Wave kinetics of drift-wave turbulence and zonal flows beyond the ray approximation

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Highlights: phase-space models for zonal flow dynamics

- Traditional wave kinetic equation (tWKE):
  - derived based on over-simplifications;
  - fails to properly capture the zonostrophic instability (ZI).
- Improved wave kinetic equation (iWKE):
  - systematically derived geometrical-optics (GO) model;
  - semi-quantitatively captures ZI;
  - does not show zonal flow (ZF) oscillations or deterioration.
- Full-wave (Wigner–Moyal, WM) model:
  - no GO (ray) approximation, subsumes iWKE;
  - produces ZF oscillations and deterioration.

Outline

1. Motivation: zonal flows and wave kinetics
2. Tutorial: quantum mechanics (wave physics) in phase space
3. Models: phase-space models for zonal flow dynamics
4. Results: simulations with different models

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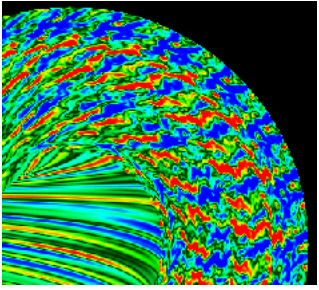
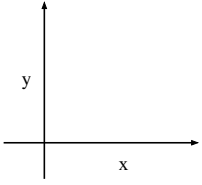
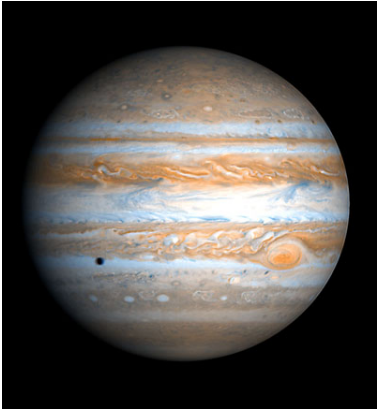
Zonal flows in nature



Magnetized plasmas
(PC: GYRO team)

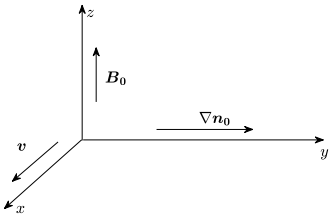


Slab coordinates



Jupiter atmosphere
(PC: NASA Cassini)

Generalized Hasegawa-Mima equation (gHME)



- Standard HME (in normalized units)

$$\partial_t w + \mathbf{v} \cdot \nabla w + \beta \partial_x \varphi = f - D, \tag{1}$$

with $\mathbf{v} = \hat{\mathbf{z}} \times \nabla\varphi$, and generalized vorticity

$$w = (\nabla^2 - \hat{a})\varphi. \tag{2}$$

$\hat{a} = 1$ for drift waves (DWs); $\hat{a} = 0$ for ZFs[1].

- Conserves enstrophy and energy (when $f, D = 0$):

$$Z = \frac{1}{2}\int \mathrm{d}^2x\, w^2,\ E = -\frac{1}{2}\int \mathrm{d}^2x\, w\varphi. \tag{3}$$

[1] G. W. Hammett et al., Plasma Phys. Control. Fusion 35, 973 (1993).

Traditional wave kinetic equation (tWKE)

- Particle-field system[2]:
  - DWs as quasi-particles (driftons). Liouville equation

$$\partial_t W = \{\mathcal{H}, W\}, \tag{4}$$

    W: "Wigner function"; $\mathcal{H}$: Hamiltonian (dispersion).
  - ZF velocity as a field.
- Intuitive and statistical derivative from gHME.
- Questionable features:
  - Small-scale ZFs observed in tWKE simulations[3].
  - Conserves DW enstrophy, **not** total enstrophy.
- Warrants **systematic** re-examination and re-derivation.

[2] A. I. Smolyakov and P. H. Diamond, Phys. Plasmas 6, 4410 (1999).
[3] R. Trines et al., Phys. Rev. Lett. 94, 165002 (2005).

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Weyl symbol and Moyal product

- Weyl transform: from (Hermitian) operator $\hat{A}$ in Hilbert space to (real) function $A(\mathbf{x}, \mathbf{p})$ (Weyl symbol) in phase space,

$$A(\mathbf{x}, \mathbf{p}) = \int \mathrm{d}^n s \, e^{-i\mathbf{p}\cdot\mathbf{s}} \left\langle \mathbf{x} + \hbar\mathbf{s}/2 \right| \hat{A} \left| \mathbf{x} - \hbar\mathbf{s}/2 \right\rangle . \tag{5}$$

- Operator product $\hat{C} = \hat{A}\hat{B}$ becomes Moyal product

$$C(\mathbf{x}, \mathbf{p}) = A(\mathbf{x}, \mathbf{p}) \star B(\mathbf{x}, \mathbf{p}) \doteq A(\mathbf{x}, \mathbf{p}) \exp\left(i\hbar\hat{\mathcal{L}}/2\right) B(\mathbf{x}, \mathbf{p}), \tag{6a}$$

$$\hat{\mathcal{L}} \doteq \overleftarrow{\partial_{\mathbf{x}}} \cdot \overrightarrow{\partial_{\mathbf{p}}} - \overleftarrow{\partial_{\mathbf{p}}} \cdot \overrightarrow{\partial_{\mathbf{x}}} = \{\cdot, \cdot\}. \tag{6b}$$

- Commutators become Moyal brackets

$$\hbar\{\{A, B\}\} \doteq -i(A \star B - B \star A) = 2A \sin\left(\hbar\hat{\mathcal{L}}/2\right) B, \tag{7a}$$

$$[[A, B]] \doteq A \star B + B \star A = 2A \cos\left(\hbar\hat{\mathcal{L}}/2\right) B. \tag{7b}$$

Wigner function and time evolution

- Schrödinger equation, (for classical waves: $\hbar \to \epsilon$, the GO parameter)

$$i\hbar\partial_t |\psi\rangle = \hat{H} |\psi\rangle . \tag{8}$$

- von Neumann equation, (density operator $\hat{\rho} \doteq |\psi\rangle \langle\psi|$)

$$i\hbar\partial_t \hat{\rho} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H}. \tag{9}$$

- Wigner function W (Weyl symbol of $\hat{\rho}$) satisfies WM equation[4,5],

$$\partial_t W = \{\{H, W\}\}. \tag{10}$$

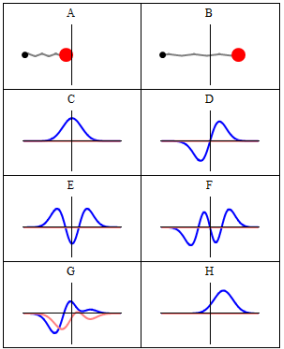
- Classical (GO) limit, $\hbar, \epsilon \to 0$: Liouville (wave kinetic) equation,

$$\partial_t W \simeq H\hat{\mathcal{L}}W = \{H, W\}. \tag{11}$$

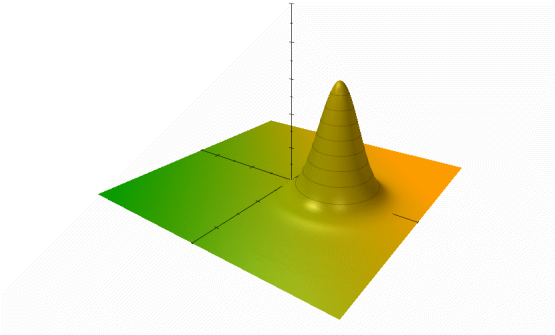
[4] E. Wigner, Phys. Rev. 40, 749 (1932).
[5] J. E. Moyal, Proc. Cambridge Philosoph. Soc. 45, 99 (1949).

Example: quantum harmonic oscillator

- Coherent state: Wigner function rotates in phase space



PC: Wikipedia



PC: T. L. Curtright

- Many (potential) applications in plasma physics[6,7]!

[6] E. R. Tracy et al., Ray Tracing and Beyond: Phase Space Methods in Plasma Wave Theory (2014).
[7] D. E. Ruiz, PhD thesis (Princeton University, 2017).

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Quasilinear approximation (standard)

- Zonal average $\bar{g} \doteq \int \mathrm{d}x\, g/L_x$, $w = \bar{w} + \tilde{w}$, $\bar{w} = -\partial_y U$,

$$\partial_t \tilde{w} + U\partial_x \tilde{w} + (\beta - U'')\partial_x \tilde{\varphi} + f_{NL} = 0, \tag{12a}$$

$$\partial_t U + \partial_y \overline{\tilde{v}_x \tilde{v}_y} = 0. \tag{12b}$$

- Neglecting eddy-eddy interaction $f_{NL} \doteq \tilde{\mathbf{v}} \cdot \nabla \tilde{w} - \overline{\tilde{\mathbf{v}} \cdot \nabla \tilde{w}}$,

$$i\partial_t \left|\tilde{w}\right\rangle = \hat{H} \left|\tilde{w}\right\rangle, \tag{13}$$

with a **non-Hermitian** Hamiltonian, ($\hat{p}_D^2 \doteq \hat{p}_x^2 + \hat{p}_y^2 + 1$)

$$\hat{H} = (\hat{U}'' - \beta)\hat{p}_x \hat{p}_D^{-2} + \hat{U}\hat{p}_x. \tag{14}$$

- Total energy and enstrophy still conserved.

Wigner-Moyal formulation: full-wave model

- (Zonal averaged) WM equations[8],

$$\partial_t W = \{\{\mathcal{H}, W\}\} + [[\Gamma, W]], \tag{15a}$$

$$\partial_t U = \partial_y \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{1}{p_D^2} \star p_x p_y W \star \frac{1}{p_D^2}, \tag{15b}$$

  with $W = W(t, y, \mathbf{p})$, and

$$\mathcal{H} = -\beta p_x / p_D^2 + p_x U + [[U'', p_x/p_D^2]]/2, \tag{16a}$$

$$\Gamma = \{\{U'', p_x/p_D^2\}\}/2. \tag{16b}$$

- Total energy and enstrophy still conserved.

[8] D. E. Ruiz et al., Phys. Plasmas 23, 122304 (2016).

GO limit: iWKE, not tWKE

- iWKE obtained in the GO limit [$\lambda_{\mathrm{ZF}} \gg \max(\lambda_{\mathrm{DW}}, 1)$],

$$\partial_t W = \{\mathcal{H}, W\} + 2\Gamma W, \tag{17a}$$

$$\partial_t U = \partial_y \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \frac{p_x p_y}{p_D^4} W, \tag{17b}$$

  with

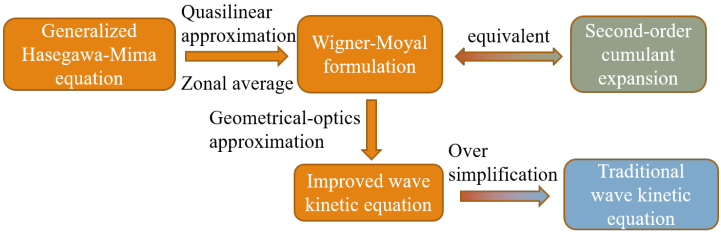
$$\mathcal{H} = (U'' - \beta) p_x / p_D^2 + p_x U, \ \Gamma = -U''' p_x p_y / p_D^4. \tag{18}$$

- Total energy and enstrophy still conserved.
- In contrast, for tWKE,

$$\mathcal{H} = -\beta p_x / p_D^2 + p_x U, \ \Gamma = 0. \tag{19}$$

- Total enstrophy **no longer** conserved.

Summary: hierarchy of models



- Second-order cumulant expansion[9] (CE2):
  - popular in geophysics;
  - correlation function: density matrix, in double-configuration space;
  - equivalent to WM, yet less intuitive.

[9] J. B. Parker, PhD thesis (Princeton University, 2014).

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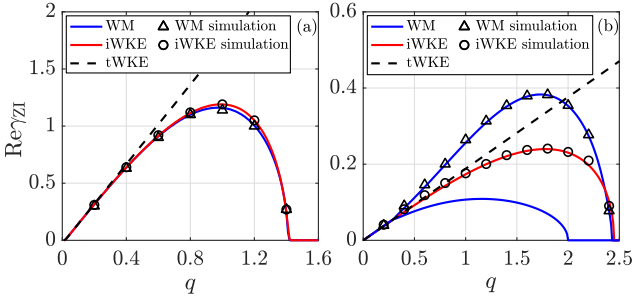
tWKE: ultraviolet catastrophe

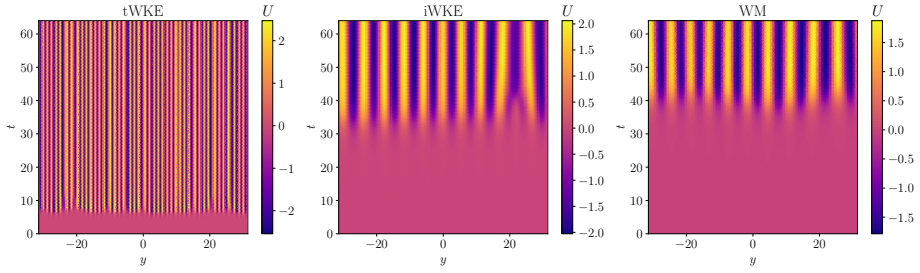
- For linear zonostrophic (secondary) instability (ZI):
  - tWKE dispersion **diverges** at high q (ZF wave number)[10];
  - iWKE and WM dispersions truncate at finite q;
  - iWKE captures WM results semi-quantitatively;
  - fastest-growing modes: $q \sim 1$, GO approximation questionable.



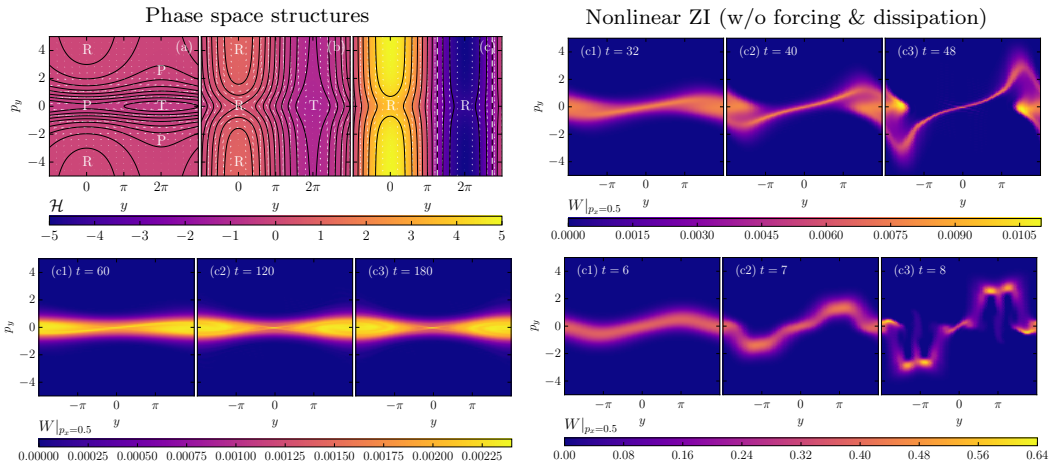
[10] J. B. Parker, J. Plasma Phys. 82, 595820602 (2016).

Nonlinear ZI: tWKE fails

- For nonlinear ZI (with forcing & dissipation):
  - tWKE: unphysical grid-scale ZFs.
  - iWKE & WM: ZFs with finite wave number $q \sim 1$.
- When do **full-wave** effects become significant?

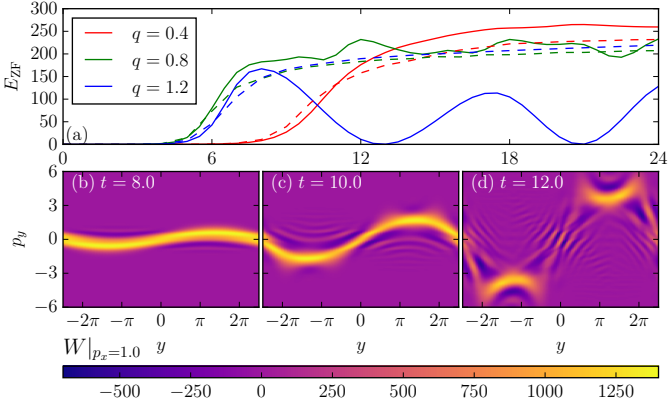


iWKE: drifton trajectories imply ZF saturation



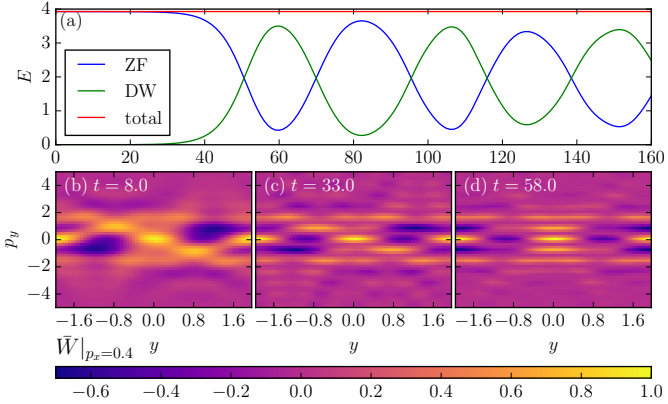
Full-wave effect: ZF oscillations

- In contrast, WM simulations show ZF oscillations for large q.
- iWKE no longer applicable for large q.



Tertiary instability: ZF deterioration

- (Strong) ZFs with large q are unstable to DW perturbations.
  - Rayleigh–Kuo criterion[11]: $U'' = \beta$ somewhere (necessary condition).
- Nonlinearly, DW and ZF show predator-prey type oscillation.



[11] H.-L. Kuo, J. Meteor. 6, 105 (1949).

Summary & outlook

- Traditional wave kinetic equation (tWKE, **caution!**):
  - over-simplified;
  - does not properly capture ZI.
- Improved wave kinetic equation (iWKE, **OK**):
  - systematically derived GO model;
  - semi-quantitatively captures ZI;
  - no ZF oscillation or deterioration.
- Full-wave (Wigner–Moyal, WM) model (**better**):
  - no ray approximation, subsumes iWKE;
  - shows ZF oscillation or deterioration.
- Future work:
  - analytical reduced model for predator-prey oscillation?
  - beyond quasilinear approximation[12]: eddy-eddy interaction?
  - more sophisticated parent model: Hasegawa–Wakatani?
- Details: [H. Zhu et al., arXiv:1712.08262 (2017).]

[12] D. E. Ruiz, M. E. Glinsky, and I. Y. Dodin, arXiv:1803.10817 (2018).