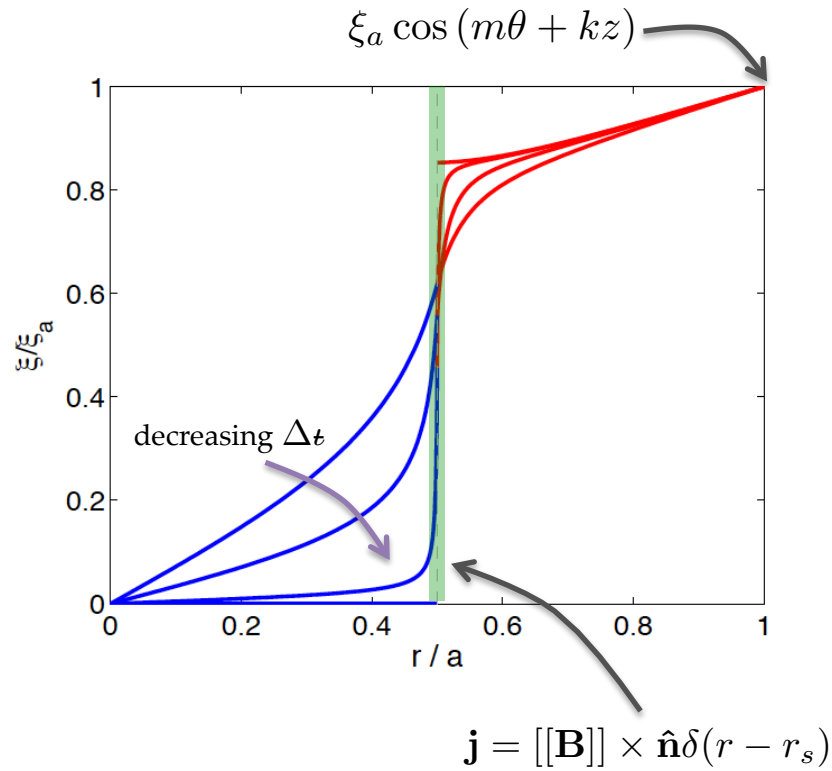
$$\delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 + \mathbf{j} \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$

- Reduces to Newcomb equation:

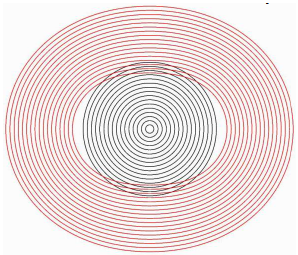
$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = B_z^2(\iota - \iota_s)^2\bar{k}r^2 \ ,$$

$$g = \frac{f}{r^2}(k^2r^2 + m^2 - 1) + B_z^2(\iota_s^2 - \iota^2)2\bar{k}^2\iota_s^2 r$$

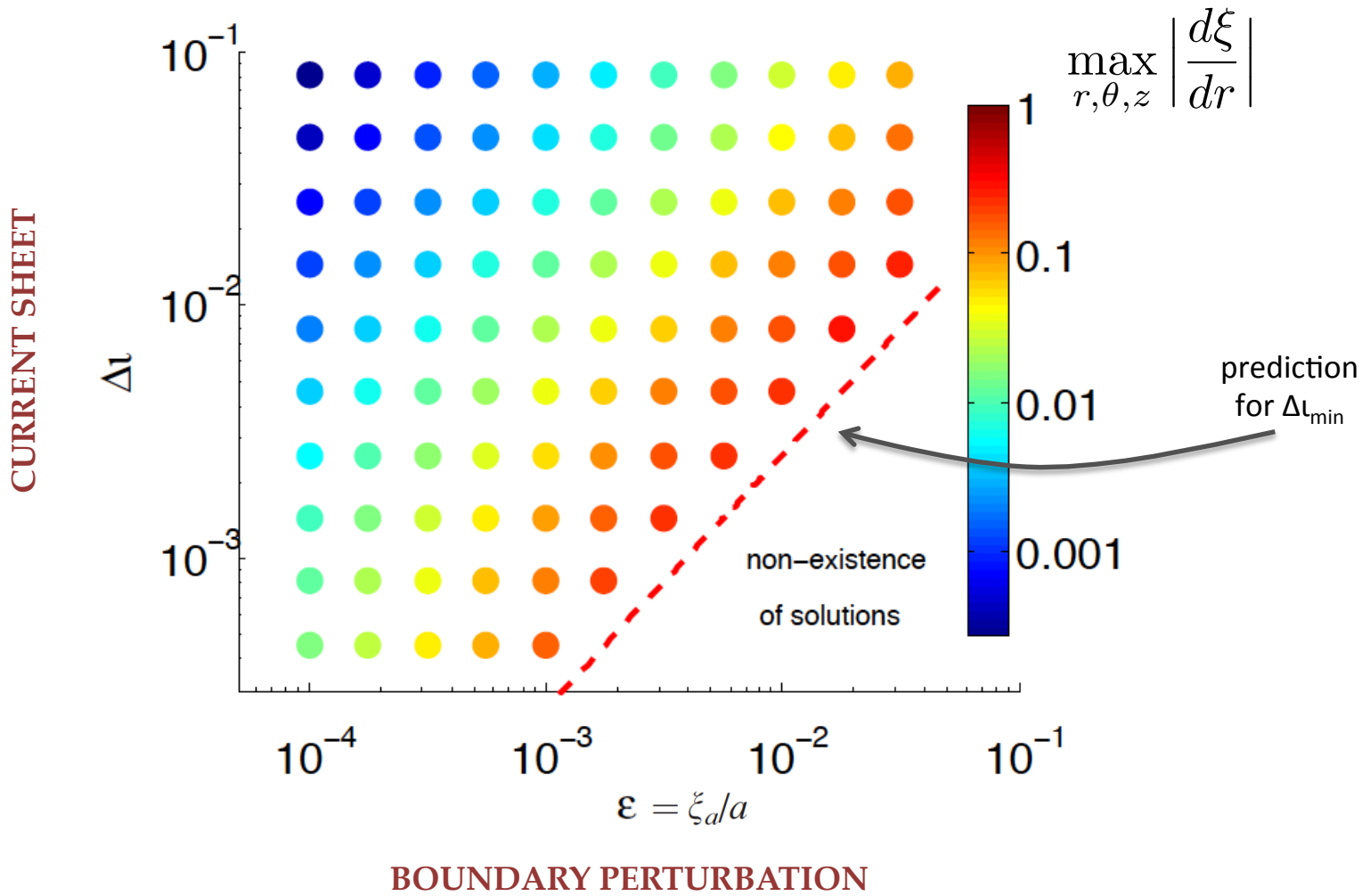


- *Sine qua non* condition for the existence of equilibria: $|\xi'| \leq 1$



- Implies **minimum current sheet**: $\Delta\iota \geq \Delta\iota_{min} = 2\iota_s'\xi_s$

Existence space is reproduced in nonlinear calculations

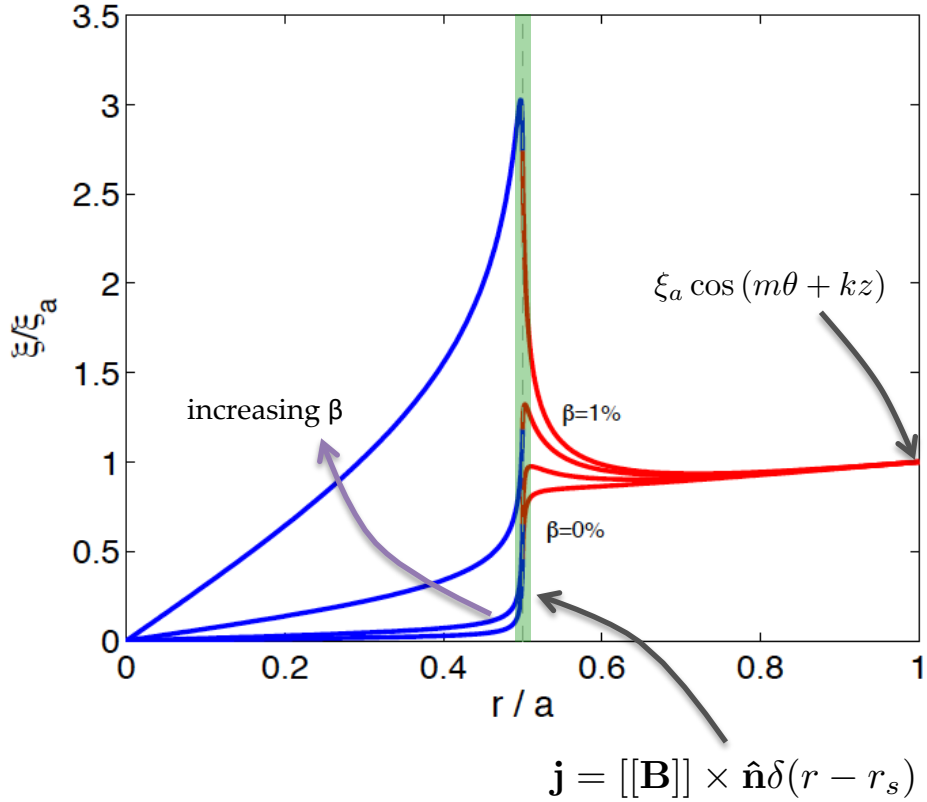


Ideal linear response to an RMP at β > 0

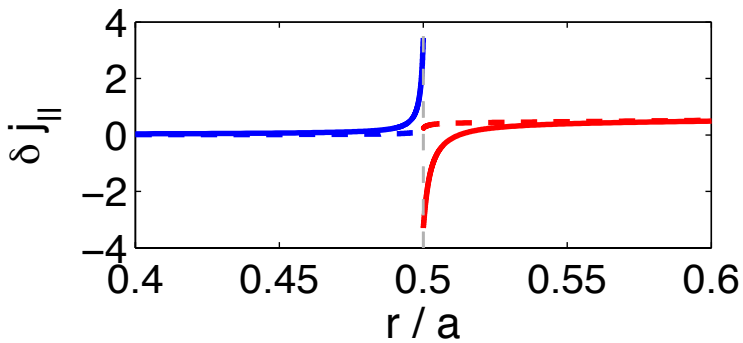
- Solve Newcomb equation:

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = f|_{\beta=0}\ ,$$
$$g = g|_{\beta=0} + \bar{k}\iota_s^2 r^3 p'$$

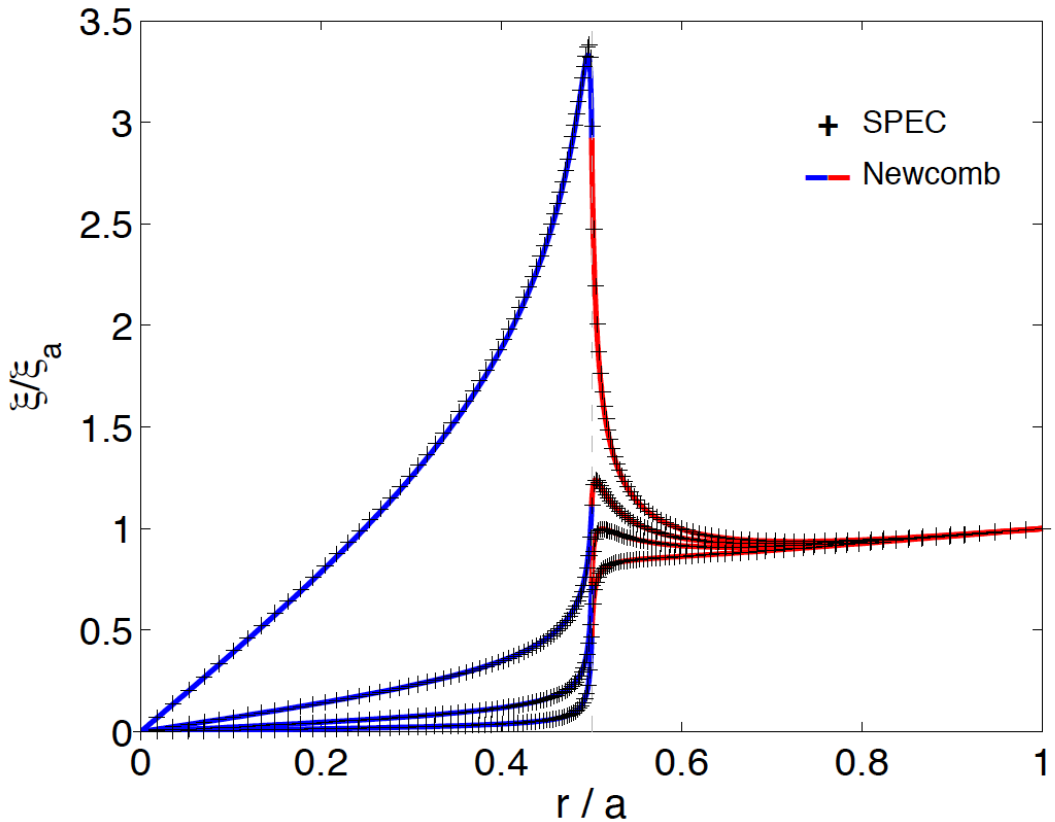


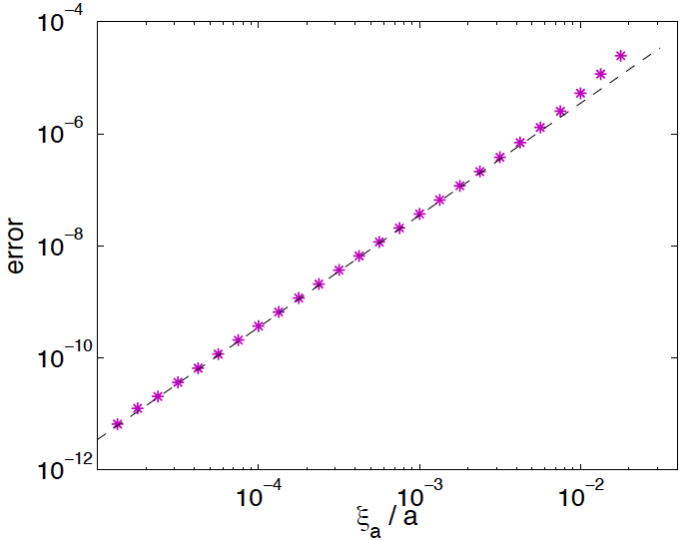
- Pfirsch-Schlutter current:



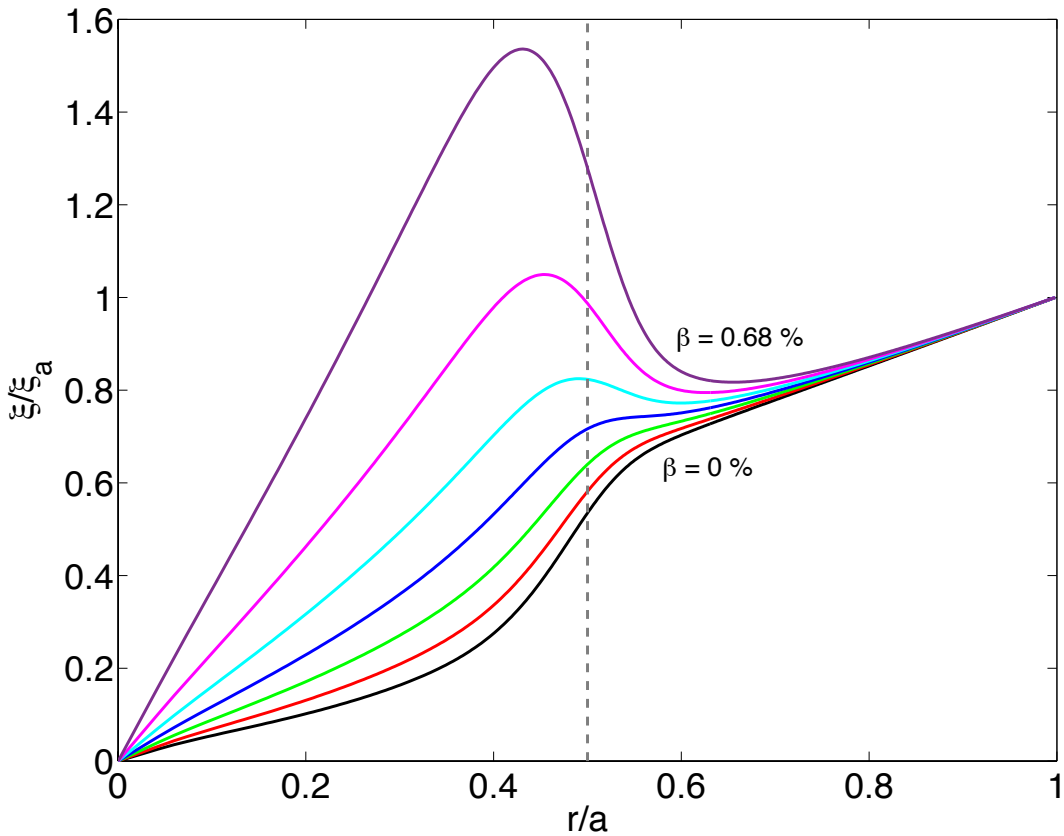
- Pressure-driven **amplification** and **penetration** of RMP in ideal MHD!

SPEC nonlinear calculations exactly verified



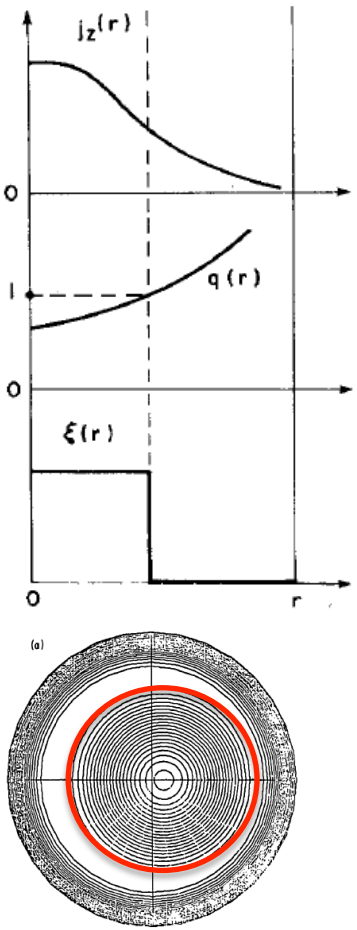


VMEC qualitatively reproduces same behaviour



An exact agreement with Newcomb's solutions may require explicit handling of discontinuities in the magnetic field.

Saturated ideal kink satisfies sine qua non condition



nonlinear solution for the displacement:

$$\xi(x,\theta) = \int_0^x dx \left(\frac{x}{[f(x)+g(\theta)]^{1/2}} - 1 \right) + h(\theta)$$

with

$$\Lambda^{(g)} \equiv \int_0^\infty df \left[\left(\oint d\theta (f+g)^{-1/2} \right) - f^{1/2} - \frac{1}{2} \oint g \, d\theta / f^{1/2} \right] + \oint g \cos\theta \, d\theta$$

$$\xi_a = \left(-\int_0^s g_1 \, dr \Big/ 3s^3 [(\mathbf{k}\cdot\mathbf{B})']_s^2 \right) \Lambda_{max}^{-1}$$

$$\oint d\theta [f(x)+g(\theta)]^{-1/2} = x^{-1}$$

[Rosenbluth et al, Physics of Fluids 16, 1973]

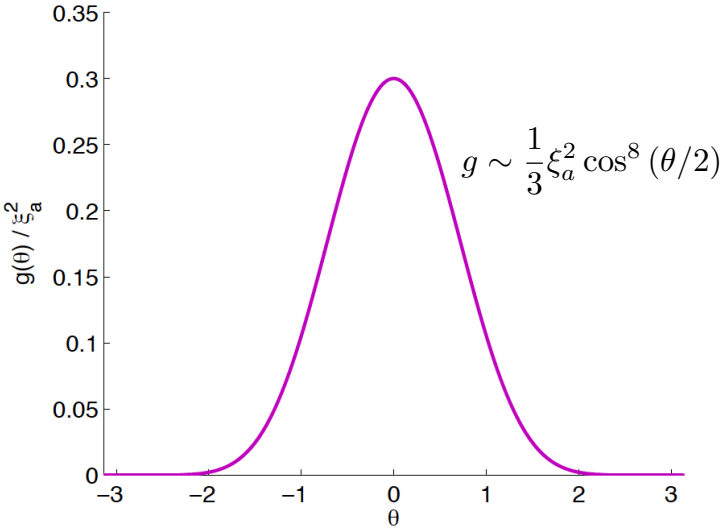
Matching to linear theory, $h(\theta) = \xi_a \cos(\theta)/2$

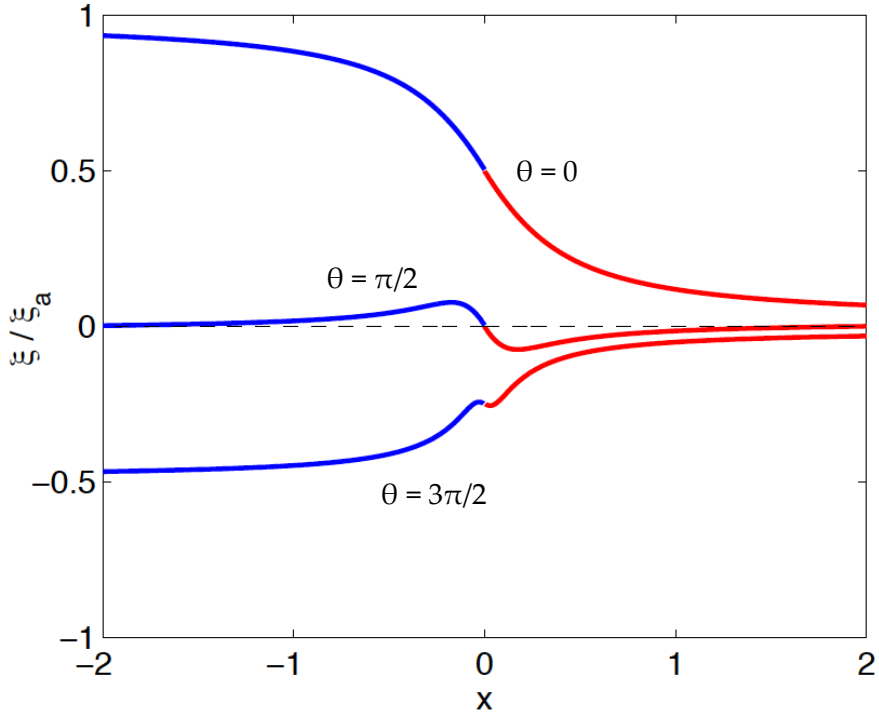
Maximize $\Lambda[g]$ to obtain $g(\theta)$

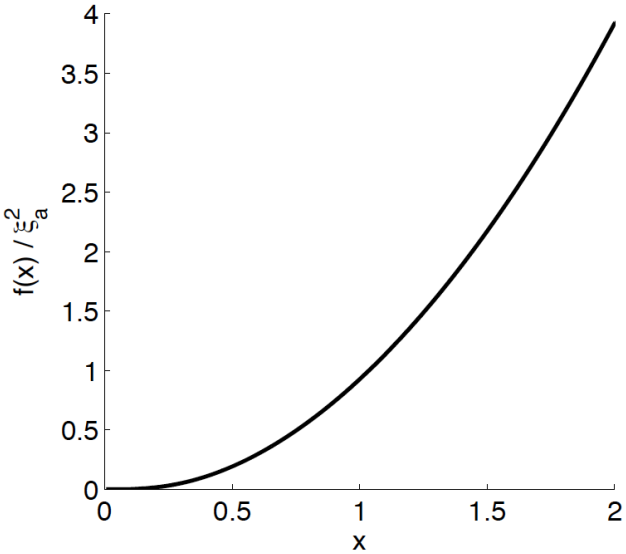
Solve for $f(x)$ given $g(\theta)$

Notice: $\lim_{x\to 0} \frac{\partial \xi}{\partial x} = -1$ thus *sine qua non* condition marginally satisfied!

Saturated ideal kink has discontinuous transform







Jump in magnetic field (current sheet):

$$[[B_\theta]] = 2\frac{r_s}{R} B_z \iota_s' \xi_a \sqrt{g(\theta)/\xi_a^2}$$

Jump in rotational transform:

$$\Delta\iota = 2\iota_s' \xi_a \oint \sqrt{g(\theta)/\xi_a^2}\, d\theta \approx 0.44\, \iota_s' \xi_a$$

Summary and perspectives

On the importance of 3D ideal-MHD

- Exact computation of singular current densities.

  [Loizu et al, Phys Plasmas 22 022501, 2015]

- New class of 3D MHD equilibria allows for nested surfaces and smooth pressure.

  [Loizu et al, Phys Plasmas 22 090704, 2015]

- Novel prediction: amplification and penetration of RMP even within ideal MHD.

  [Loizu et al, Phys Plasmas 23, 055703, 2016]

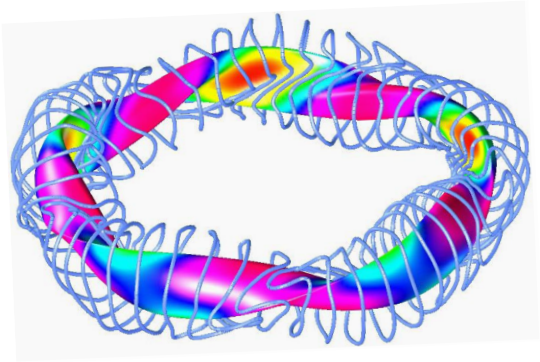
- Ideal kink states [Rosenbluth et al, 1973] belong to this class of equilibria.

- What generally determines the value of $\Delta\iota$ remains to be investigated.

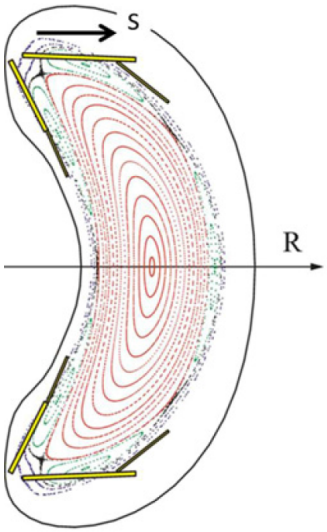
On partially relaxed 3D MHD equilibria

Wendelstein 7-X

- W7-X was optimised with respect to [Grieger et al, Phys Fluids B, 1992]
  - Good MHD stability
  - Improved neoclassical confinement
  - Good fast-particle confinement
  - Low bootsrap current



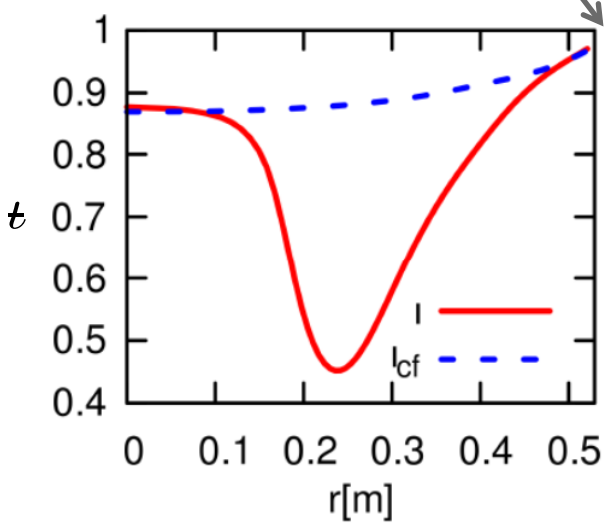
- Main goals of W7-X:
  - Large triple product $nT\tau_E \sim 10^{20}$ m^{-3} keV s
  - Quasi-stationary discharges *c.a.* 30 min
  - Operation with an island divertor *e.g.* ι_a=5/5



Two problems require physics understanding

Effect of finite bootstrap current

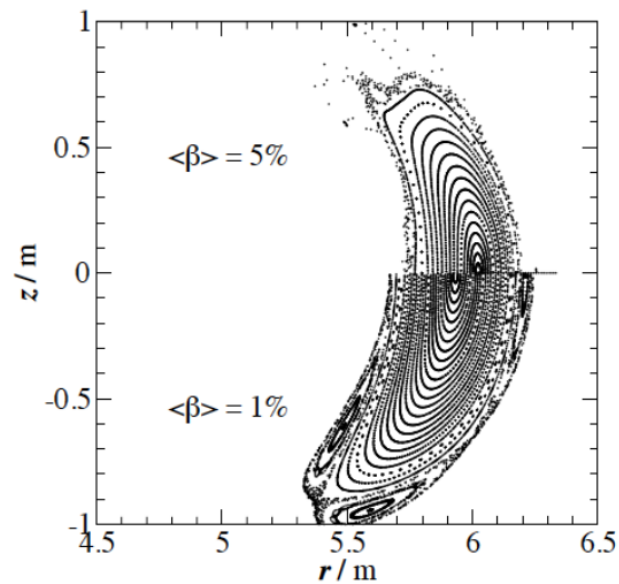
- Perturbation in ι_a is harmful for the SOL.
- ECCD clamping of ι_a may open islands.
- Need equilibrium code with islands.



[J. Geiger et al, NF, 2015]

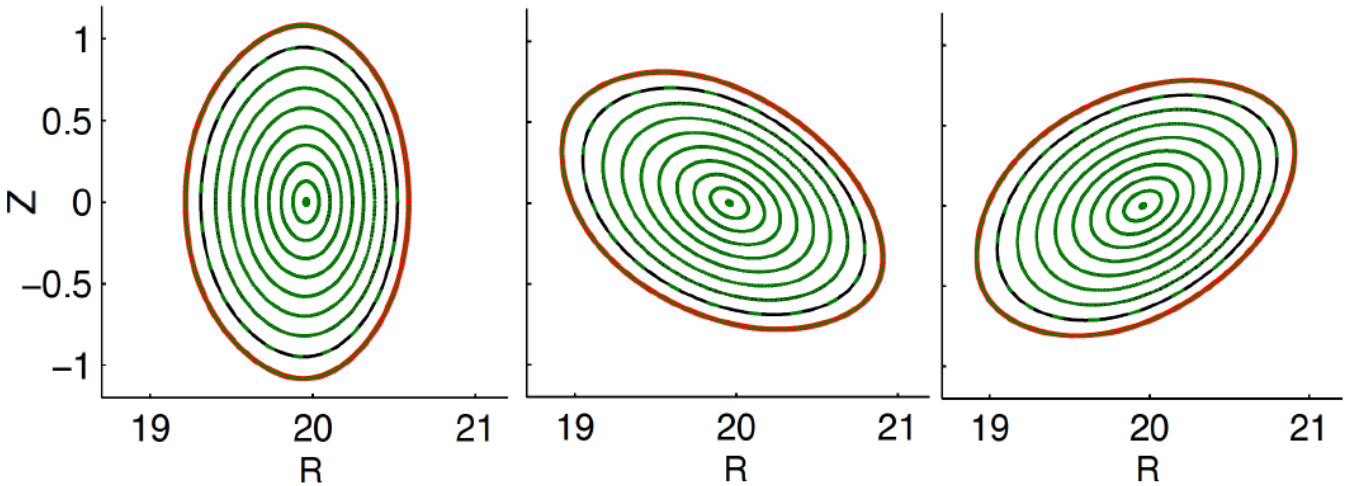
Unknown β-limit

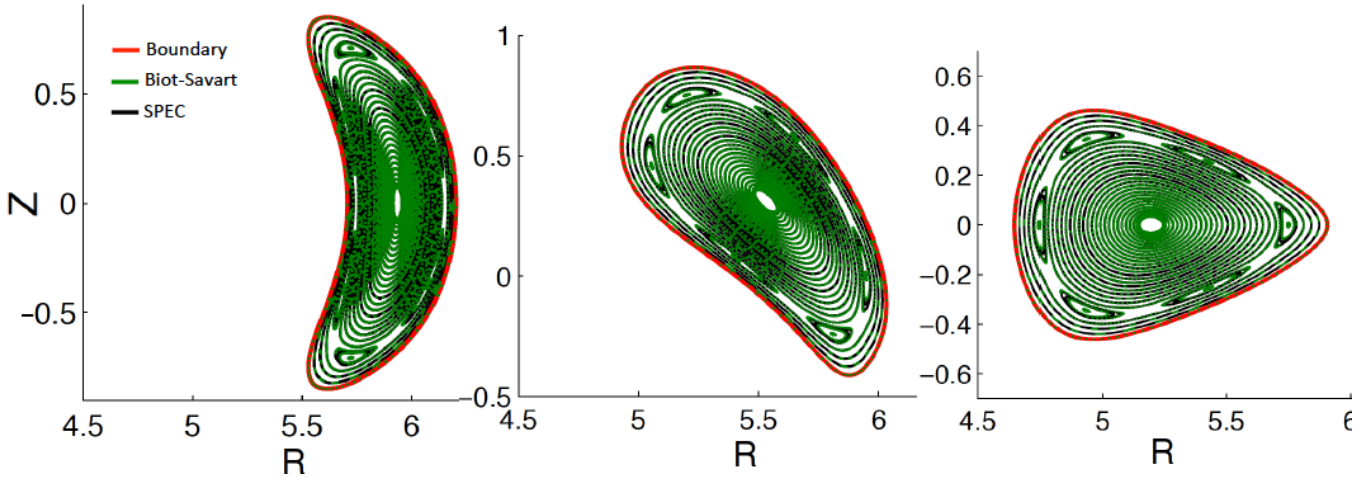
- Probably determined by the equilibrium.
- Possible degradiation at high β.
- Need for a robust, reliable, and fast code.

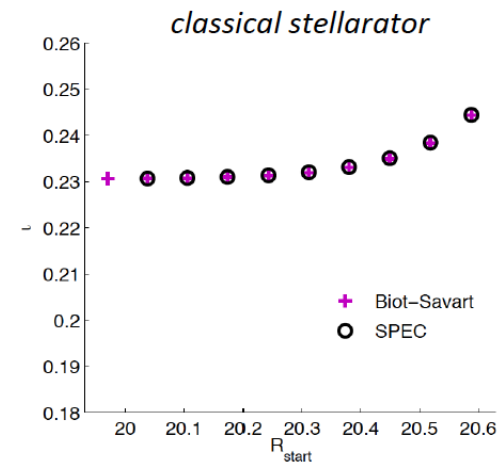


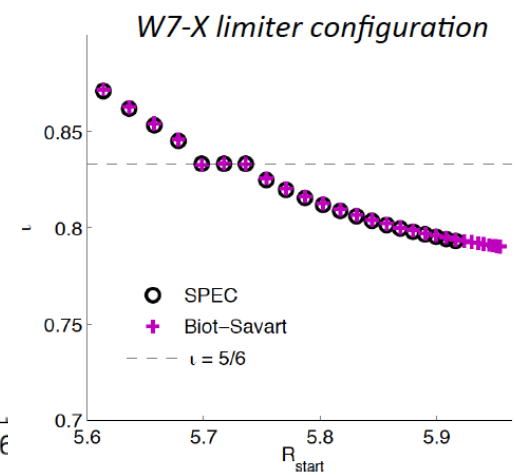
[M. Drevlak et al, NF, 2005]

SPEC used first to compute stellarator vacuum fields





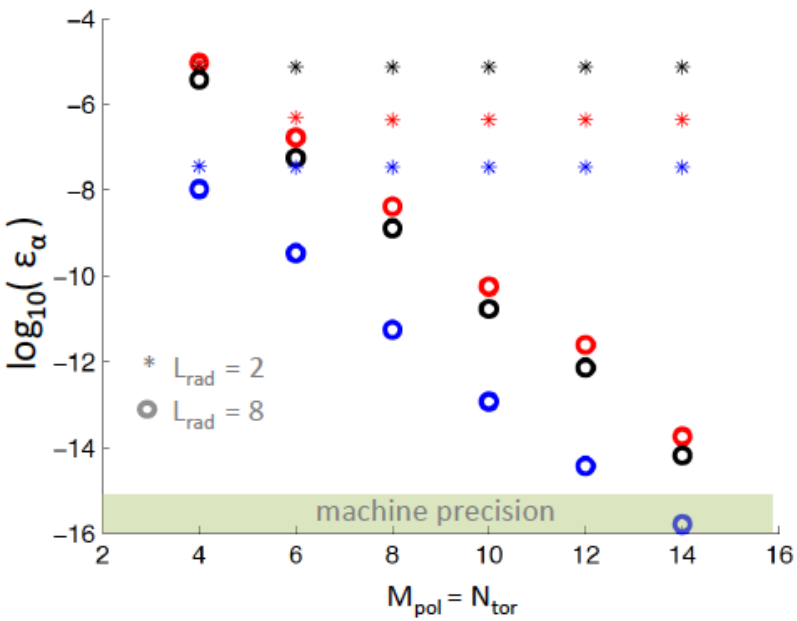




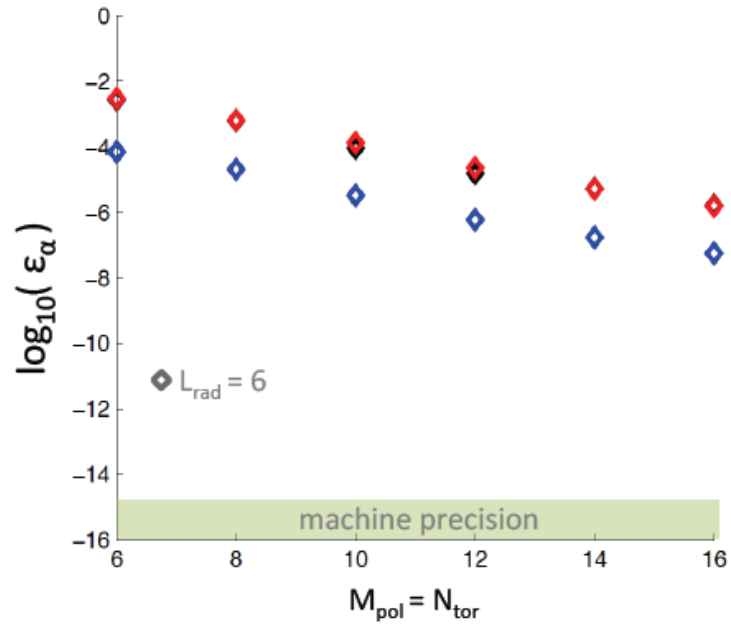
SPEC verified for stellarator vacuum fields

$$\varepsilon_\alpha = \frac{1}{V}\int ds \oint d\theta \oint d\varphi\ \mathcal{J}(s,\theta,\varphi)|(\nabla\times\mathbf{B}-\mu\mathbf{B})\cdot\nabla\alpha|$$

classical $l = 2$ stellarator

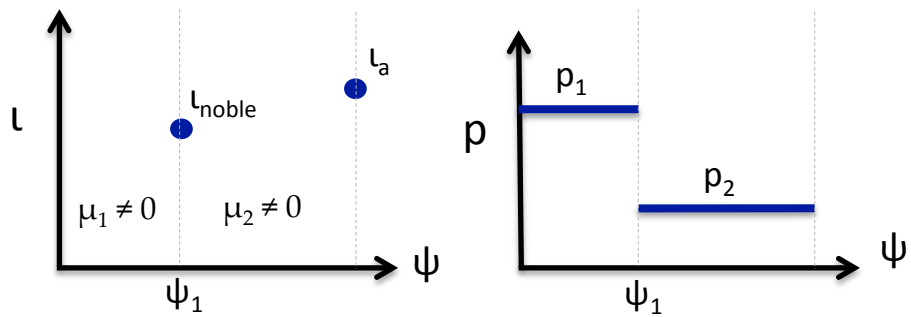


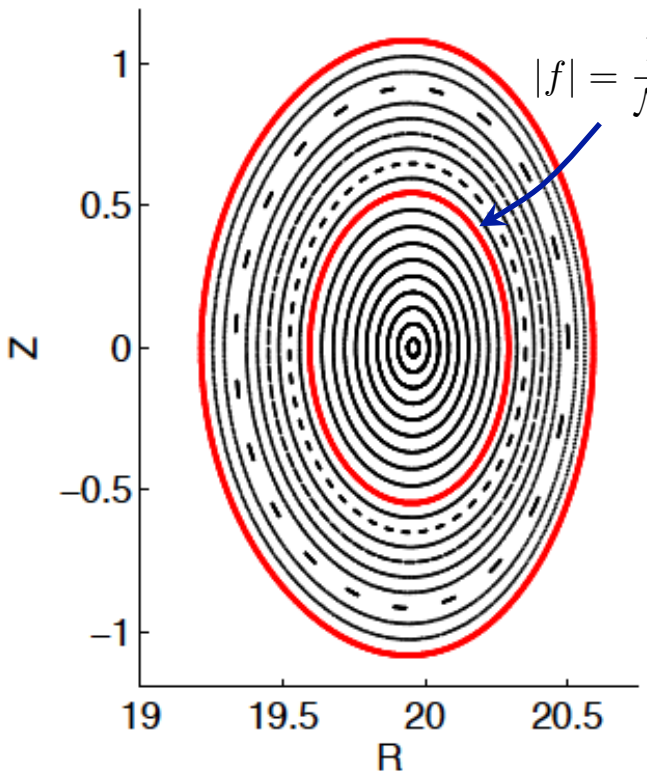
W7-X limiter configuration



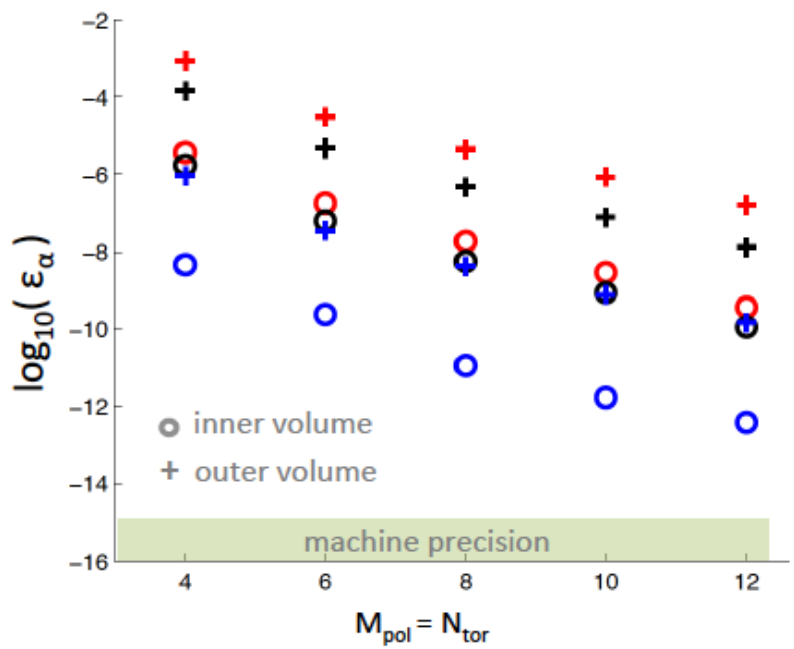
SPEC verified for multi-volume stellarator equilbria

- N = 2 volumes
- Impose $\iota^{+/-} = \iota_{noble}$
- Impose edge ι_a and ψ_1
- Impose $p_1 = p_2 = 0$



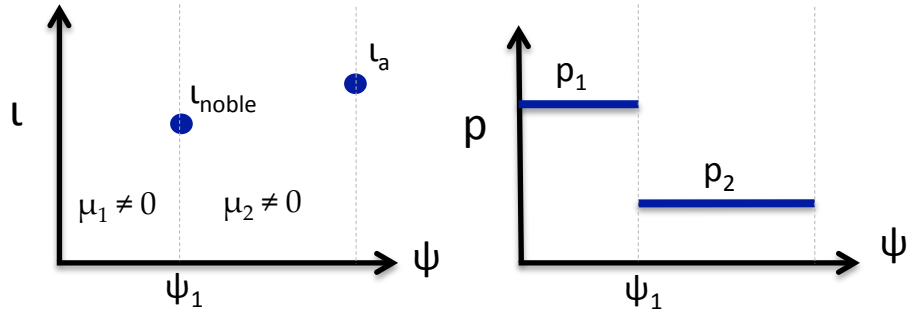


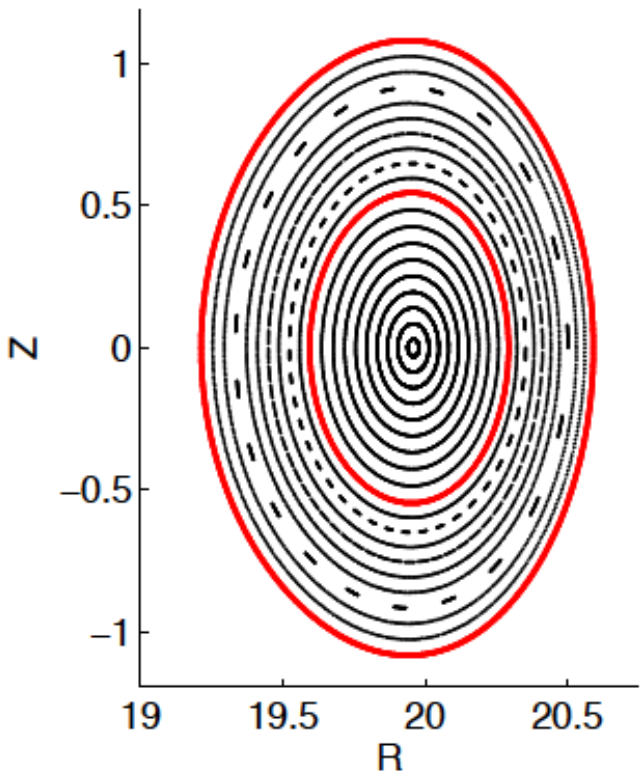
$$|f| = \frac{1}{\mathcal{N}} \sum_{mn} [[p + \frac{B^2}{2}]]_{mn} \sim 10^{-16}$$

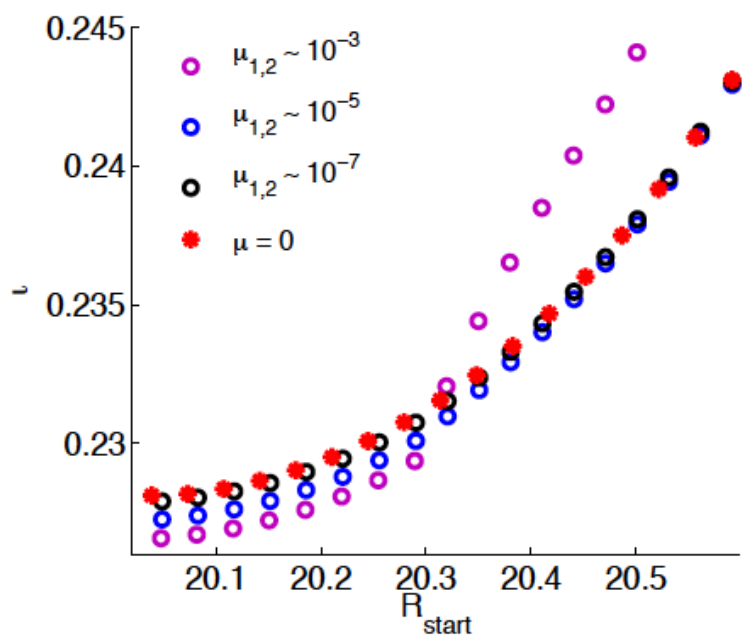


SPEC can find zero-current stellarator equilibria

- Iterate on ι_a and ψ_1 to constraint current $\mu_{1,2}$
- Example: recover vacuum state







Magnetic islands and chaos can appear at finite β

- Ideal β-limit in an *l*=2 stellarator

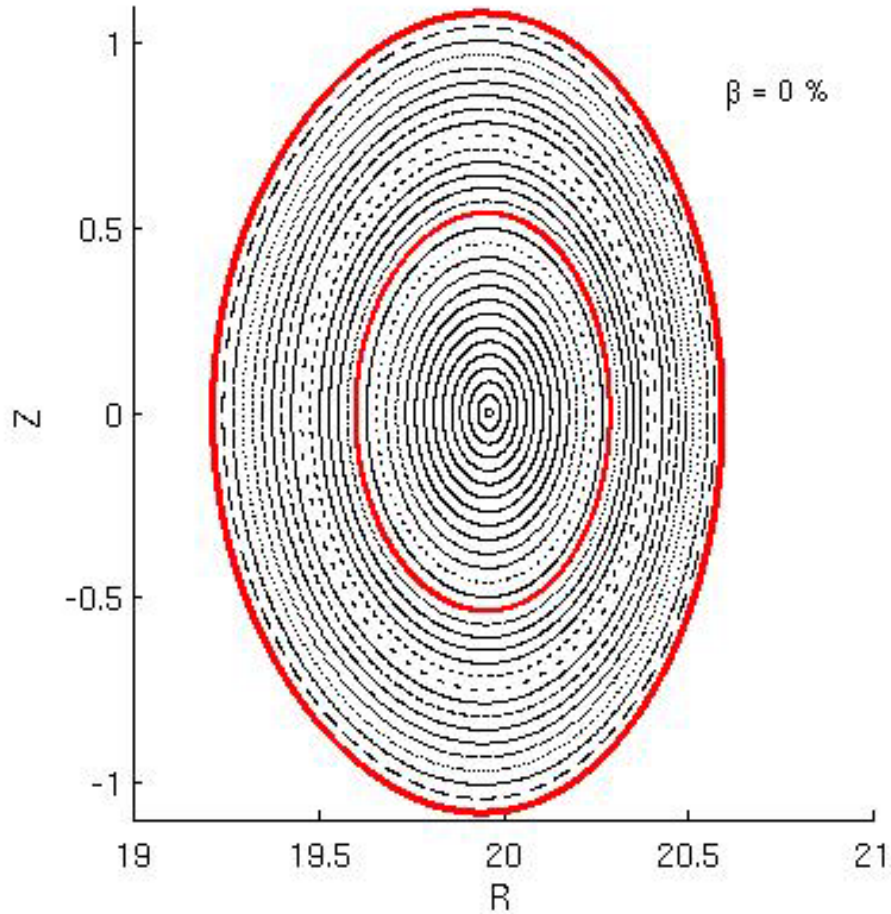
$$\beta \sim \epsilon\, \iota^2$$

here gives β ≈ 0.2% .

[J. P. Freidberg, Ideal MHD, p.273, 2014]

- Shafranov shift is clearly observed.
- For β > 0.1%, islands start to appear.
- For β > 0.2%, chaos emerges.

Free-boundary SPEC will allow studying β-limit in stellarators



Summary and perspectives

On the importance of 3D ideal-MHD

- First numerical proof of the existence of singular current densities.

  [Loizu et al, Phys Plasmas 22 022501, 2015]

- New class of 3D MHD equilibria allows for nested surfaces and smooth pressure.

  [Loizu et al, Phys Plasmas 22 090704, 2015]

- Novel prediction: amplification and penetration of RMP even within ideal MHD.

  [Loizu et al, Phys Plasmas 23, 055703, 2016]

- Ideal kink states [Rosenbluth et al, 1973] belong to this class of equilibria.

- What determines the value of $\Delta\iota$ remains to be investigated.

On partially relaxed 3D MHD equilibria

- Stellarator equilibria with islands have been computed and verified with SPEC.

  [Loizu et al, Phys Plasmas, submitted]

- Free-boundary version of SPEC will allow to study, e.g., stellarator β-limit.