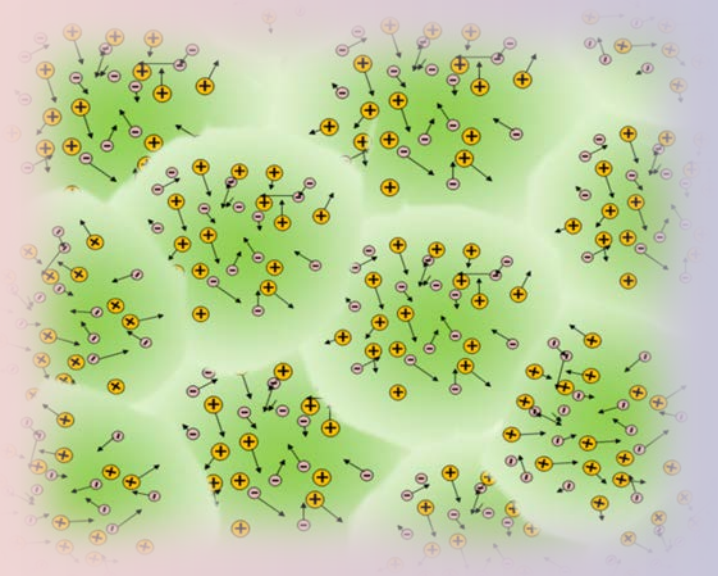
PPPL – *July 7, 2016*

Statistical origin and properties of kappa distributions



George Livadiotis

Southwest Research Institute

San Antonio, TX, USA

Summary

1. Introduction: What is the kappa distribution?
2. Mathematical Motivation
3. Connection with Thermodynamics
4. Properties of kappa distributions
5. Applications in space plasmas
6. Large-Scale Quantization
7. Non-Euclidean Kappa Distributions

1. Introduction:

What is the kappa distribution?

Space plasmas:
Systems out of thermal equilibrium

- Thermal equilibrium: flow of heat in balance (thermal conduction, transfer, radiation).

- Stationary: no explicit time-dependence e.g., stationary distribution, $(\partial / \partial t) P(\vec{u}) \sim 0$

- Systems at *thermal equilibrium* have their distribution function stabilized into a Maxwellian.

- Space plasmas are stationary but *out of thermal equilibrium*, with their distribution functions stabilized into a kappa distribution.

Stationary states in/out of thermal equilibrium

Space plasmas reside in a **stationary state**

$$\frac{\partial}{\partial t} P(\varepsilon) \sim 0$$

Distribution of energy is quasi-invariant

in thermal equilibrium

$$P(\varepsilon) \sim \exp\{-\varepsilon / k_{\mathrm{B}} T\}$$

Boltzmann-Maxwell Distribution

or

out of thermal equilibrium

$$P(\varepsilon) \sim \left[1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \varepsilon / k_{\mathrm{B}} T \right]^{-\kappa - 1}$$

Kappa Distribution

Kappa distribution of velocities

$$P(\vec{u};\theta;\kappa) = \pi^{-\frac{3}{2}} \cdot \mathrm{A}(\kappa) \cdot \theta^{-3} \cdot \underbrace{\left[1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{(\vec{u} - \vec{u}_{\mathrm{b}})^2}{\theta^2}\right]^{-\kappa-1}}$$

$$\mathrm{A}(\kappa) \equiv (\kappa - \tfrac{3}{2})^{-3/2} \cdot \frac{\Gamma(\kappa+1)}{\Gamma(\kappa - \frac{1}{2})}$$

$\kappa \to \infty$

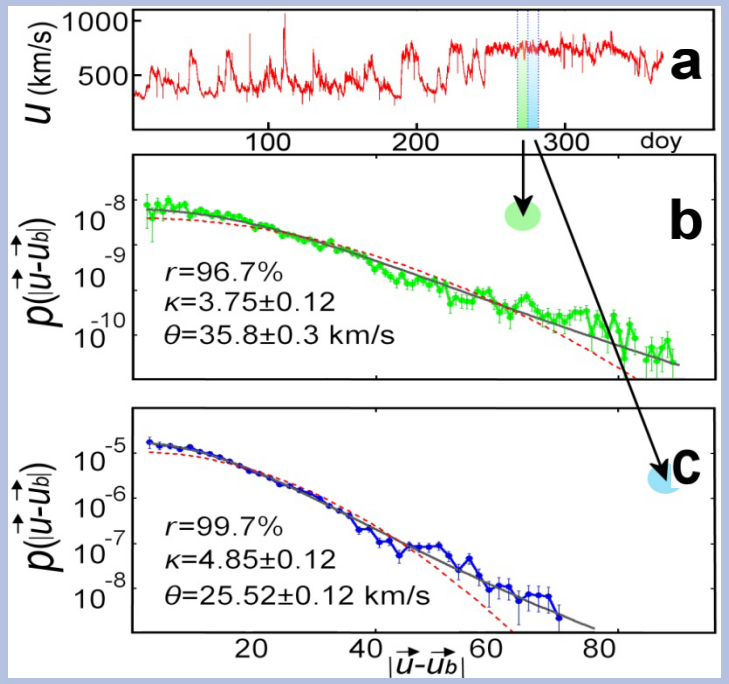
$$\mathrm{A}(\kappa) \to 1 \qquad \exp\left\{-\frac{(\vec{u} - \vec{u}_{\mathrm{b}})^2}{\theta^2}\right\}$$

$$P(\vec{u};\theta;\kappa) = \pi^{-\frac{3}{2}} \cdot \theta^{-3} \cdot \exp\left\{-\frac{(\vec{u} - \vec{u}_{\mathrm{b}})^2}{\theta^2}\right\}$$

$\theta = \sqrt{2k_B T / m}$

Temperature (speed units)

$\vec{u}_{\mathrm{b}}$ *Bulk velocity*



Ulysses ion speeds (2001)

Livadiotis & McComas 2013, JGR, 118, 2863

2. Non-Extensive Statistical Mechanics/ Mathematical Motivation

Mathematical Motive (1)

The exponential is defined by

$$e \equiv \lim_{\kappa \to \infty} \left(1 - \frac{1}{\kappa}\right)^{-\kappa}$$

or

exp(x): exponential function

$$e^x = \lim_{\kappa \to \infty} \left(1 - \frac{x}{\kappa}\right)^{-\kappa}$$

$$e^x = \lim_{q \to 1} \left[1 + (1-q)x\right]^{\frac{1}{1-q}}$$

ln(x): Inverse function

$$\ln x = \lim_{\kappa \to \infty} \left[\kappa \cdot \left(1 - x^{-\frac{1}{\kappa}}\right)\right]$$

$$\ln x \equiv \lim_{q \to 1} \left(\frac{1 - x^{1-q}}{q-1}\right)$$

$$\boldsymbol{\kappa} = \frac{1}{q-1} \Leftrightarrow \boldsymbol{q} = 1 + \frac{1}{\kappa}$$

Livadiotis & McComas, 2009, *JGR*, 114, A11105, 22pp

Mathematical Motive (2)

exp(x) function

$$e^{x} = \lim_{\kappa\to\infty}\left(1-\frac{x}{\kappa}\right)^{-\kappa}$$

$$p(\varepsilon) \sim e^{-\frac{\varepsilon}{kT}} = \lim_{\kappa\to\infty}\left(1+\frac{1}{\kappa}\cdot\frac{\varepsilon}{kT}\right)^{-\kappa}$$

Boltzmann-Gibbs Distribution

ln(x) function

$$\ln x = \lim_{\kappa\to\infty}\left[\kappa\cdot\left(1-x^{-\frac{1}{\kappa}}\right)\right]$$

$$\ln(1/p) = \lim_{\kappa\to\infty}\left[\kappa\cdot\left(1-p^{\frac{1}{\kappa}}\right)\right]$$

or

$$p\cdot\ln(1/p) = \lim_{\kappa\to\infty}\left[\kappa\cdot\left(p-p^{1+\frac{1}{\kappa}}\right)\right]$$

$$S \equiv \sum p\cdot\ln(1/p) = \lim_{\kappa\to\infty}\left[\kappa\cdot\left(1-\sum p^{1+\frac{1}{\kappa}}\right)\right]$$

Boltzmann-Gibbs Entropy

Non-Extensive Statistical Mechanics

Use of exp(*x*) :
DISTRIBUTION

$$p(\varepsilon) \sim e^{-\frac{\varepsilon}{kT}} \quad = \quad \lim_{\kappa \to \infty} \left[\left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon}{kT} \right)^{-\kappa} \right]$$

Use of ln(*x*) :
ENTROPY

$$S \equiv \sum p \cdot \ln(1/p) \quad = \quad \lim_{\kappa \to \infty} \left[\kappa \cdot \left(1 - \sum p^{1+\frac{1}{\kappa}} \right) \right]$$

$$[\,Boltzmann - Gibbs\,] \quad = \quad \lim_{\kappa \to \infty} \quad [\;\; Tsallis \;\;]$$

$$\boxed{\boldsymbol{\kappa} = \frac{1}{q-1} \quad \Leftrightarrow \quad \boldsymbol{q} = 1 + \frac{1}{\kappa}}$$

3. Connection to Thermodynamics

Temperature ~ Mean Energy

***Temperature T* : Mean kinetic energy** $< \varepsilon > = \frac{1}{2} m < (\vec{u} - \vec{u}_b)^2 > = \frac{3}{2} k_B T$

Kinetic Definition: $T \equiv \frac{2}{3} k_{\mathrm{B}}^{-1} \cdot < \varepsilon >$

Thermodynamic Definition: $T \equiv (\partial S / \partial U)^{-1} \cdot [1 - \frac{1}{\kappa} \cdot S / k_{\mathrm{B}}]$

Sumiyoshi Abe, Phys Rev E, 2001

(S: Entropy , U: internal energy, here $U = < \varepsilon >$)

→ The two fundamental definitions coincide,
- at thermal equilibrium ($\kappa \rightarrow \infty$),
- out of thermal equilibrium (any κ).

→ T can be defined at, or out of, thermal equilibrium

CORONA plasma	+	LAB plasma	=	Mixed plasma
L~10m, n~10^{13}m^{-3}, T~10^6K		L~10m, n~10^{18}m^{-3}, T~100K		L~10m, n~10^{18}m^{-3}, T~110K

Livadiotis, 2015, *JGR*, 120, 1607, 13pp

1 / Kappa ~ Energy Correlation

$\varepsilon_1, \varepsilon_2, \ldots$ energies of different particles.

Correlation between particles 1 and 2 :

$$\rho = \frac{\langle \varepsilon_1 \cdot \varepsilon_2 \rangle - \langle \varepsilon_1 \rangle \cdot \langle \varepsilon_2 \rangle}{\sqrt{\langle \varepsilon_1^2 \rangle - \langle \varepsilon_1 \rangle^2} \cdot \sqrt{\langle \varepsilon_2^2 \rangle - \langle \varepsilon_2 \rangle^2}}$$

- Thermal equilibrium: κ→∞

Maxwell distribution can be factorized:

$$P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f) = \prod_{i=1}^{f} P(\varepsilon_i)$$

$$\langle \varepsilon_1 \cdot \varepsilon_2 \rangle = \int \varepsilon_1 \varepsilon_2 \cdot P(\varepsilon_1, \varepsilon_2) d\varepsilon_1 d\varepsilon_2 = \int \varepsilon_1 \cdot P(\varepsilon_1) d\varepsilon_1 \cdot \int \varepsilon_2 \cdot P(\varepsilon_2) d\varepsilon_2 = \langle \varepsilon_1 \rangle \cdot \langle \varepsilon_2 \rangle$$

$\Longrightarrow$ No Correlation $\rho = 0$

- Out of thermal equilibrium: Any κ<∞

Livadiotis & McComas 2011, ApJ, 741, 88

Kappa distribution cannot be factorized:

$$P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f) \neq \prod_{i=1}^{f} P(\varepsilon_i) \qquad \langle \varepsilon_1 \cdot \varepsilon_2 \rangle \neq \langle \varepsilon_1 \rangle \cdot \langle \varepsilon_2 \rangle \Longrightarrow \text{Correlation} \quad \rho = \frac{1.5}{\kappa}$$

$$\kappa = \frac{\frac{1}{2} f}{\rho}$$

κ→∞ , ρ→0% ← ***κ→f/2 , ρ→100%***

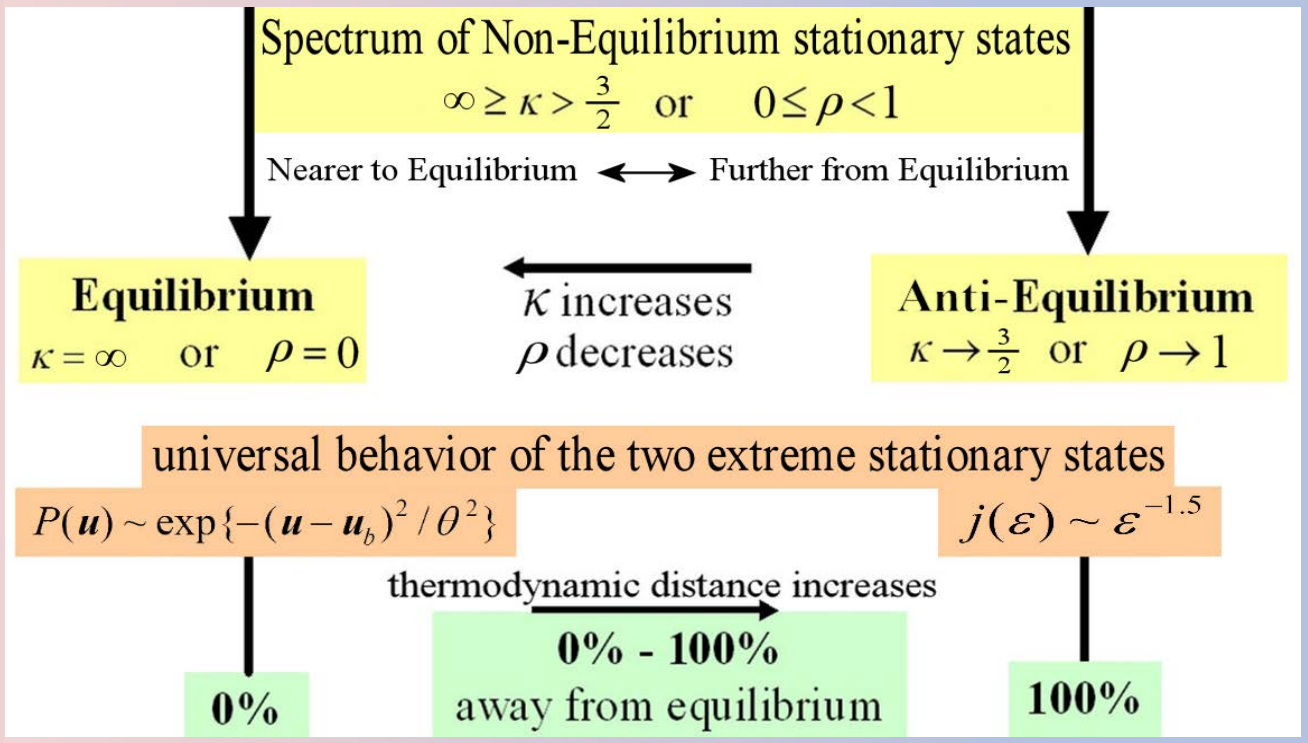
Livadiotis, McComas, 2011, *Astrophys J* 741, 88, 28pp

The Kappa spectrum

κ : Measure of the "thermodynamic distance" from thermal equilibrium

- Thermal equilibrium: $\kappa \rightarrow \infty$
- Stationary States, Out of thermal equilibrium: any $\kappa < \infty$
- Anti-Equilibrium, Furthest from thermal equilibrium: $\kappa \rightarrow 3/2$

The κ-spectrum is the whole set of different states, labeled by the values of κ :



$$\rho = \frac{1.5}{\kappa}$$

Livadiotis et al, 2013, Space Sci Rev, 75, 183

4. Properties of kappa distribution

Velocities or kinetic energies

Distribution of velocities

$$P(\vec{u};\theta;\kappa) \sim \left[1+\frac{1}{\kappa-\frac{3}{2}}\cdot\frac{(\vec{u}-\vec{u}_{\mathrm{b}})^2}{\theta^2}\right]^{-\kappa-1}$$

$$\int P(\vec{u};\theta;\kappa)\,d^3\vec{u}=1$$

Distribution of kinetic energies $\varepsilon=\frac{1}{2}m\cdot(\vec{u}-\vec{u}_{\mathrm{b}})^2$

$$P(\varepsilon;T;\kappa) \sim \left(1+\frac{1}{\kappa-\frac{3}{2}}\cdot\frac{\varepsilon}{k_{\mathrm{B}}T}\right)^{-\kappa-1}$$

$$\int P(\varepsilon;T;\kappa)\,\varepsilon^{\frac{1}{2}}\,d\varepsilon=1$$

Unique distribution for any exponent

(1) $$P(\varepsilon) \sim \left(1 + \frac{1}{\kappa} \cdot \frac{\varepsilon - \langle \varepsilon \rangle}{k_B T}\right)^{-\kappa-1}$$

$$\langle \varepsilon \rangle = \tfrac{3}{2} k_B T$$

$$P(\varepsilon) \sim \left(1 + \frac{1}{\kappa - \frac{\langle \varepsilon \rangle}{k_B T}} \cdot \frac{\varepsilon}{k_B T}\right)^{-\kappa-1}$$

$$P(\varepsilon) \sim \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{\varepsilon}{k_B T}\right)^{-\kappa-1}$$

$$\tfrac{3}{2} < \kappa \leq \infty$$

Different exponents? No… It's equivalent!

(2) $$P(\varepsilon) \sim \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{\varepsilon}{k_B T}\right)^{-\kappa-a}$$

$$< \varepsilon > \equiv \tfrac{3}{2} k_B T_{real} = \frac{\kappa - \frac{3}{2}}{\kappa + a - \frac{5}{2}} \cdot \tfrac{3}{2} k_B T$$

$$P(\varepsilon) \sim \left[1 + \frac{1}{(\kappa + a - 1) - \frac{3}{2}} \cdot \frac{\varepsilon}{k_B T_{real}}\right]^{-(\kappa+a-1)-1}$$

$$\tfrac{3}{2} < \kappa + a - 1 \leq \infty$$

$$\kappa + a - 1 \rightarrow \kappa$$

$$P(\varepsilon) \sim \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{\varepsilon}{k_B T_{real}}\right)^{-\kappa-1}$$

Livadiotis, 2015, *JGR*, 120, 1607, 13pp

Kappa distribution of the Hamiltonian (1)

- Kappa Distribution of a Hamiltonian

$$P(\vec{r},\vec{u}) \sim \left\{1+\frac{1}{\kappa-\frac{3}{2}}\cdot\frac{H(\vec{r},\vec{u})}{k_{\mathrm{B}}T}\right\}^{-\kappa-1}$$

- Kappa Distribution when there is a potential Φ

$$P(\vec{r},\vec{u}) \sim \left\{1+\frac{1}{\kappa-\frac{3}{2}}\cdot\left[\frac{\frac{1}{2}m(\vec{u}-\vec{u}_{\mathrm{b}})^2}{k_B T}+\frac{1}{k_{\mathrm{B}}T}\cdot\Phi_{\mathrm{c}}(\vec{r})\right]\right\}^{-\kappa-1}$$

$$H(\vec{r},\vec{u}) = \tfrac{1}{2}m(\vec{u}-\vec{u}_{\mathrm{b}})^2 + \Phi_{\mathrm{c}}(\vec{r})$$

where $\Phi_{\mathrm{c}}(\vec{r}) \equiv \Phi(\vec{r}) - \langle\Phi\rangle$ (Independence of the zero level)

- Kappa Distribution of the velocities, when there is no external force

$$P(\vec{u}) \sim \left[1+\frac{1}{\kappa-\frac{3}{2}}\frac{(\vec{u}-\vec{u}_{\mathrm{b}})^2}{\theta^2}\right]^{-\kappa-1}$$

$$H(\vec{u}) = \tfrac{1}{2}m(\vec{u}-\vec{u}_{\mathrm{b}})^2$$

$$\theta = \sqrt{2k_B T/m}$$

Livadiotis, 2015, *JGR*, 120, 880-903, 24pp

Kappa distribution of the Hamiltonian (2)

- Kappa Distribution when there is a potential Φ

$\Phi(z) \sim mg\,z$ or $\Phi(z) \sim eE\,z$

$$H(z,\vec{u}) = \tfrac{1}{2}m(\vec{u}-\vec{u}_{\mathrm{b}})^2 + mg\cdot(z-\langle z\rangle)$$

Phase space distribution (z,ε_{K}):

$$P(z,\varepsilon_{\mathrm{K}};\kappa_0,T) = \frac{\Gamma(\kappa+1)}{(\kappa-1-\frac{1}{2}d)^{\frac{1}{2}d+1}\Gamma(\frac{1}{2}d)\Gamma(\kappa-\frac{1}{2}d)}\cdot(k_B T)^{-\frac{1}{2}d} z_0^{-1}\cdot\left[1+\frac{1}{\kappa-\frac{d}{2}-1}\cdot\left(\frac{\varepsilon_{\mathrm{K}}}{k_{\mathrm{B}}T}+\frac{z}{z_0}\right)\right]^{-\kappa-1}\varepsilon_{\mathrm{K}}^{\frac{1}{2}d-1}$$

Positional distribution (z)
(after the integration over the kinetic energy or velocities)

$$z_0 \equiv \frac{k_{\mathrm{B}}T}{mg}$$

$$P(z) \sim \left(1+\frac{1}{\kappa-\frac{5}{2}}\cdot\frac{z}{z_0}\right)^{-\kappa+\frac{1}{2}} \sim z^{-\kappa+\frac{1}{2}} \longrightarrow n(z) = n(0)\left(1+\frac{1}{\kappa-\frac{5}{2}}\cdot\frac{z}{z_0}\right)^{-\kappa+\frac{1}{2}}$$

$$\tfrac{d}{2}k_{\mathrm{B}}T_{local}(z) = \int_0^\infty P(z,\varepsilon_{\mathrm{K}})\varepsilon_{\mathrm{K}}\,d\varepsilon_{\mathrm{K}} = \tfrac{d}{2}k_{\mathrm{B}}T\cdot\left(1+\frac{1}{\kappa-\frac{5}{2}}\cdot\frac{z}{z_0}\right) \longrightarrow T_{loc}(z) = T_{loc}(0)\cdot\left(1+\frac{1}{\kappa-\frac{5}{2}}\cdot\frac{z}{z_0}\right)$$

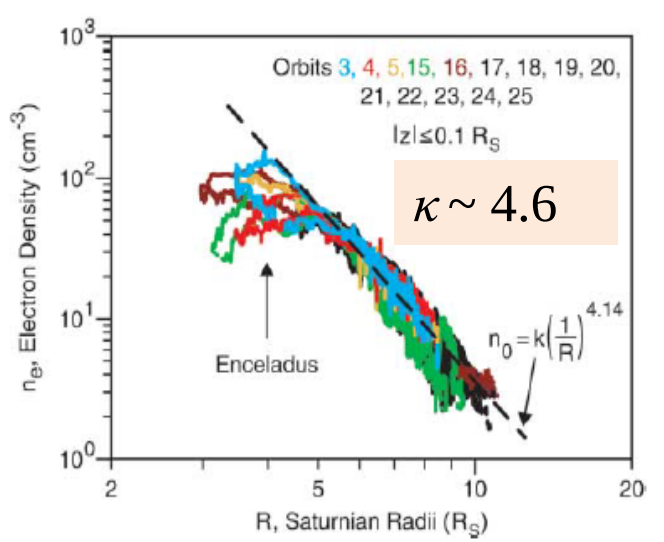
$$\longrightarrow p(z) = p(0)\left(1+\frac{1}{\kappa-\frac{5}{2}}\cdot\frac{mg\,z}{k_{\mathrm{B}}T}\right)^{-\kappa+\frac{3}{2}} \quad \kappa\to\infty \quad p(z) = p(0)\exp\left(-\frac{mg\,z}{k_{\mathrm{B}}T}\right)$$

Barometric formula

Livadiotis, 2015, *JGR*, 120, 880-903, 24pp

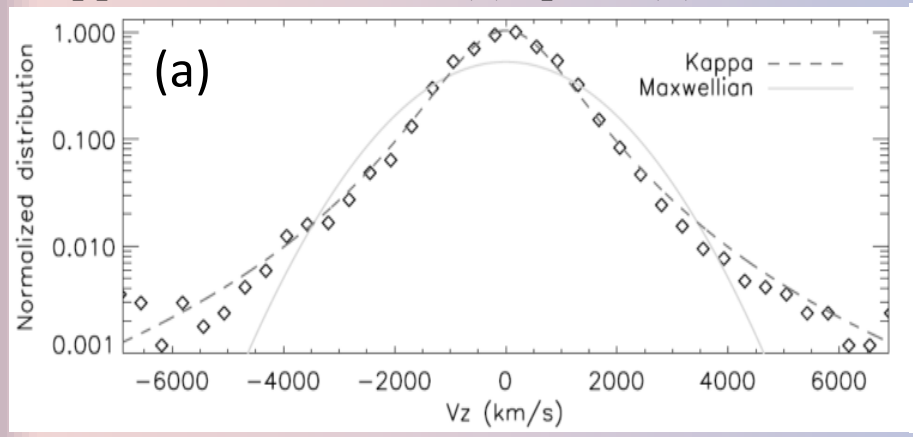
Kappa distribution of the Hamiltonian (3)

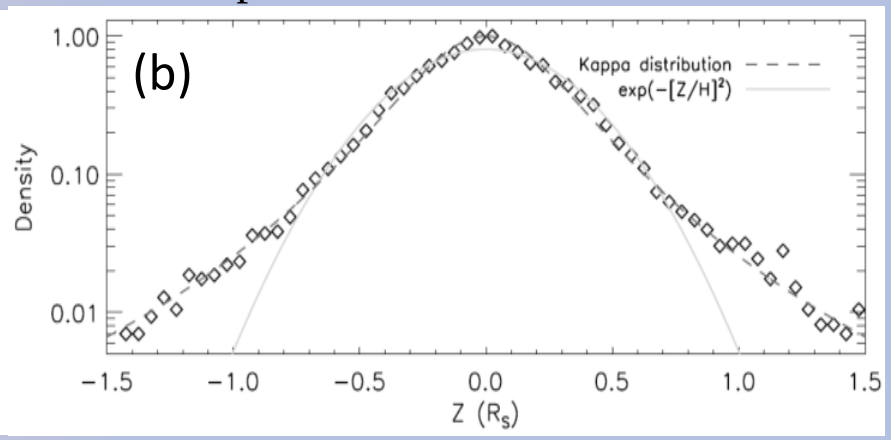
$$P(z) \sim \left[1 + \frac{1}{\kappa - \frac{5}{2}} \cdot \frac{mg}{kT} z\right]^{-\kappa+\frac{1}{2}} \sim z^{-\kappa+\frac{1}{2}}$$



The positional electron distribution at high altitude of the Saturnian magnetosphere

Kappa distributions of (a) speed (b) altitude for OH in the plasma near Enceladus





Kappa distribution of the Hamiltonian: Magnetization – Curie constant (1)

- Kappa Distribution when there is a potential: $m\Phi(\vartheta) = -\mu B \cos\vartheta$ $\quad \beta \equiv \mu B/(k_B T)$

i.e., Hamiltonian: $H(\vartheta,\vec{u}) = \frac{1}{2} m(\vec{u} - \vec{u}_b)^2 - \mu B \cos\vartheta$

- Phase-Space Distribution:

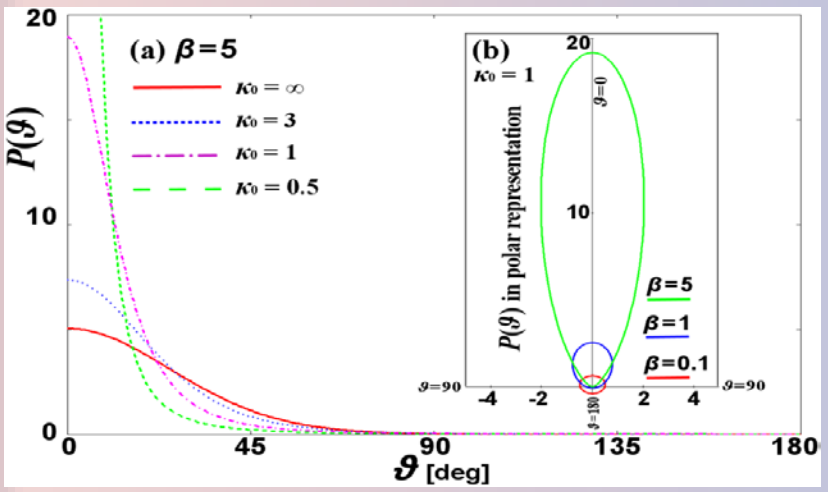
$$P(\varepsilon_K,\vartheta)d\varepsilon_K d\cos\vartheta \propto \left[1 + \frac{1}{\kappa - \frac{d}{2} + \beta\langle\cos\vartheta\rangle}\cdot\left(\frac{\varepsilon_K}{k_B T} - \beta\cos\vartheta\right)\right]^{-\kappa-1} \varepsilon_K^{\frac{1}{2}d-1} d\varepsilon_K d\cos\vartheta$$

$\int d\cos\vartheta$

- Marginal Distributions:

$$P(\varepsilon_K)d\varepsilon_K \propto \left\{\left[1 + \frac{\varepsilon_K - \beta k_B T}{k_B T(\kappa - \frac{d}{2} + \beta\langle\cos\vartheta\rangle)}\right]^{-\kappa} - \left[1 + \frac{\varepsilon_K + \beta k_B T}{k_B T(\kappa - \frac{d}{2} + \beta\langle\cos\vartheta\rangle)}\right]^{-\kappa}\right\}\cdot \varepsilon_K^{\frac{1}{2}d-1} d\varepsilon_K$$

$\int d\varepsilon_K$



$$P(\cos\vartheta)d\cos\vartheta \propto \left(1 - \frac{\beta\cos\vartheta}{\kappa - \frac{d}{2} + \beta\langle\cos\vartheta\rangle}\right)^{-\kappa+\frac{1}{2}} d\cos\vartheta$$

Livadiotis, 2016, *EPL*, 113, 10003, pp6

Kappa distribution of the Hamiltonian: Magnetization – Curie constant (2)

$$P(\cos\vartheta) \propto \left(1 - \frac{\beta\cos\vartheta}{\kappa_0 + \beta\langle\cos\vartheta\rangle}\right)^{-\kappa_0 - 1 - \frac{d}{2}}$$

$$\kappa = \kappa_0 + \tfrac{d}{2}$$

$$\beta \equiv \mu B/(k_B T)$$

$$\langle\cos\vartheta\rangle = \int_{-\infty}^{\infty} P(\cos\vartheta)\cos\vartheta \, d\cos\vartheta$$

$$\frac{1+\beta\langle\cos\vartheta\rangle+\beta}{1+\beta\langle\cos\vartheta\rangle-\beta} = \left(\frac{\kappa_0+\beta\langle\cos\vartheta\rangle+\beta}{\kappa_0+\beta\langle\cos\vartheta\rangle-\beta}\right)^{\kappa_0}$$

$$M = \mu \cdot \langle\cos\vartheta\rangle$$

$$\boxed{\tfrac{1}{\mu}M = \langle\cos\vartheta\rangle = L_{\kappa_0}\left(\frac{\beta}{1+\frac{1}{\kappa_0}\beta\cdot\frac{1}{\mu}M}\right)}$$

κ-Langevin function

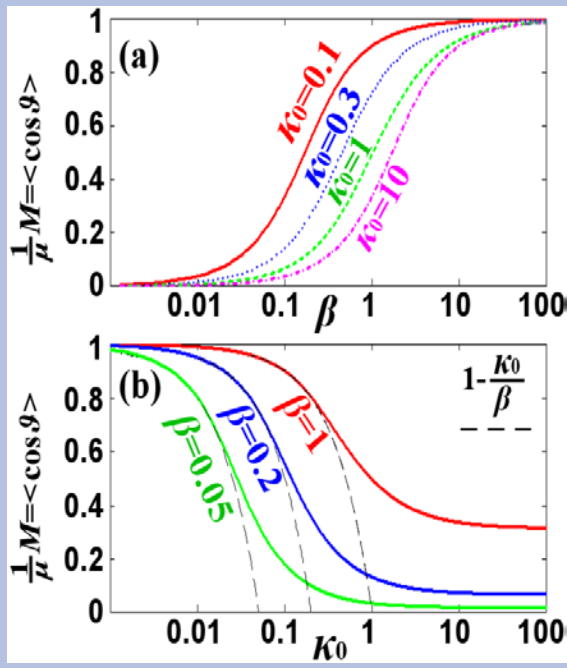
$$L_{\kappa_0}(x) \equiv (1-1/\kappa_0)^{-1}\cdot\left[\coth_{\kappa_0}(x) - \frac{1}{x}\right]$$

$$\kappa_0 \to \infty \qquad L(x) \equiv \coth(x) - \frac{1}{x}$$

$$\coth_{\kappa_0}(x) = \frac{\exp_{\kappa_0}(x) + \exp_{\kappa_0}(-x)}{\exp_{\kappa_0}(x) - \exp_{\kappa_0}(-x)} \qquad \exp_{\kappa_0}(-x) = [1 + \tfrac{1}{\kappa_0}\cdot x]^{-\kappa_0}$$

$$\tfrac{1}{\mu}M \cong \tfrac{1}{3}(1+\tfrac{1}{\kappa_0})\cdot\left(\frac{\mu B}{k_B T}\right)\left[1 + \tfrac{1}{15}(1+\tfrac{5}{\kappa_0}+\tfrac{1}{\kappa_0^2})\cdot\left(\frac{\mu B}{k_B T}\right)^2\right] + O\left(\frac{\mu B}{k_B T}\right)^4$$

$$\boxed{C \equiv \lim_{(B/T)\to 0}\frac{\partial M}{\partial(B/T)} \cong \tfrac{1}{3}k_B^{-1}\mu^2\cdot(1+\tfrac{1}{\kappa_0})}$$



Livadiotis, 2016, *EPL*, 113, 10003, pp6

N-particle Kappa distribution (1)

N-particle Kappa Distribution

$$P(\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_N) \sim \left[1+\frac{1}{\kappa-\frac{3N}{2}}\frac{1}{\theta^2}\sum_{i=1}^{N}(\vec{u}_i-\vec{u}_\mathrm{b})^2\right]^{-\kappa-1}$$

integrated over $\vec{u}_2,\vec{u}_3,\ldots,\vec{u}_N$, so that

$$P(\vec{u}_1) \sim \left[1+\frac{1}{\kappa-\frac{3N}{2}}\frac{1}{\theta^2}(\vec{u}_1-\vec{u}_\mathrm{b})^2\right]^{-\kappa-1+\frac{3(N-1)}{2}}$$

1-particle Kappa Distribution

$$P(\vec{u}_1) \sim \left[1+\frac{1}{\kappa-\frac{3}{2}}\frac{1}{\theta^2}(\vec{u}_1-\vec{u}_\mathrm{b})^2\right]^{-\kappa-1}$$

Different formulae → The kappa index depends on the degrees of freedom:

$$P(\vec{u}_1) \sim \left[1+\frac{1}{\kappa(3N)-\frac{3N}{2}}\frac{1}{\theta^2}(\vec{u}_1-\vec{u}_\mathrm{b})^2\right]^{-[\kappa(3N)-\frac{3N}{2}]-1-\frac{3}{2}}$$

$$P(\vec{u}_1) \sim \left[1+\frac{1}{\kappa(3)-\frac{3}{2}}\frac{1}{\theta^2}(\vec{u}_1-\vec{u}_\mathrm{b})^2\right]^{-[\kappa(3)-\frac{3}{2}]-1-\frac{3}{2}}$$

$f=3N$

$$\kappa(f)-\tfrac{1}{2}f \equiv \kappa_0$$

$f=3$

$$\kappa(3N)-\frac{3N}{2}=\kappa_0$$

$$\kappa(3)-\frac{3}{2}=\kappa_0$$

$$P(\vec{u}_1) \sim \left[1+\frac{1}{\kappa_0}\frac{1}{\theta^2}(\vec{u}_1-\vec{u}_\mathrm{b})^2\right]^{-\kappa_0-1-\frac{3}{2}}$$

Livadiotis, 2015, *Entropy*, 17, 2062

N-particle Kappa distribution (2)

$$\kappa(f) - \tfrac{1}{2} f \equiv \kappa_0$$

- Kappa Distribution of the Hamiltonian

$$P(\vec{r}_1,\vec{u}_1,\ldots,\vec{r}_N,\vec{u}_N) \sim \left[1+\frac{1}{\kappa_0}\cdot\frac{H(\vec{r}_1,\vec{u}_1,\ldots,\vec{r}_N,\vec{u}_N)}{k_\mathrm{B}T}\right]^{-\kappa_0-1-\frac{3N}{2}}$$

- Kappa Distribution of the velocities, when there is no external force

$$P(\vec{u}_1,\ldots,\vec{u}_N) \sim \left[1+\frac{1}{\kappa_0}\frac{1}{\theta^2}\sum_{i=1}^{N}(\vec{u}_i-\vec{u}_\mathrm{b})^2\right]^{-\kappa_0-1-\frac{3N}{2}}$$

$$H(\vec{u}_1,\ldots,\vec{u}_N) = \tfrac{1}{2}m\sum_{i=1}^{N}(\vec{u}_i-\vec{u}_\mathrm{b})^2$$

- Kappa Distribution of the velocities, d-dimensional

$$P(\vec{u};T;\kappa_0) = (\pi\theta^2)^{-\frac{3}{2}}\cdot\kappa_0^{-\frac{3}{2}}\frac{\Gamma(\kappa_0+1+\frac{d}{2})}{\Gamma(\kappa_0+1)}\cdot\left(1+\frac{1}{\kappa_0}\cdot\frac{u^2}{\theta^2}\right)^{-\kappa_0-1-\frac{d}{2}}$$

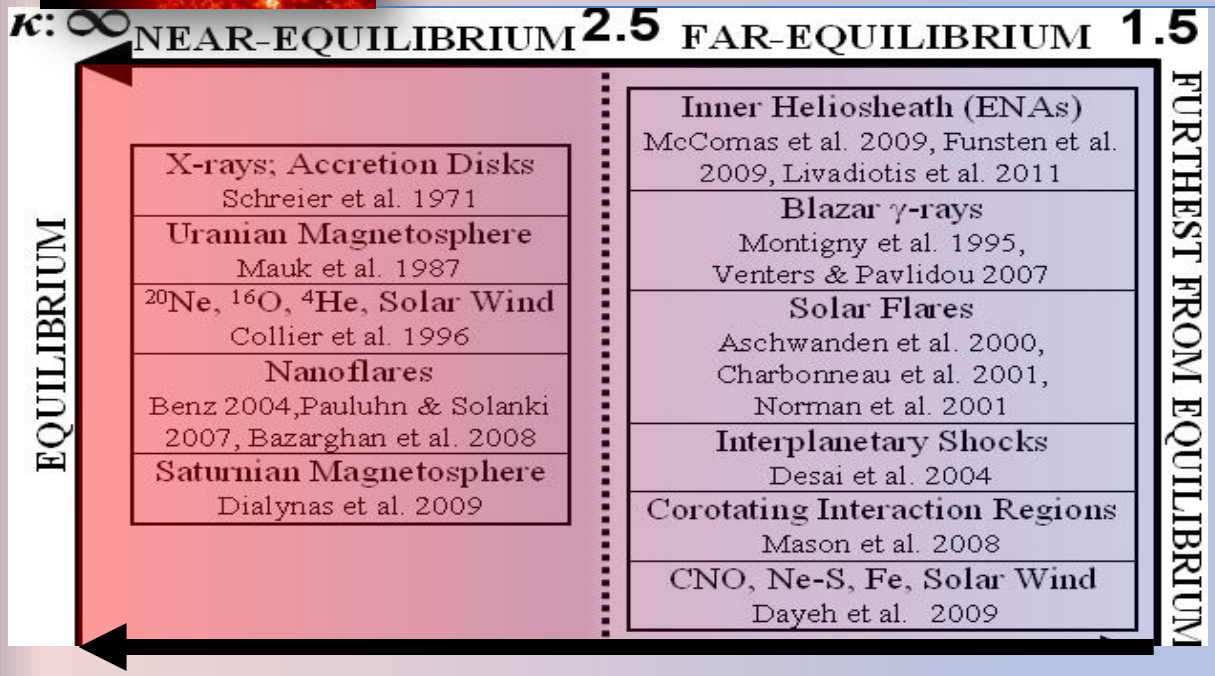
- Kappa Distribution when having a potential

$$P(\vec{r}_1,\vec{u}_1,\ldots,\vec{r}_N,\vec{u}_N) \sim \left\{1+\frac{1}{\kappa_0}\cdot\left[\frac{1}{\theta^2}\sum_{i=1}^{N}(\vec{u}_i-\vec{u}_\mathrm{b})^2+\frac{1}{k_\mathrm{B}T}\cdot\Phi_\mathrm{c}(\vec{r}_1,\ldots,\vec{r}_N)\right]\right\}^{-\kappa_0-1-\frac{3N}{2}}$$

$$H(\vec{r}_1,\vec{u}_1,\ldots,\vec{r}_N,\vec{u}_N) = \tfrac{1}{2}m\sum_{i=1}^{N}(\vec{u}_i-\vec{u}_\mathrm{b})^2+\Phi_\mathrm{c}(\vec{r}_1,\ldots,\vec{r}_N)$$

5. Applications in space plasmas

Applications: Near-equilibrium vs. Far-equilibrium

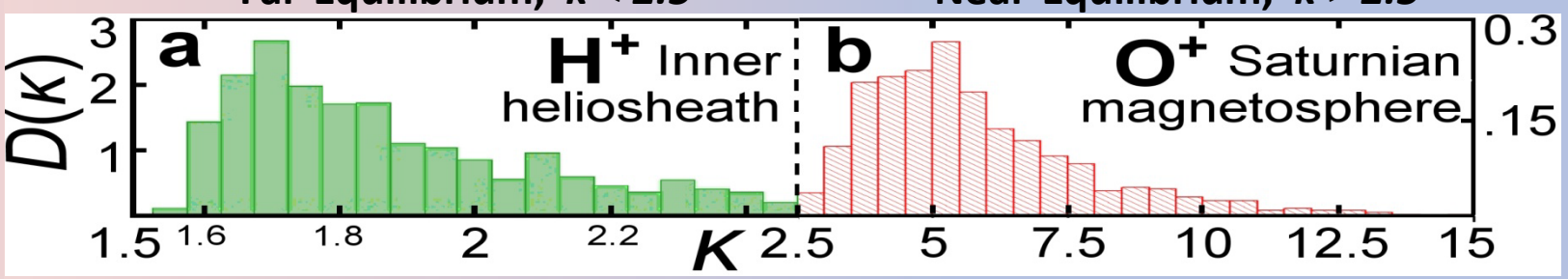


Space plasmas reside
- Near-Equilibrium region with kappa indices κ>2.5,
- Far-Equilibrium region with kappa indices κ<2.5

Livadiotis & McComas 2010, ApJ, 714, 971

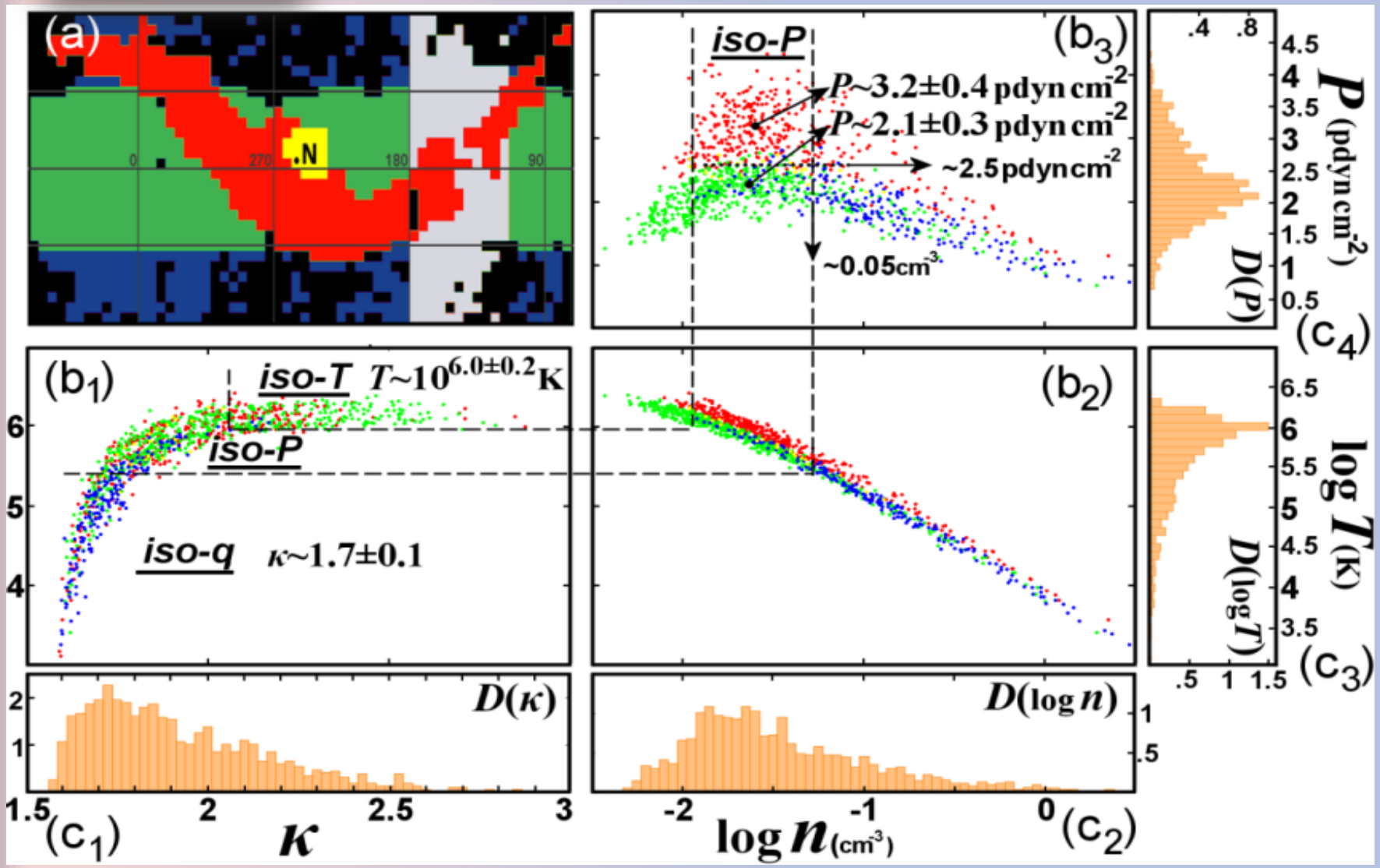
In the inner heliosheath κ-indices are distributed only in the <u>*Far-Equilibrium*</u> region, while in the inner heliosphere κ-indices are distributed in the <u>*Near-Equilibrium*</u> region

Far-Equilibrium, $\kappa < 2.5$ **Near-Equilibrium, $\kappa > 2.5$**



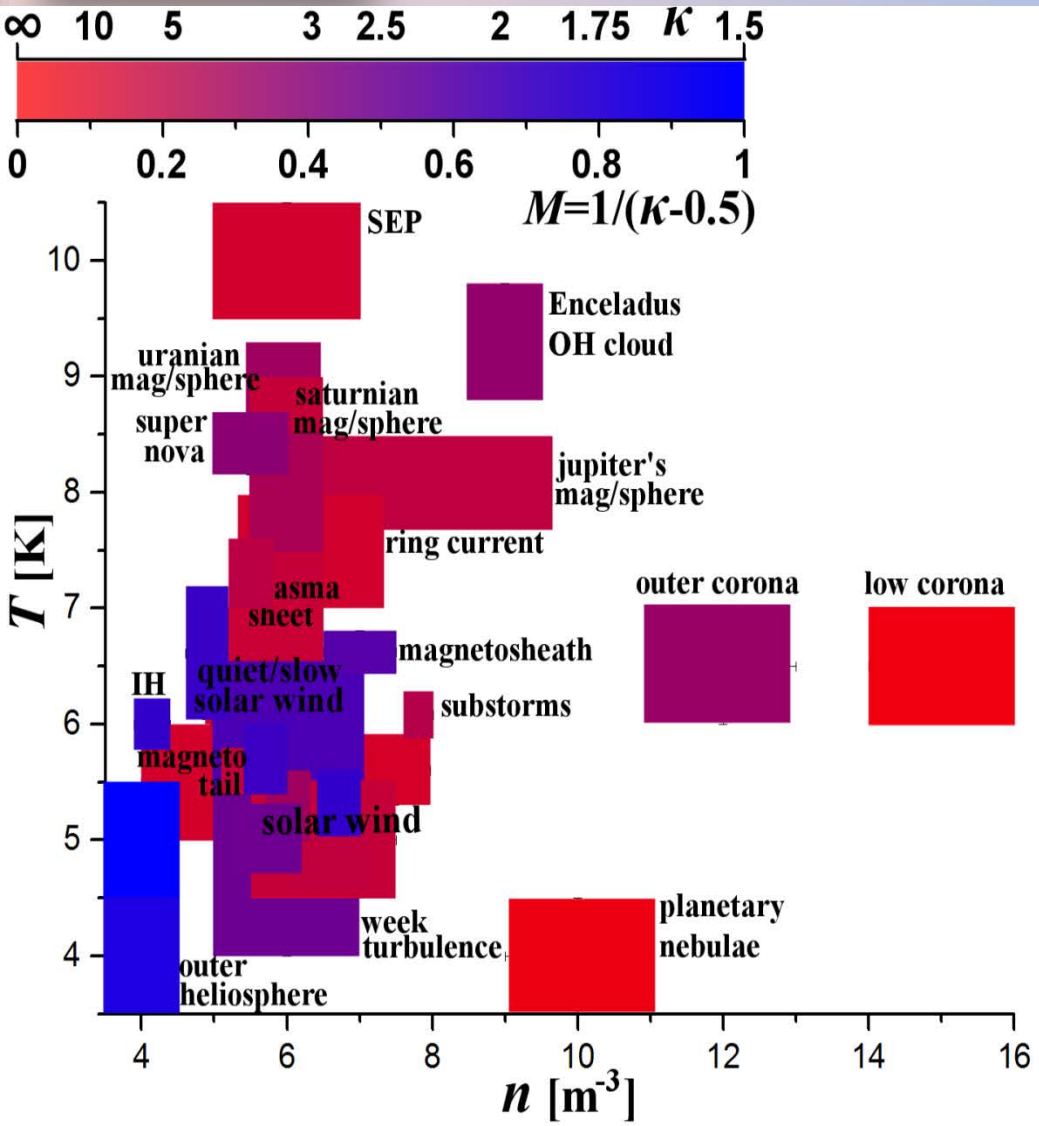
Livadiotis, McComas, 2011, *Astrophys J* 741, 88, 28pp

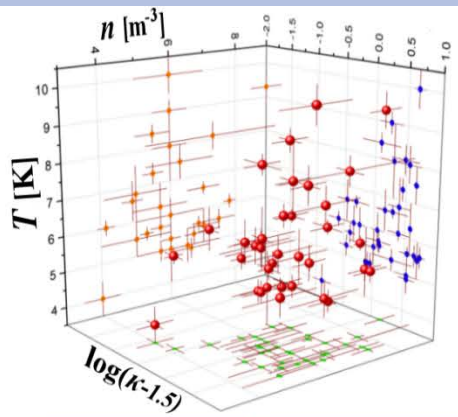
Applications: Inner Heliosheath

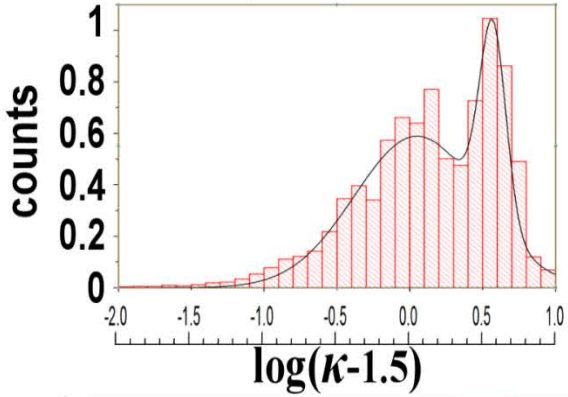


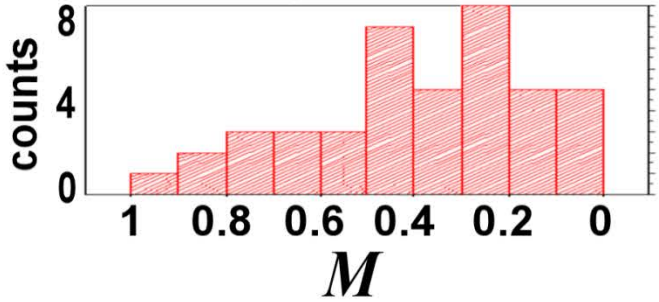
Livadiotis, et al., 2011, *ApJ*, 734, 1, 19pp

Applications: ~50 space plasmas









Livadiotis, 2015, *JGR*, 120, 1607, 13pp

6. Applications: ***New Quantization Constant***

Applications: 2nd Quantization Constant

Space plasmas reveal that the Planck constant is not unique. A second quantization constant exist:

$$\Delta E \, \Delta t \geq \tfrac{1}{2} \hbar_*$$

1) $\hbar \cong 1.1 \times 10^{-34}\,\mathrm{J\,s}$ (Planck quantization)

2) $\hbar_* \cong 1.2 \times 10^{-22}\,\mathrm{J\,s}$ (Plasma quantization)

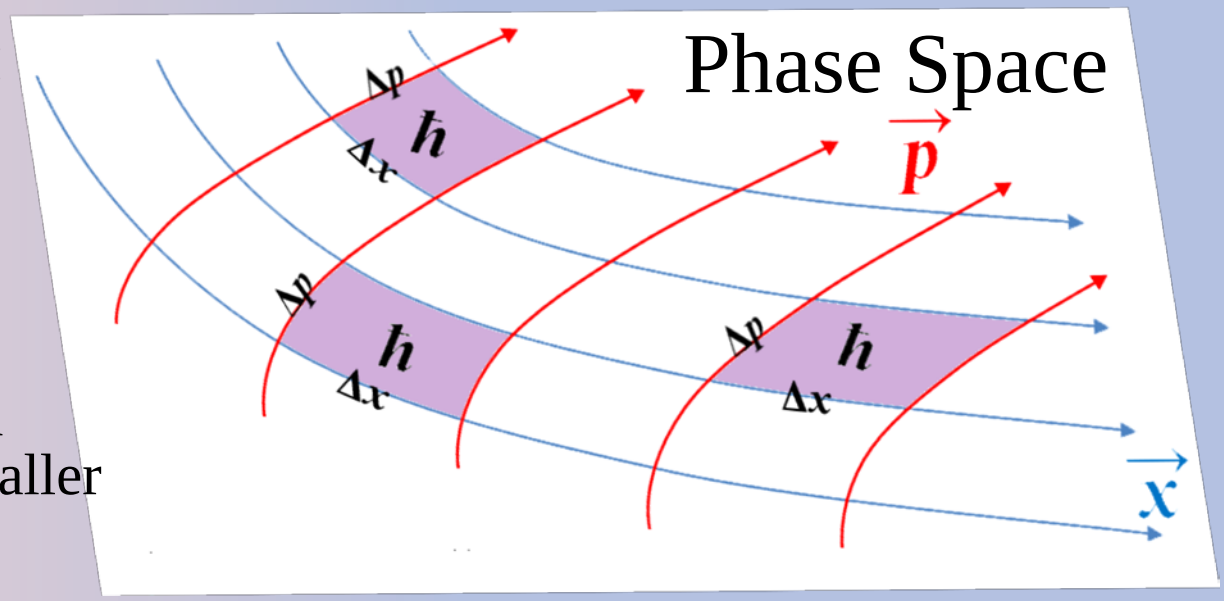
- Livadiotis & McComas, 2013, *Entropy*, 15, 1118
- Livadiotis & McComas, 2014, *ASP Conf Ser*, 484, 131, pp6
- Livadiotis & McComas, 2014, *J Geophys Res*, 119, 3247
- Livadiotis, 2014, *Entropy*, 16, 4290-4308, pp19
- Livadiotis, 2015, *J Phys Conf Ser*, 577, 012018, pp7
- Livadiotis, 2016, *ApJ Suppl Ser*, 223, 13, pp13
- Livadiotis & Desai, 2016, *ApJ*, In Press
- *Nature News*, doi:10.1038/nature.2013.13159

Systems with Local Correlations

Phase Space cell:

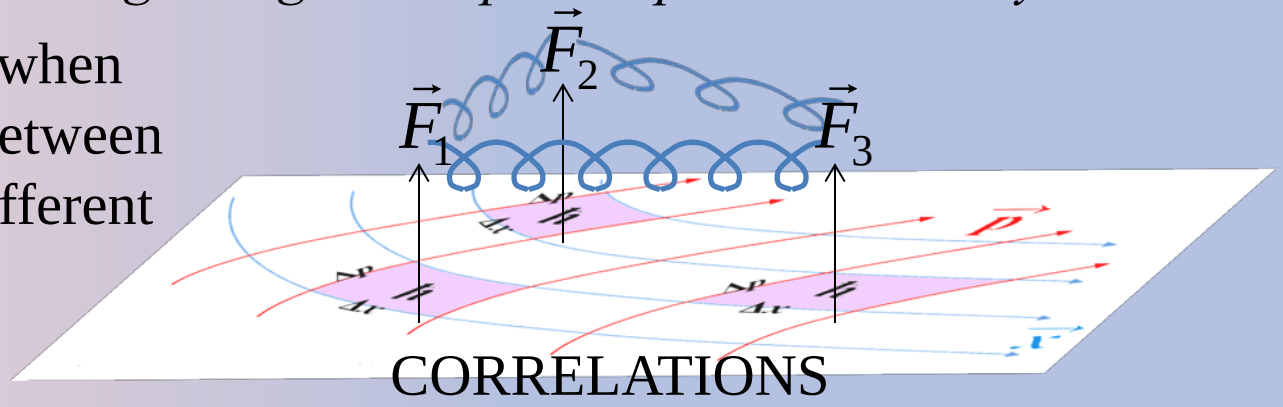
$$\Delta p\, \Delta x \sim \hbar$$

How would the laws of physics act in systems, where the information cannot be exchanged in phase-space parcels smaller than some $\hbar_*$ ($>\hbar$) ?



These long-lived correlations (collisionless plasmas) could be responsible for generating a large scale phase space elementary cell: $\hbar_*$.

This can be realized when there is connection between the information of different $\hbar$-cells 1, 2, 3…

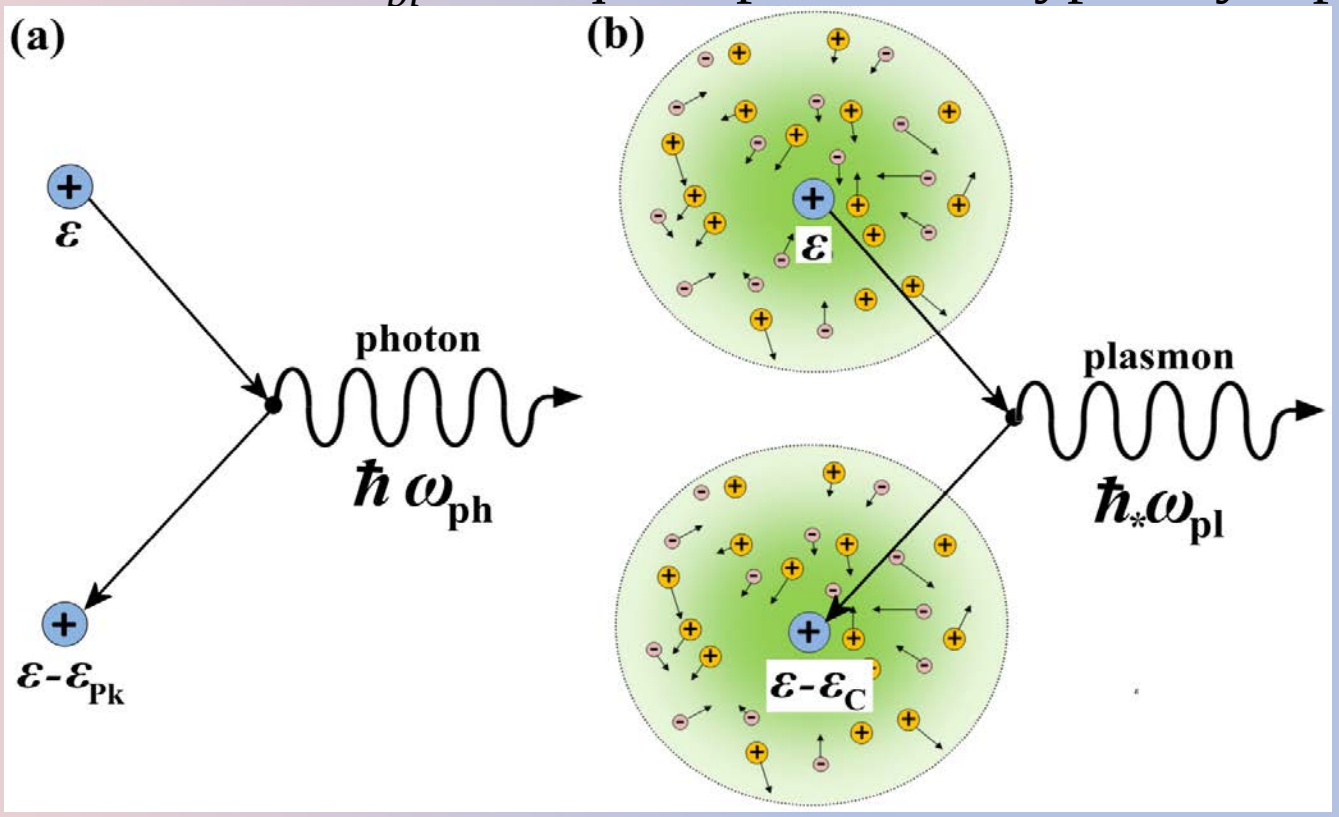


Livadiotis, 2015, *J Phys Conf Ser*, 577, 012018, pp7

Large-Scale Quantization: The energy-frequency relation

(a) The smallest energy that is transferred from uncorrelated particles is $\sim\hbar\cdot\omega_{ph}$. This creates a quasi-particle, e.g., a photon.

(b) For correlated particles, the smallest energy that can be transferred is $\sim\hbar_*\cdot\omega_{pl}$. The quasi-particle is typically a plasmon.



Livadiotis & McComas, 2014, *J Geophys Res*, 119, 3247

Methods for finding the value of $\hbar_*$
Observation methods - data analyses

$$\hbar_* = \varepsilon_C / \omega_{pl}$$

Correlated particle mean energy / plasmon frequency

$$\varepsilon_C \sim \tfrac{1}{2} m u_{ms}^{\ 2} \quad \& \quad t_C \sim \tfrac{1}{2} \omega_{pl}^{\ -1} \sim \tfrac{1}{2} \lambda_D / \theta$$

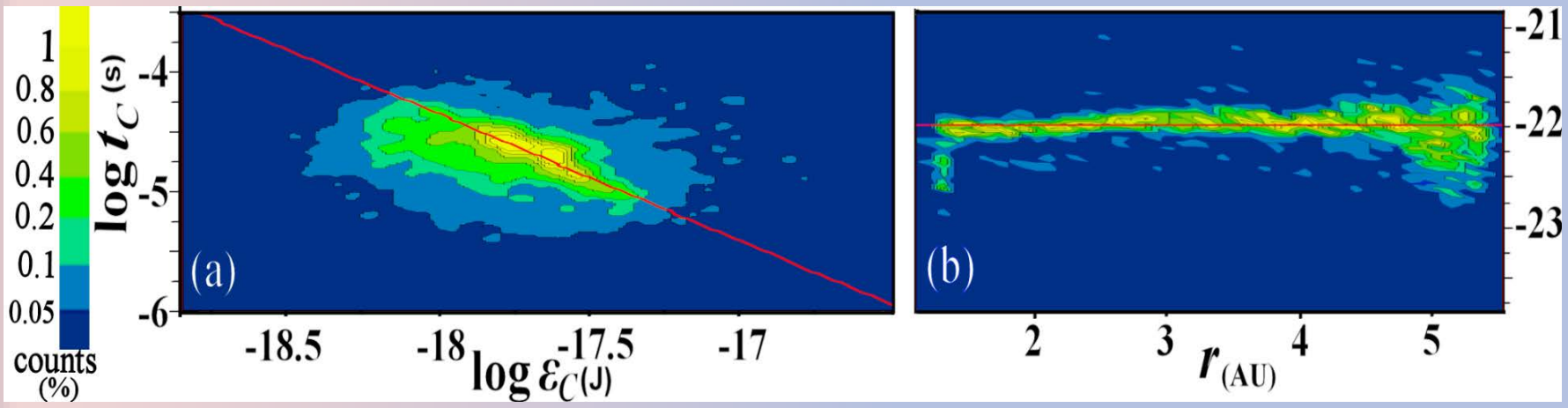
i.e., $$\hbar_* = \varepsilon_C / \omega_{pl} \sim (\tfrac{1}{2} m u_{ms}^{\ 2})\,(\tfrac{1}{2} \lambda_D / \theta)$$

or … $$\hbar_* \sim \sqrt{\frac{\omega}{m_i^{-1} + m_e^{-1}}}\,\frac{k_B}{2e} \cdot n^{-\frac{3}{2}} \left[\frac{B^2}{\mu_0 k_B} + n(\gamma_i T_i + \gamma_e T_e) \right]$$

$$\hbar_* \cong 0.071 n^{-\frac{3}{2}} B^2 + 2.45 \gamma\, n^{-\frac{1}{2}} T$$

Example: Ulysses data

Ulysses measurements of the solar wind, a largely proton-electron plasma, reveal a constant $\hbar_*$,
even over a broad range of heliocentric distances and heliolatitudes from -80º to +80º.



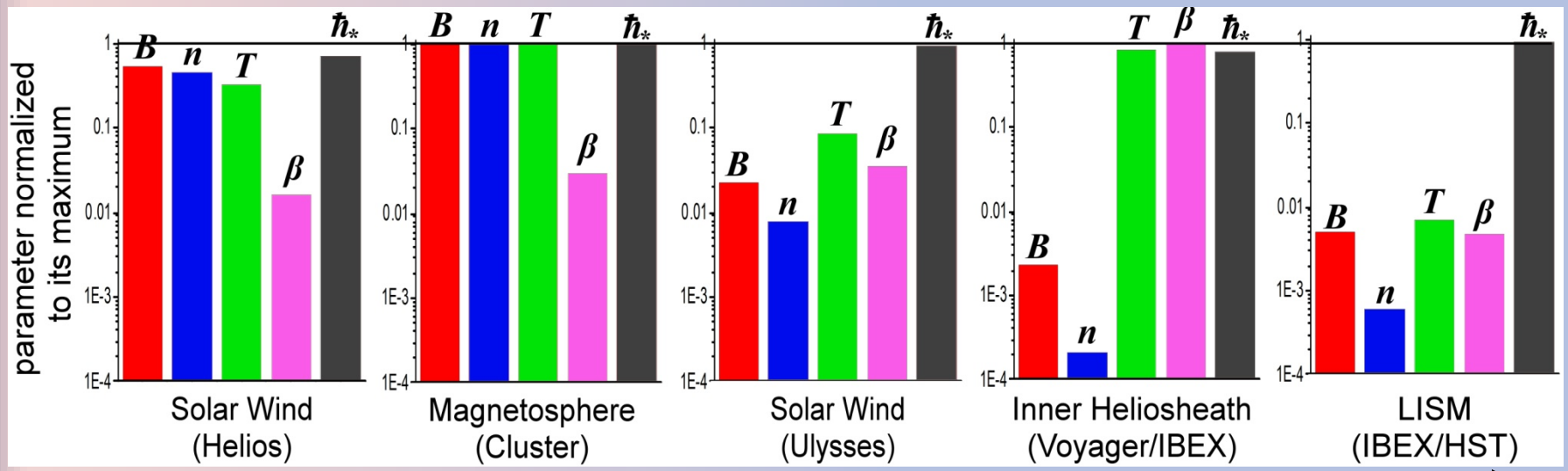
$$\log \hbar_* [\mathrm{J} \cdot \mathrm{s}] \cong -21.92 \pm 0.15$$

Livadiotis & McComas, 2013, *Entropy*, 15, 1118

Constant $\hbar_*$ throughout the heliosphere

Typical plasma parameters and $\hbar_*$ values for various space plasmas of different r. Constancy of the values of $\varepsilon_c/\omega_{pl}=\hbar_*$ in contrast to other plasma parameters is apparent.

(Each parameter is normalized to its maximum of the five samples)



Heliocentric Distance, r

Livadiotis & McComas, 2014, *J Geophys Res*, 119, 3247

Constant $\hbar_*$ throughout the heliosphere

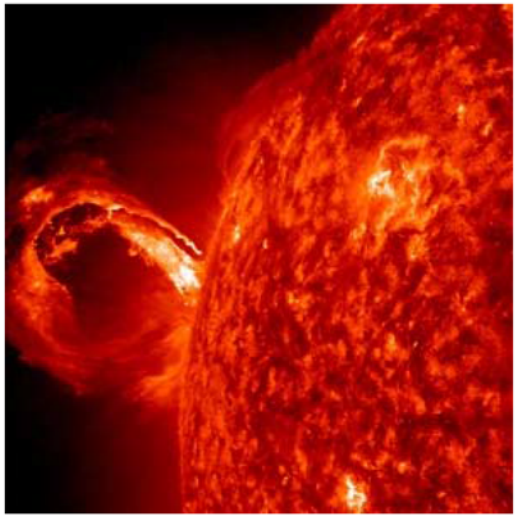
NATURE | NEWS

Space plasmas share a secret

Superhot clumps of matter behave according to a surprisingly universal rule.

Physicists have discovered a fundamentally new way to describe how information and energy flow within plasmas — the seething hot seas of electrically charged particles that make up the Sun, other stars and much of the rest of the visible Universe.

The controversial idea comes from George Livadiotis and David McComas, two physicists at the Southwest Research Institute (SwRI) in San Antonio, Texas. They had been exploring basic parameters of different plasmas across the Universe, such as their temperature, density and the strength of their magnetic fields. From that information, the researchers calculated a new parameter, called *h**, that characterizes each plasma.



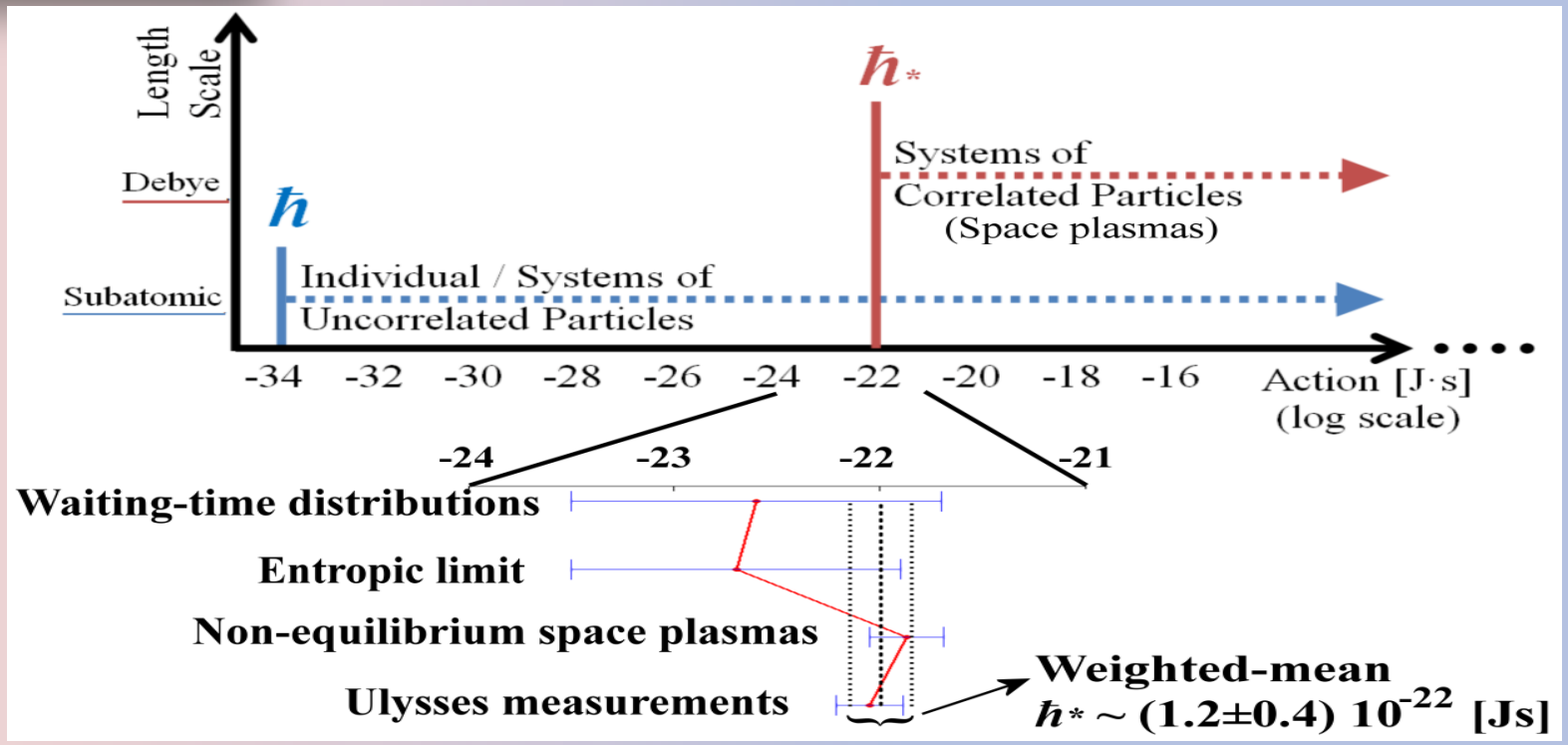
Scientists have found a new way to describe the conditions that exist in space plasmas, the seas of electrically charged particles that make up the Sun.

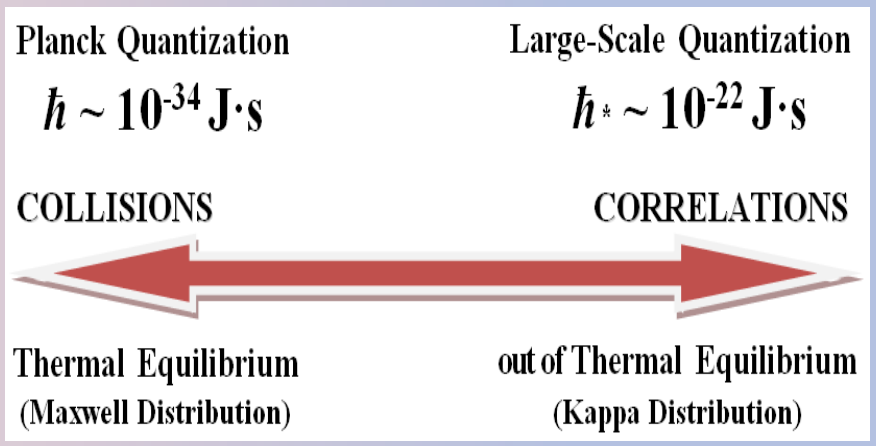
Using *h**, the authors have set out the minimum energy contained in each group of particles and the period of time for which that group exists.

"I really think George is on to something very significant," says Nathan Schwadron, a space-plasma physicist at the University of New Hampshire in Durham who was initially sceptical. "The question at the moment is whether the idea is correct. I believe it is."

Nature doi:10.1038/nature.2013.13159

The value of $\hbar_*$

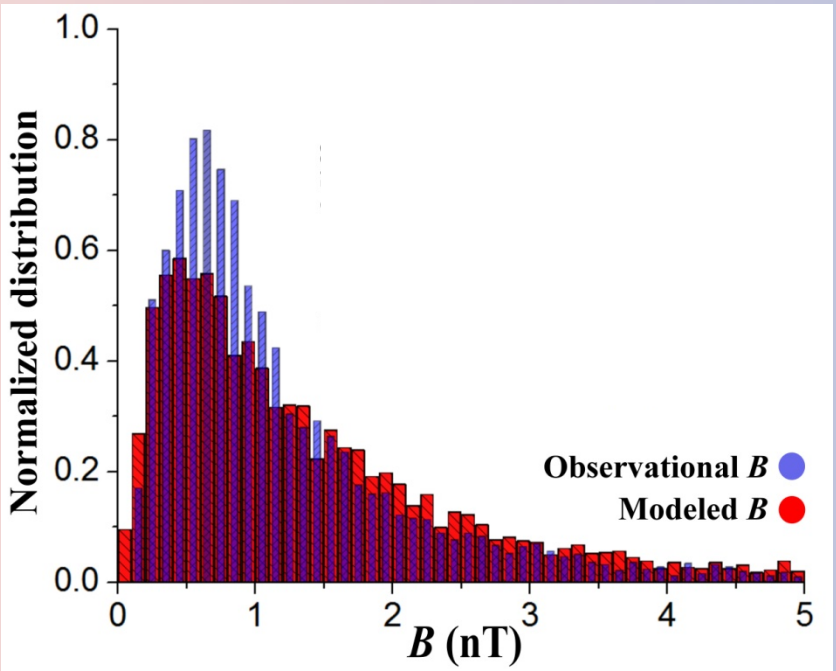




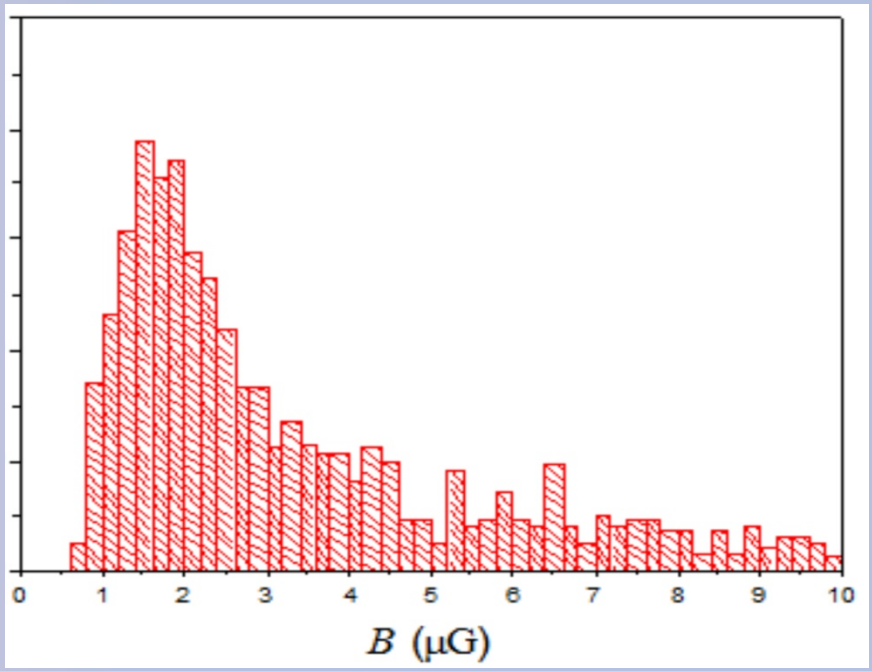
Derivation of the magnetic field

$$\hbar_* \cong 0.071 n^{-\frac{3}{2}} B^2 + 2.45\gamma n^{-\frac{1}{2}} T \Rightarrow B \cong \sqrt{14.19\hbar_* n^{\frac{3}{2}} - 34.7\gamma n T_i}$$

Solar wind
- Ulysses measurements -

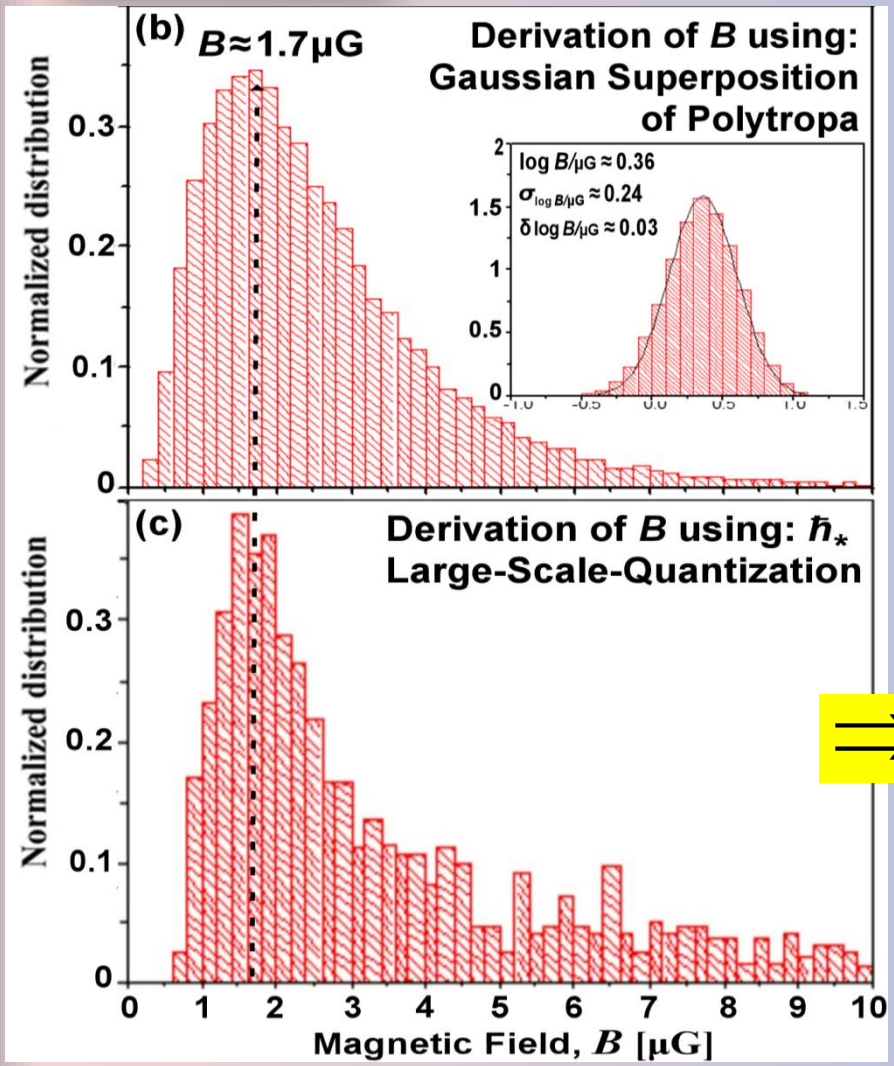


Inner Heliosheath
- IBEX measurements -



Livadiotis, 2015, *J Phys Conf Ser*, 577, 012018, pp7

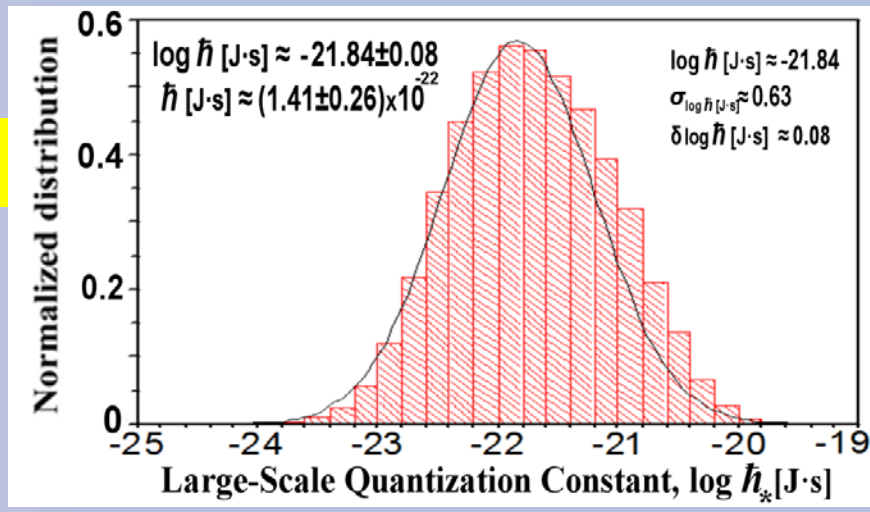
Derivation of the magnetic field (2)



Gaussian superposition of polytropic processes

$$T(n) = T_* \cdot (n/n_*)^{\bar{a}-1} \cdot \exp\left[-\tfrac{1}{2}\sigma^2 \ln^2(n/n_*)\right]$$

Applying the Bernoulli integral



Livadiotis, 2016, *ApJ Suppl Ser*, 223, 13, pp13

A new type of quantization

Planck Quantization

- Fourier Analogue:

$$\omega_{ph}\Delta t \geq \tfrac{1}{2}$$

- Uncertainty Analogue:

$$\Delta E\,\Delta t \geq \tfrac{1}{2}\hbar$$

- Quasiparticle:

(*photons*) $$\varepsilon = \hbar\,\omega_{ph}$$

- Quantization:

$$\hbar \cong 1.1\times 10^{-34}\ \mathrm{J\ s}$$

Plasma Quantization

$$\omega_{pl}\Delta t \geq \tfrac{1}{2}$$

(for some ΔE, Δt)

$$\Delta E\,\Delta t \geq \tfrac{1}{2}\hbar_*$$

$$\varepsilon = \hbar_*\omega_{pl}$$ (*plasmons*)

$$\hbar_* \cong (1.2 \pm 0.1)\times 10^{-22}\ \mathrm{J\ s}$$

7. Non-Euclidean Kappa Distributions

Generalized kappa distribution

Non-Euclidean-Normed Statistical Mechanics

George Livadiotis

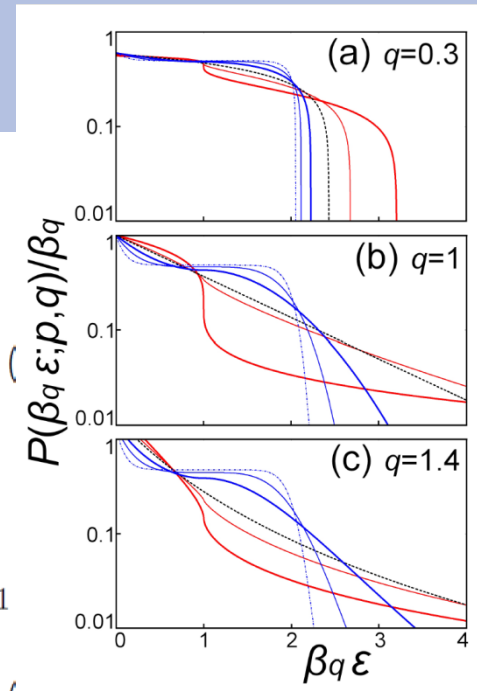
Southwest Research Institute
San Antonio, TX-78238, USA

$$P(\varepsilon; T_q; p, \kappa_0) = \frac{(k_B T_q / x_*)^{-f/2}}{A_{(p,\kappa_0)}}$$

$$\times \left\{ 1 + \frac{1}{\kappa_0 (p-1)} \left[\frac{f}{2} + x_*^{p-1} \left| \frac{\varepsilon}{k_B T_q} - \frac{f}{2} \right|^{p-2} \left(\frac{\varepsilon}{k_B T_q} - \frac{f}{2} \right) \right] \right\}^{-\kappa_0 - 1 - \frac{f/2}{p-1}} \varepsilon^{f/2-1},$$

where at the limit of high energy becomes

$$P(\varepsilon; T_q; p, \kappa_0) \cong \frac{(k_B T_q / x_*)^{-f/2}}{A_{(p,\kappa_0)}} \left[1 + \frac{x_*^{p-1}}{\kappa_0 (p-1)} \left(\frac{\varepsilon}{k_B T_q} \right)^{p-1} \right]^{-\kappa_0 - 1 - \frac{f/2}{p-1}} \varepsilon^{f/2-1}$$



Livadiotis, 2016, *Physica A*. 445, 240–255, pp16.

Conclusions

1. Introduction: What is the kappa distribution?
It replaces the classical Maxwell distribution for particle systems out of thermal equilibrium.

2. Statistical Framework:
Tsallis statistics: q-Entropy & escort probability

3. Connection with Thermodynamics:
Well-defined temperature, thermal pressure, etc.

4. Properties of kappa distribution:
Velocity/Kin.Energy, Hamiltonian, N-particle, etc.

5. Applications in space plasmas
Heliosheath, Heliosphere, LQC, etc.

Recent References

Livadiotis, G., **2016**, Law of Large Numbers for non-Euclidean L_p means, *Entropy*. Submitted.
Livadiotis, G., & Desai, M.I., **2016**, Plasma-field coupling at small length scales in solar wind near 1AU, *Astrophys. J*. Submitted.
Livadiotis, G., **2016**, Curie law for systems described by kappa distributions, *Europhys Lett.* 113, 10003, pp6.
Livadiotis, G., **2016**, Superposition of polytropa in the inner heliosheath, *Astrophys J Suppl. Ser.* 223, 13, pp13.
Livadiotis, G., **2016**, Non-Euclidean-normed Statistical Mechanics, *Physica A*. 445, 240–255, pp16.

Livadiotis, G., Dennis, B., Elaydi, S., & Kwessi, E. **2015**, Kappa function as a unifying framework for discrete population modeling, *Nat Res Mod*. 29, 130–144.
Livadiotis, G. **2015**, Shock strength in space and astrophysical plasmas, *Astrophys J*, 809, 111, 21pp.
Livadiotis, G. **2015**, Kappa & *q* indices: dependence on the degrees of freedom, *Entropy*, 17, 2062-81, 20pp.
Livadiotis, G. **2015**, Statistical background and properties of kappa distributions in space plasmas, *JGR*, 120, 1607, 13pp.
Livadiotis, G. **2015**, Kappa distribution in the presence of a potential energy, *J Geophys Res*, 120, 880-903, 24pp.
Livadiotis, G. **2015**, Application of the theory of Large-Scale Quantization to the inner heliosheath, *JPCS*, 577, 012018, 7pp.
Livadiotis, G., Assas., L., Dennis, B., Elaydi, S., & Kwessi, E. **2015**, A discrete time host-parasitoid model with an Allee effect, *J Biol Dyn*, 9, 34-51, 18pp.

Livadiotis, G. **2014**, Lagrangian temperature: Derivation and physical meaning for systems described by kappa distributions, *Entropy*, 16, 4290-4308, 19pp.
Livadiotis, G., & McComas, D.J. **2014**, Large-scale phase-space quantization from local correlation, *JGR*, 119, 3247, 12pp.
Livadiotis G. **2014**, Chi-p distribution: Characterization of the goodness of the fitting using L^p norms, *J Stat Distr Appl*, 1, 4, 14pp.
Livadiotis, G., & McComas, D.J. **2014**, Large-scale quantization in space plasmas: Summary and applications, *ASP Conf Ser*, 484, 131, 6pp.
Livadiotis, G., & McComas, D.J. **2014**, Electrostatic shielding in plasmas and the physical meaning of the Debye length, *J Plasma Phys*, 80, 341-378, 38pp.

Livadiotis, G., & McComas, D.J. **2013**, Near-equilibrium heliosphere – Far-equilibrium heliosheath, *AIP Conf Proc*, 1539, 344-350, (7pp).
Livadiotis, G., & McComas, D.J. **2013**, Fitting method based on correlation maximization: Applications in Astrophysics, *J Geophys Res*, 118, 2863-2875, 13pp.
Livadiotis, G., & McComas, D.J. **2013**, Understanding kappa distributions: A toolbox for space science and astrophysics, *Space Sci Rev*, 75, 183–214, 32pp.
Livadiotis G, & McComas DJ. **2013**, Evidence of large scale phase space quantization in plasmas, *Entropy* 15, 1116-32, 17pp.
Livadiotis, G., McComas DJ, Schwadron N, Funsten HO & Fuselier SA **2013**, Pressure of the proton plasma in the inner heliosheath, *Astrophys J*, 762, 134, 19pp.

Livadiotis, G., McComas, D.J., Randol, B., et al., **2012**, Pick-up ion distributions and their influence on ENA spectral curvature, *Astrophys J*, 751, 64, 21pp.
Livadiotis, G., & McComas, D.J. **2012**, Non-equilibrium thermodynamic processes: Space plasmas and the inner heliosheath, *Astrophys J*, 749, 11, 4pp.

Livadiotis G, McComas DJ **2011**, Invariant kappa distribution in space plasmas out of equilibrium, *Astrophys J* 741, 88, 28pp.
Livadiotis, G., McComas, D.J. **2011**, The influence of pick-up ions on space plasma distributions, *Astrophys J*, 738, 64, 13pp.
Livadiotis, G., McComas D.J, Dayeh M.A., Funsten H.O. & Schwadron N.A. **2011**, First sky map of the inner heliosheath temperature, *Astrophys J*, 734, 1, 19pp.

Livadiotis, G., & McComas, D.J. **2010**, Non-equilibrium stationary states in the Heliosphere: The influence of pick-up ions, *AIP Conf Proc*, 1302, 70-76, 7pp.
Livadiotis, G., & McComas, D.J. **2010**, Measure of the departure of the *q*-metastable stationary states from equilibrium, *Phys Scr*, 82, 035003, 9pp.
Livadiotis G. & McComas DJ. **2010**, Exploring transitions of space plasmas out of equilibrium, *Astrophys J*, 714, 971, 17pp.

Livadiotis, G., & McComas D.J. **2009**, Beyond kappa distributions: Exploiting Tsallis statistical mechanics in space plasmas, *J Geophys Res*, 114, A11105, 22pp.
Livadiotis, G. **2009**, Approach on Tsallis statistical interpretation of hydrogen-atom by adopting the radial distribution function, *J Math Chem*, 45, 930-939, 10pp.
Livadiotis, G. **2009**, Definition and applications of the ascent-probability distribution in 1-dimensional maps", Int J Bifurcat Chaos, 19, 3567-3591, 25pp.
Livadiotis, G. **2009**, Approach to the block entropy modeling and optimization, *Physica A*, 387, 2471-2494, (24pp).

AGU FALL MEETING

San Francisco | 12–16 December 2016

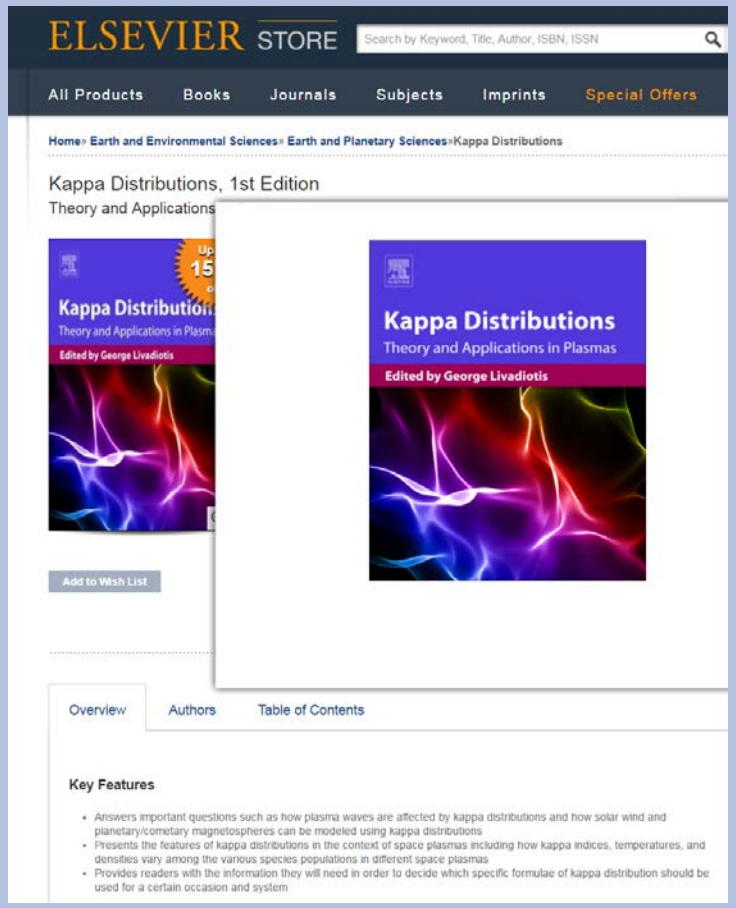
SH010: Kappa Distributions: Theory and Applications in Space Plasmas

Session ID#: 13396
Session Description:
Classical particle systems reside at thermal equilibrium with their velocity/energy distribution function stabilized into a Maxwell/Boltzmann form. On the contrary, collisionless and correlated particle systems, such as space and geophysical plasmas, are out of thermal equilibrium and characterized by a non-Maxwellian/Boltzmannian behavior, typically described by kappa distributions and combinations thereof. We welcome abstracts reporting on the progress of three major topics of contents. (A) Theoretical methods: Connection of kappa distributions with the background of non-extensive statistical mechanics, relation to thermodynamic processes, formulations and properties of isotropic and anisotropic kappa distributions. (B) Analytical plasma methods: effects of kappa distributions on various basic plasma topics, spanning linear/nonlinear plasma waves, solitons, shock waves, and dusty plasmas. (C) Applications in space plasmas: Usage and importance of kappa distributions in theoretical, simulation, and data analyses of space plasmas throughout the heliosphere and beyond.

Primary Convener:
George Livadiotis, Southwest Research Institute, San Antonio, TX, United States
Conveners:
Adolfo F. Vinas, NASA Goddard Space Flight Center, Greenbelt, MD, United States, **Peter H Yoon**, University of Maryland College Park, College Park, MD, United States and **Thomas W Broiles**, Southwest Research Institute, San Antonio, TX, United States

SM015: KAPPA DISTRIBUTIONS: ORIGIN AND EFFECTS ON PLANETARY MAGNETOSPHERES

Session ID#: 13399
Session Description:
Particle distributions in the terrestrial and planetary magnetospheres deviate from the classical Maxwellian forms, and are typically described by the velocity/energy kappa distributions. These distributions have become increasingly widespread in magnetospheric plasma physics, while a breakthrough in the field came with their connection with non-extensive statistical physics, a generalization of the classical Maxwell-Boltzmann statistical theory. The origin of magnetospheric kappa distributions is attributed to various particle acceleration and plasma transport processes, e.g., magnetic reconnection, dipolarizations and substorm injections, ionospheric outflow, shock acceleration, turbulent transport, and mixture of hot-cold plasmas. Kappa distributions have also important effect on magnetospheric plasma waves and instabilities, properties and features (e.g., field-aligned conductance, inverted-V events, sounder stimulated resonances), and interplay of particle sources (e.g., solar wind, volcanic moons) and sinks (e.g., collisions, moon/ring surfaces). We welcome abstracts reporting on the progress of the origin of magnetospheric kappa distributions and their effect on corresponding plasma processes.

Primary Convener:
George Livadiotis, Southwest Research Institute, San Antonio, TX, United States
Conveners:
Mitsuo Oka, University of California Berkeley, Berkeley, CA, United States, **Kostas Dialynas**, Office for Space Research and Technology, Academy of Athens, Athens, Greece and **Keiichi Ogasawara**, Southwest Research Institute, San Antonio, TX, United States

Kappa Distributions, 1st Edition
Theory and Applications in Plasmas