Study of Explosive Solutions of Time Delayed Nonlinear Cubic Equation Derived in Fluids (Hickernal) and Plasmas (Berk-Breizman)

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Introduction

- The physical scenario under consideration is that of a kinetic system such as a plasma, perturbed by an electric field wave. The evolution of the wave near its instability threshold is studied.
- The normalized amplitude *A* of the wave has been found to obey the following cubic equation near the instability threshold[1,2,3,4].

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = A - \frac{e^{i\phi}}{2}\int_0^{\tau/2} \mathrm{d}z\, z^2 A(\tau - z)\int_0^{\tau-2z} \mathrm{d}x\, \exp\Big[\underbrace{-\hat{\nu}^3 z^2 (2z/3 + x)}_{\text{phase-space diffusion}} \underbrace{-\,\hat{\beta}(2z + x)}_{\text{Annihilation}} \overbrace{-\,i\hat{\alpha}^2 z(z + x)}^{\text{Drag}}\Big] A(\tau - z - x)\, A^*(\tau - 2z - x)$$

τ is the time measured in units of the linear growth time $1/\gamma$;

$\hat{\nu}$, $\hat{\beta}$ and $\hat{\alpha}$ are all in units of γ; ϕ measures the contribution of the hot particles to the mode frequency.

1. F. J. Hickernell, Journal Fluid Mechanics, **142**, 431 (1984)
2. Berk H.L., Breizman B.N., Pekker M.S., 1996 *Phys. Rev. Lett.* **76** 1256.
3. Breizman B.N., Berk H.L.,et. al., 1997 *Phys. Plasmas* **4** 1559.
4. Lilley M.K., Breizman B.N., Sharapov S.E., 2009 *Phys. Rev. Lett.* **102** 195003.

Wave Equation

1. For a wave of a plasma, linear theory can be used to give global spatial relation of the electro-magnetic field, if the frequency is close enough to an eigen-frequency.

$$\mathbf{E}(\vec{r},t) = C(t)\exp\left[-i\omega_0 t\right]\mathbf{E}(\vec{r},\omega(t))$$

where $\mathbf{E}(\mathbf{r};\omega_0)$ is the spatial form of eigen-mode amplitude and C(t) the mode amplitude.

$$G_0(\omega_0) = \frac{1}{2}\int d^3r\left[\omega_0\mathbf{E}^*(\vec{r},\omega_0)\cdot\overline{\overline{\varepsilon}}(\vec{r},\omega_0)\cdot\mathbf{E}(\vec{r},\omega_0) - \frac{1}{\omega}\nabla\times\mathbf{E}^*(\vec{r},\omega_0)\cdot\nabla\times\mathbf{E}(\vec{r},\omega_0)\right] = 0,$$

2. Resulting wave evolution equation responding to linear and nonlinear resonant particles and background dissipation

$$iG_{0\omega_0}\left(\partial C(t)/\partial t + \gamma_d C\right) = -i\int d\vec{r}\,\vec{j}_{res}(\vec{r},t)\cdot\mathbf{E}(\vec{r},\omega_0)e^{i\omega_0 t}$$

$$= -i\int d\Gamma e f_{res}(\vec{r},\vec{p},t)\vec{\mathrm{v}}\cdot\mathbf{E}(\vec{r},\omega_0)e^{i\omega_0 t}$$

$\omega G_{0\omega}|C|^2 \equiv$ wave energy of the eigenmode

The perturbed resonant distribution function, $f_{res} = f(\vec{r},\vec{p},t) - F_0(E,\mu,P_\phi)$ expressed in to third order in the field amplitude $C(t)\mathbf{E}(\vec{r},\omega_0)$
and substituted into above field equation, to produce the
"time delayed cubic nonlinear equation" [Hickernell (1984), shear flow in fluids] and [Berk-Breizman (1997), resonant kinetic response in plasmas] shown previous slide.

Brief description of the model

- What does it do? It describes the evolution of a wave in a kinetic system near the instability threshold.
- An inverted energetic-particle distribution function causes the instability drive $\gamma_L \propto \mathrm{d}F_0/\mathrm{d}v$.
- Wave instability happens when γ_L surpasses the dissipation rate γ_d. The difference is assumed small:

$$\gamma \equiv \gamma_L - \gamma_d \ll \gamma_L \sim \gamma_d,$$

  leading to a perturbative expansion to cubic order in the mode amplitude.
- This theory is valid for time scales shorter than the bounce period $\sim \omega_b^{-1}$ ($\omega_b \propto |\mathbf{E}|^{1/2}$ is the bounce frequency),

  and stochastic correlation time ~ $1/\nu_{corr}$

Evolution of the wave

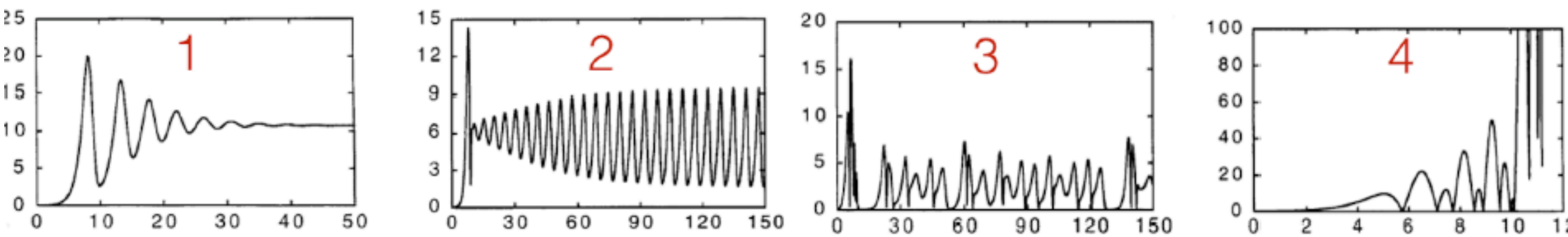
- The evolution of the normalized mode amplitude A is determined by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = A(t) - \frac{e^{i\phi}}{2}\int_0^{t/2} \mathrm{d}z\, z^2 \int_0^{t-2z} \mathrm{d}y\, K\, A(t-z)A(t-z-y)A^*(t-2z-y).$$

Diffusion: $K = e^{-\hat{\nu}^3 z^2(2z/3+y)}$; Annihilation: $K = e^{-\hat{\beta}(2z+y)}$;
Drag: $K = e^{-i\hat{\alpha}^2 z(z+y)}$. *or* $K=\exp[-\hat{\nu}^3 z^2(2z/3+y)-\hat{\beta}(2z+y)-i\hat{\alpha}^2 z(z+y)]$

- For small drag ($\hat{\alpha}$), in order of decreasing values of $\hat{\nu}$ or $\hat{\beta}$:

1. Saturation, 2. Periodic amplitude modulation, 3. Chaos, 4. Explosion.



Extreme possibilities; steady or oscillatory

1. steady; established with enough background relaxation
2. Oscillatory; Cannot hover near marginal stability; can be precursor to chirping

Point 1 was covered previous talks by Vinicius Duarte and applied to experimental data in D-III-D and NSTX, to determine the compatibility of model's predictions [i.e. (a) chirping or (b) steady oscillations] with experimental data
Part 2 of talk will discuss structure of 'explosive' solutions as a function of the system parameter, ϕ, when transport parameters are neglected.

Evolution of the wave

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$$K = 1$$

Now transport coefficients taken to be negligible.
What is the nature of the resulting explosive solution as a function of the system parameter, ϕ ?

Explosive regime

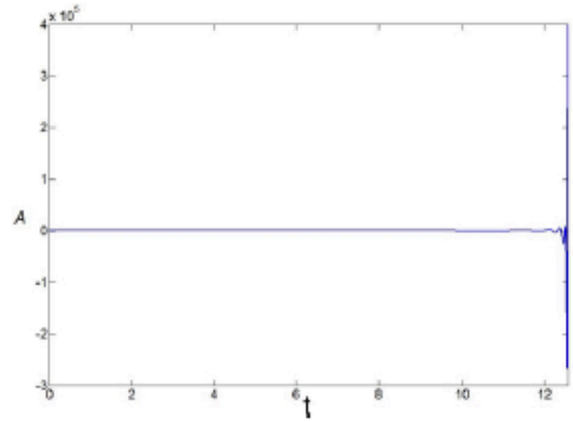
- Two main features: Blows up in a finite time t_0 ($A \sim (t_0 - t)^{-5/2}$), and oscillates with ever-increasing frequency.
- For the evolution equation to be valid, we require $|A| \ll (\gamma_L/\gamma)^{5/2}$. The cubic term competes with the linear term when $|A| \sim 1$. Hence, we have the window

$$1 < |A| < (\gamma_L/\gamma)^{5/2}$$

where the explosive behavior can be studied with this theory.

Simulations of the explosive regime

- An appropriate time variable is needed to resolve the oscillations. We seek solutions of the form $A = g(x)/(t_0 - t)^{5/2}$, where $g(x)$ is an oscillatory function of $x = -\ln(t_0 - t)$.



- Neglect $\hat{\nu}, \hat{\beta}, \hat{\alpha}$.
- We use a uniformly-spaced grid in x, which forces us to know t_0 a priori. The number of data points one obtains is *very* sensitive to the choice of t_0. We run the code iteratively, correcting t_0.

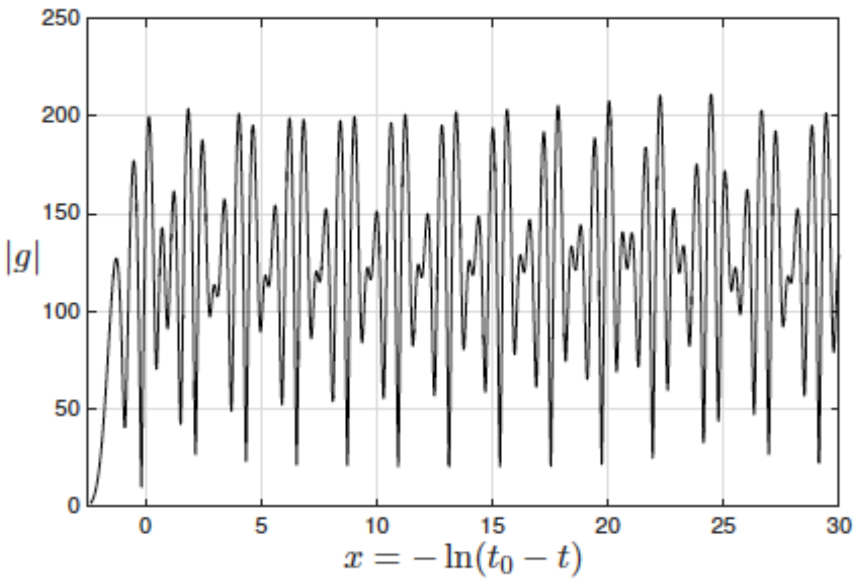
Reducing the evolution equation

- Substitute the form $A = g(x)/(t_0 - t)^{5/2}$ into the evolution equation.
- By ignoring the linear term in the evolution equation, and setting the transport parameters to zero, we obtain a reduced equation for the oscillatory function $g(x)$:

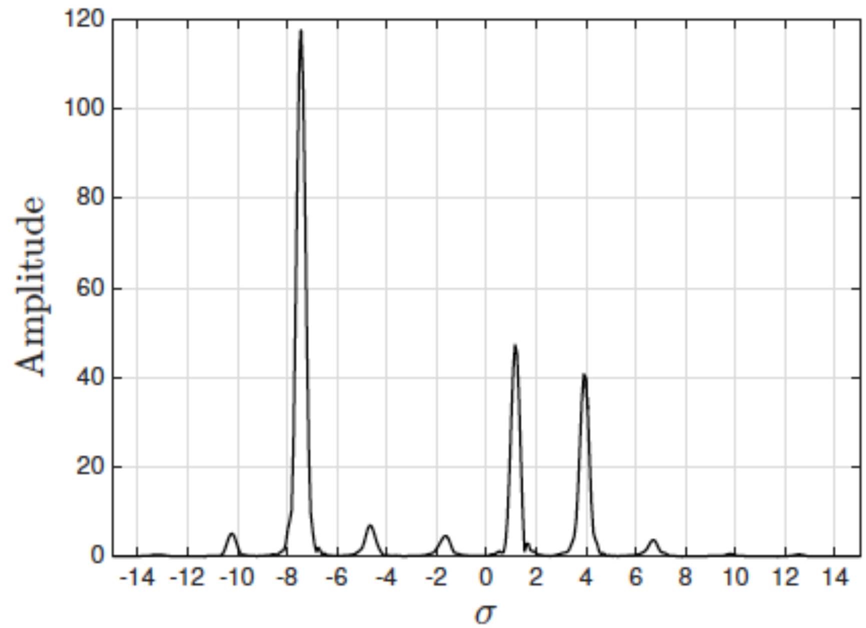
$$\frac{\mathrm{d}g}{\mathrm{d}x} + \frac{5}{2}g \;=\; -\frac{\mathrm{e}^{i\phi}}{2}\int_0^U \mathrm{d}\xi \int_0^{2(U-\xi)} \mathrm{d}\eta\, V(\xi,\eta)\, g(x_1)g(x_2)g^*(x_3),$$

where $V(\xi,\eta) = \xi^2/[(1+\xi)(1+\xi+\eta)(1+2\xi+\eta)]^{5/2}$ and the limit $U = (t/2)/(t_0 - t)$ is large in the explosive regime.

After a sufficient number of iterations (for $\phi = 0.3$):

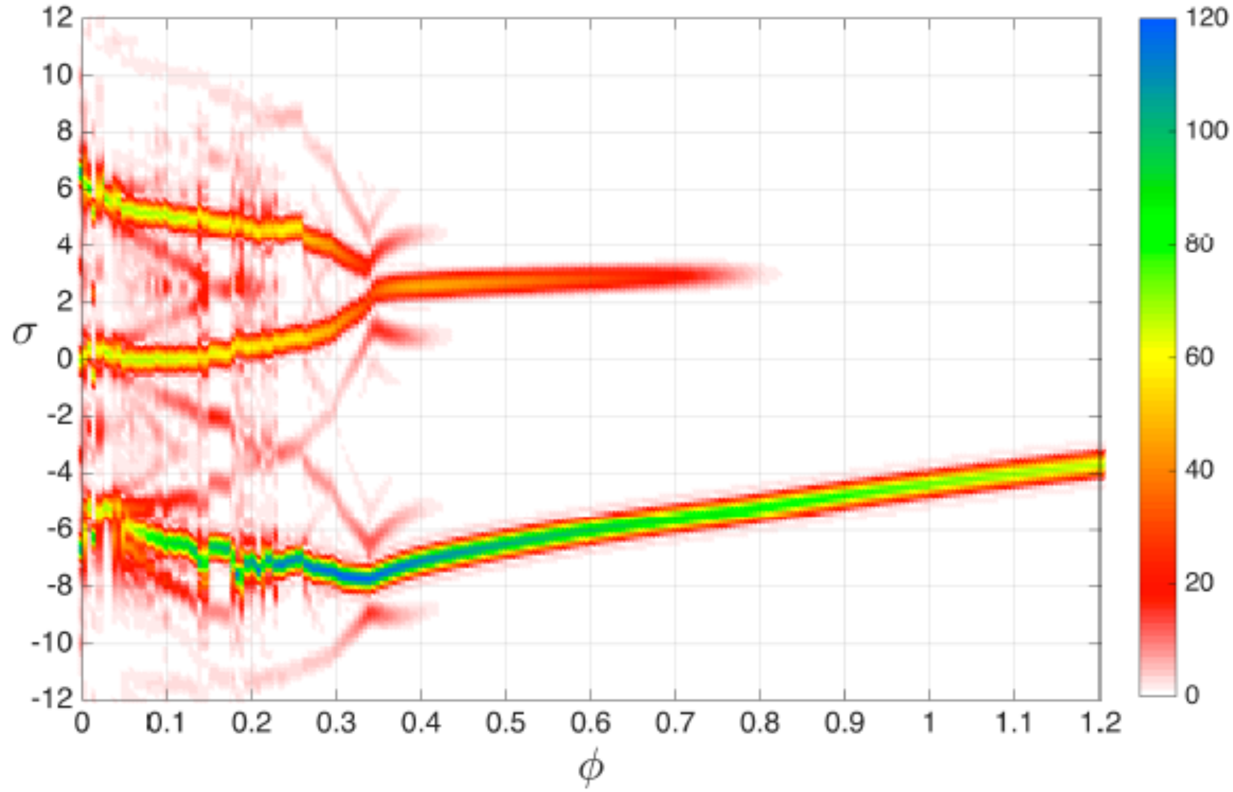


Evolution of $|g| = |A|(t_0 - t)^{5/2}$.



Fourier spectrum from $x = 5$ to $x = 30$.

Repeat for different values of ϕ and we can observe the variation of the spectrum as a function of ϕ:



- Color indicates Fourier amplitude.
- For $\phi = 0$ we find a response that is close to a solution proposed in Berk et al., PRL, 1996. For $\phi > 0.782 \approx \pi/4$ we observe the exact, one-component solution found in Breizman et al., Phys. Plas., 1997.
- No significant transients except for very small ϕ ($\lesssim 0.05$).

Solving the reduced equation

- Previous attempts assumed a Fourier series expansion

$$g = \sum_m c_m e^{im\sigma x},$$

  or even a simple one-term oscillation

$$g = c e^{i\sigma x},$$

  which is actually an exact solution.
- We propose a superposition of incommensurate pseudo frequencies:

$$g = e^{i\sigma_0 x} \sum_{\vec{m}} c_{\vec{m}}\, e^{i\vec{m}\cdot\vec{\sigma}x}.$$

Predicted chirping

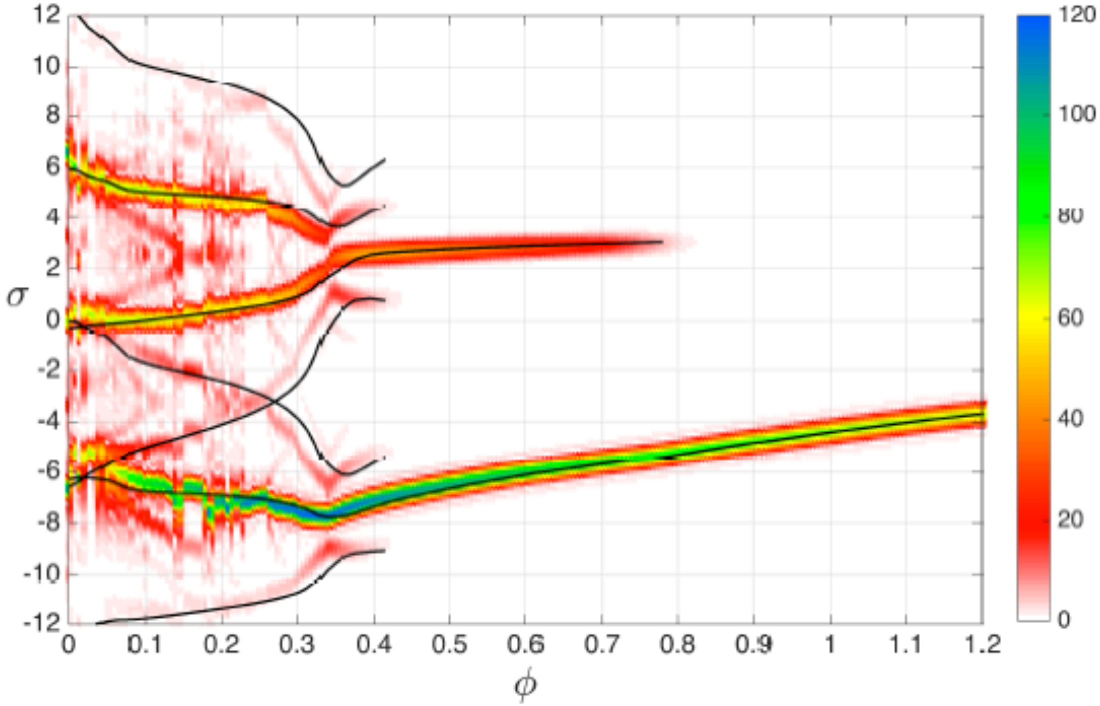
- In view of our form ($g = \mathrm{e}^{i\sigma_0 x} \sum c_{\vec{m}} \mathrm{e}^{i\vec{m}\cdot\vec{\sigma}x}$), an arbitrary term in g oscillates as

$$\exp[i(\sigma_0 + \vec{m}\cdot\vec{\sigma})x] = \exp[-i(\sigma_0 + \vec{m}\cdot\vec{\sigma})\ln(t_0 - t)].$$

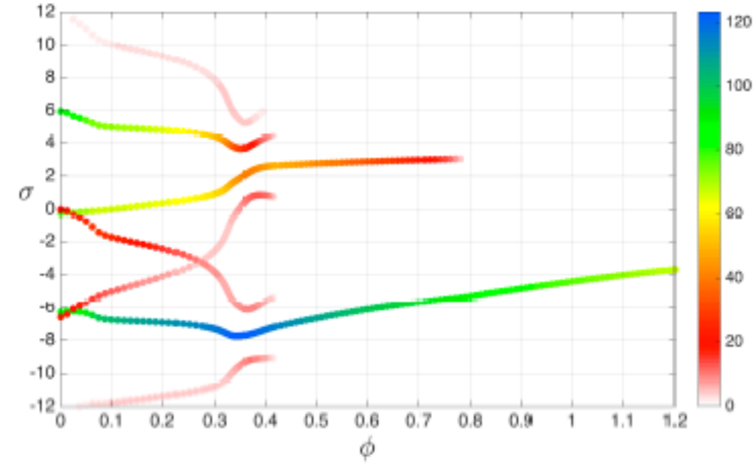
The (real) frequencies of the wave in the explosive regime are thus

$$\frac{\mathrm{d}}{\mathrm{d}t}[-(\sigma_0 + \vec{m}\cdot\vec{\sigma})\ln(t_0 - t)] = \frac{\sigma_0 + \vec{m}\cdot\vec{\sigma}}{t_0 - t}.$$

The pseudo-frequency $\sigma_0 + \vec{m}\cdot\vec{\sigma}$ of each component determines the direction and strength of the chirping effect.



Simulation spectra (color) overlaid with our theoretical prediction (black curves).

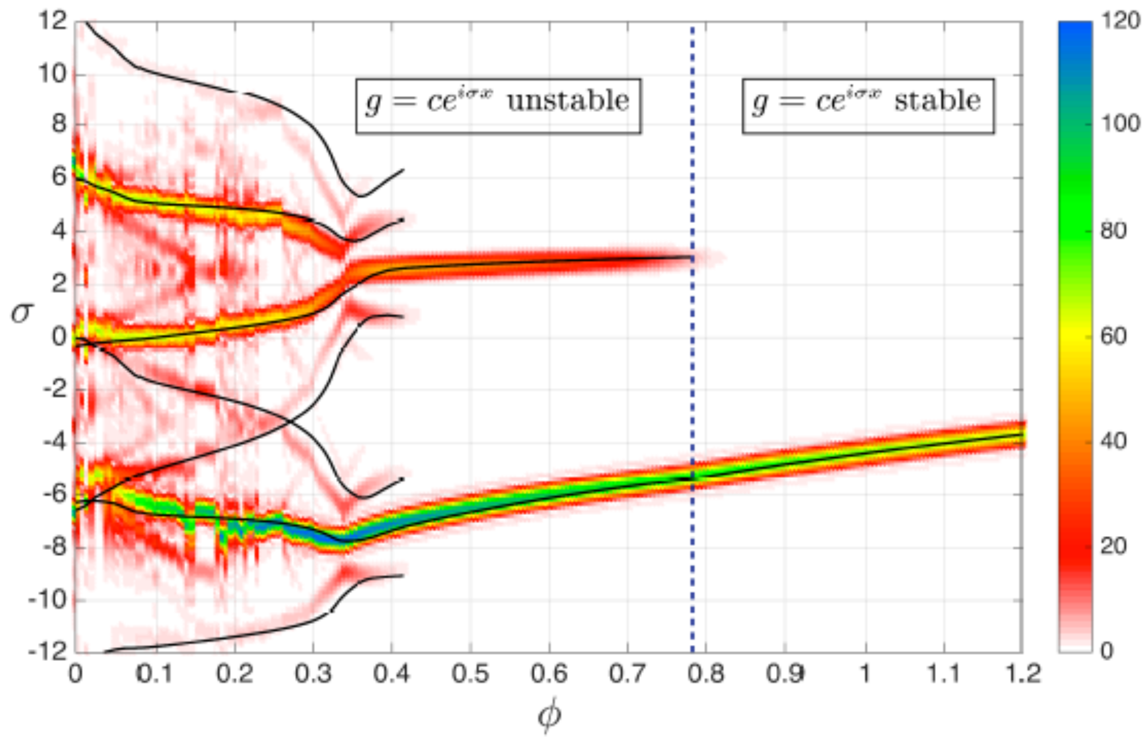


Amplitude of our theoretical prediction.

- No significant change in simulation spectra with different pseudo time step size Δx (recall $x = -\ln(t_0 - t)$). The match with the theoretical prediction improves with smaller Δx, this being most evident around $\phi = 0.35$.

Stability

- Our form replicates the simulations spectra. Why do the other solutions not show up?
- We find that for $\phi < 0.782$ the exact solution, $g = ce^{i\sigma x}$, is unstable to an oscillation at a frequency that agrees well with the component that emerges at this point in our simulations.

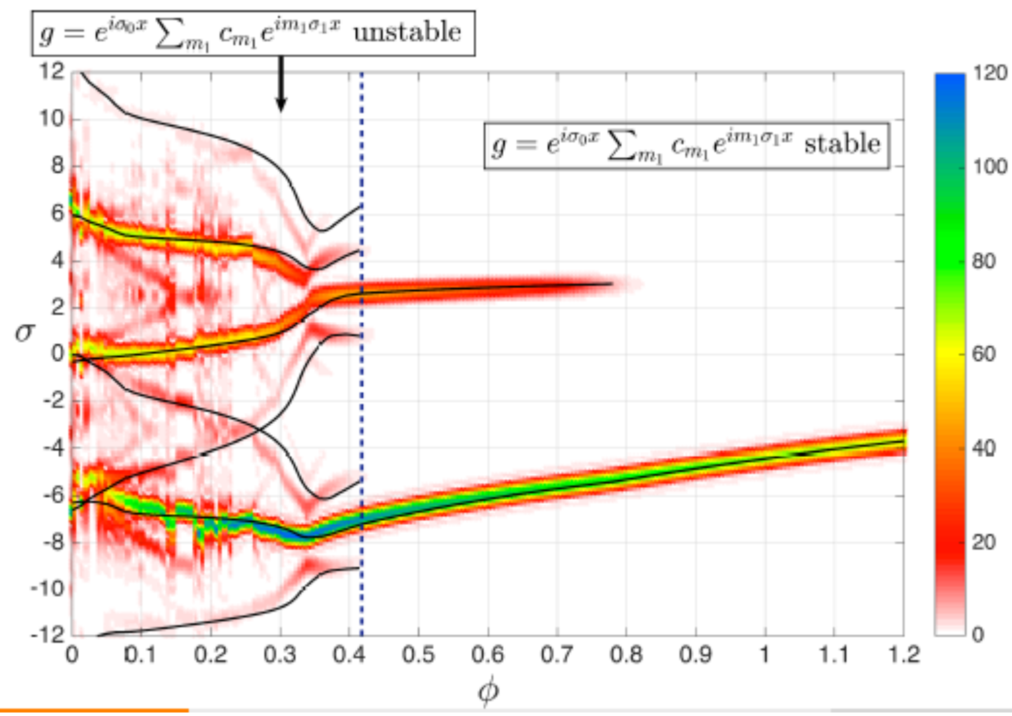


Stability

- Thus, for $\phi < 0.782$, we require two frequency variable, σ_0 and σ_1:

$$g = e^{i\sigma_0 x} \sum_{m_1} c_{m_1} e^{i m_1 \sigma_1 x}.$$

- Then, a stability analysis of this solution shows that it is unstable for $\phi < 0.418$, at which point side bands develop.



Stability

- We need yet another frequency variable for $\phi < 0.418$:

$$g = \mathrm{e}^{i\sigma_0 x} \sum_{m_1,m_2} c_{m_1,m_2} \mathrm{e}^{i(m_1\sigma_1 + m_2\sigma_2)x}.$$

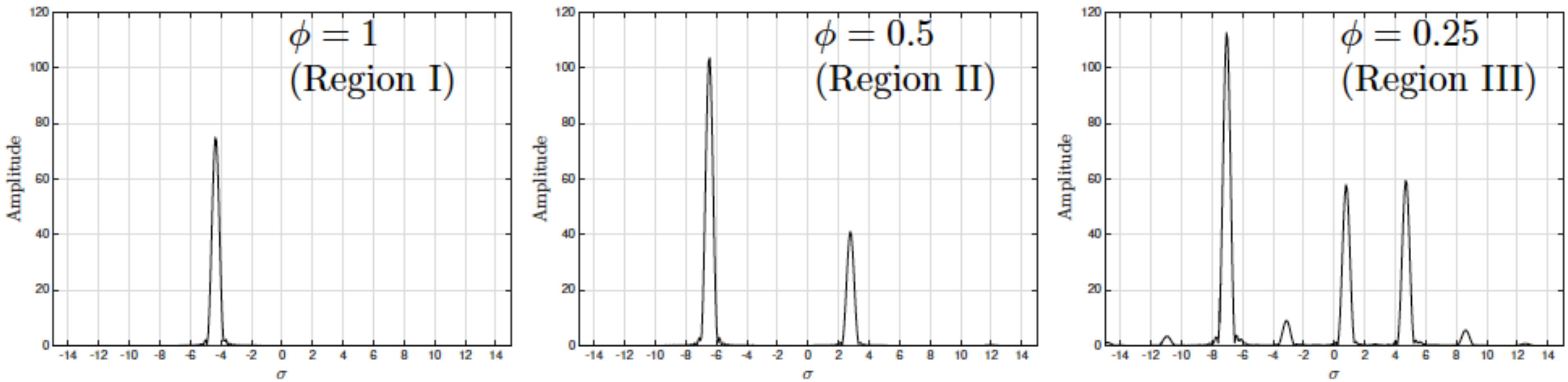
- This form reproduces the observed side bands which then evolve to the observed spectral components.
- This explains the whole ϕ domain, although some issues remain for $\phi \lessapprox 0.05$.

Simulation vs. analytical approximations

- We have implicitly defined three regions (in increasing level of complexity):

$$\text{I. } \phi > 0.782, \qquad \text{II. } 0.418 < \phi < 0.782, \qquad \text{III. } \phi < 0.418.$$

- For comparison with simulations, we submit our analytic solutions to the same processes as the simulations.



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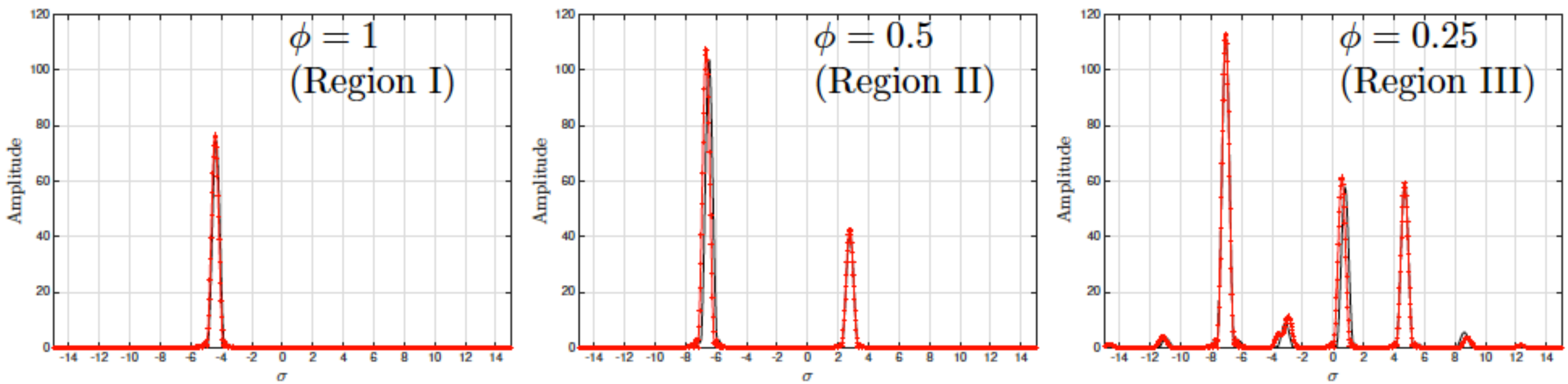
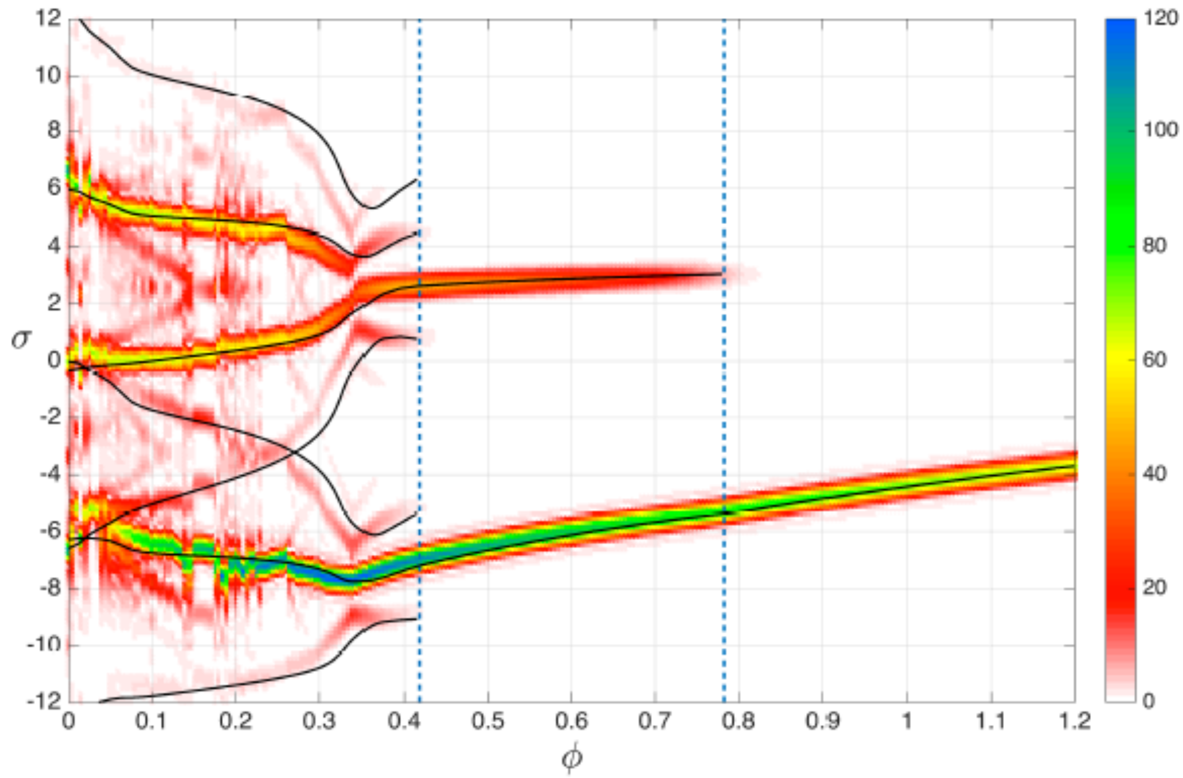


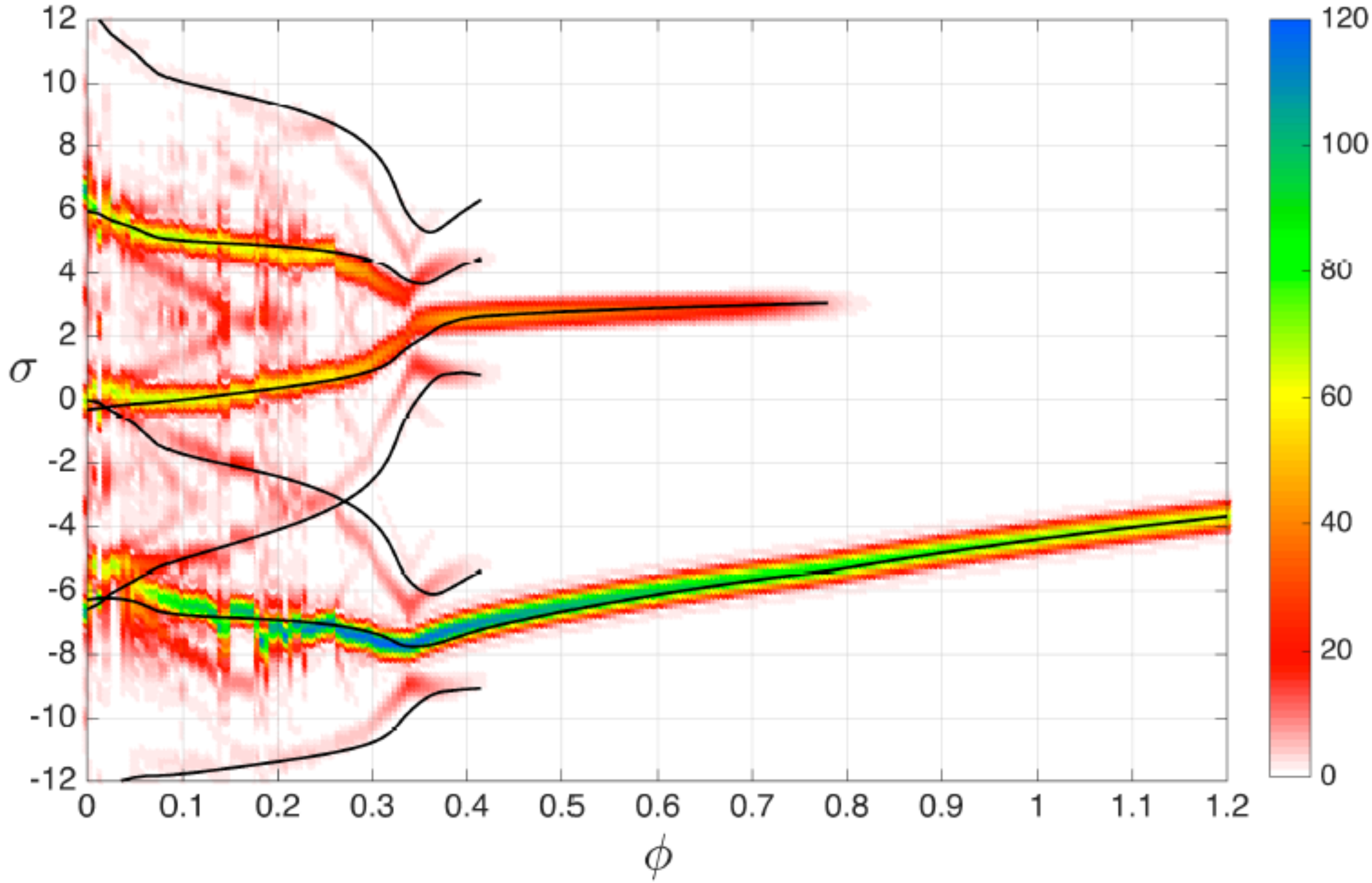
Some remarks

- When more than one equilibria are available to the system, it attracts to the more complex one.
- Strong trend towards complexity as ϕ decreases.



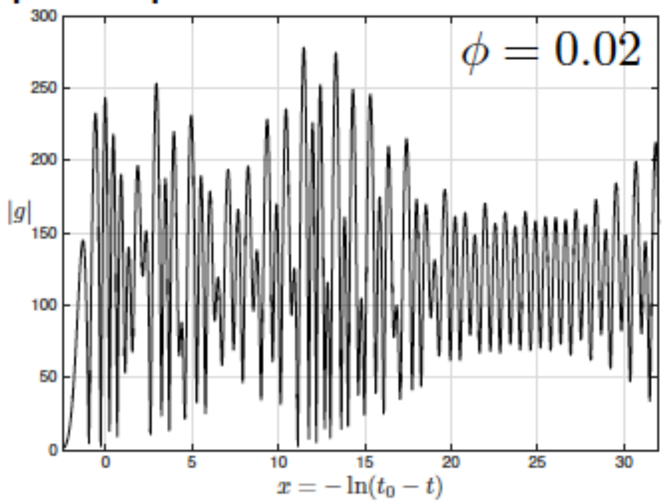
Issues in the small ϕ ($\lessapprox 0.05$) region

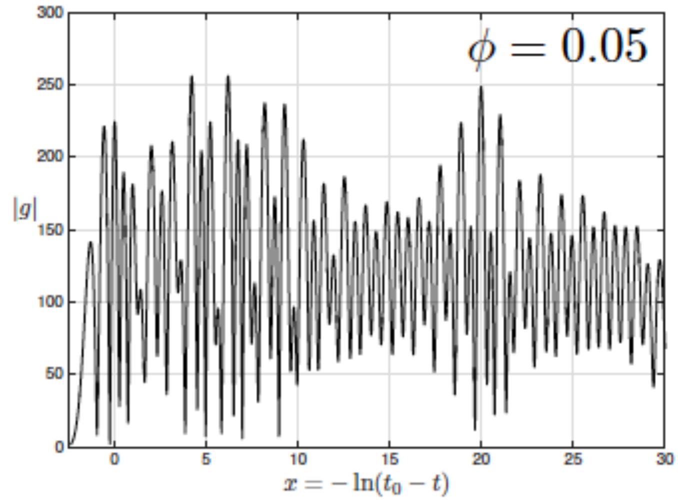
Note jaggedness for $\phi \lessapprox 0.05$ in simulation spectrum. The match with predictions is also somewhat worse than for other ϕ values.



Issues in the small ϕ ($\lesssim 0.05$) region

- Transient behavior observed in this region: our solution may still be correct but it may be taking longer for an equilibrium to establish. Or perhaps it has a more complicated form that the one we assumed.





- Number of doubling times n can be estimated from

$$2^n = \left(\frac{t_0 - t_i}{t_0 - t_f} \right)^{5/2} \Rightarrow n \approx 112.$$

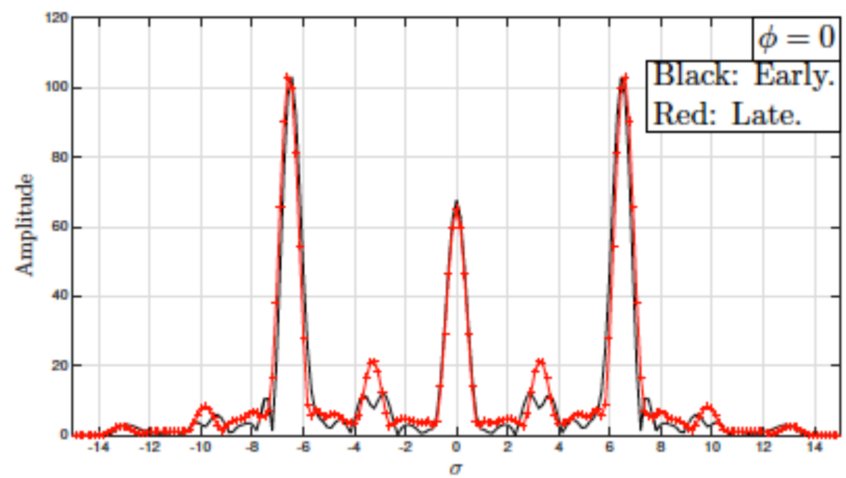
Apparently, this is enough to obtain a converged solution everywhere in the ϕ domain, except in this small ϕ region.

Other attempts at explaining $\phi \lessapprox 0.05$

- The response for $\phi = 0$ is close to being a simple Fourier series expansion, as was proposed by Berk et al., PRL, 1996:

$$g = \sum_m c_m e^{im\sigma x},\ \sigma \approx 6.58.$$

- Analytically, we find that this solution is unstable to side bands that develop halfway in-between the preexisting components. This has correspondence with the simulations.

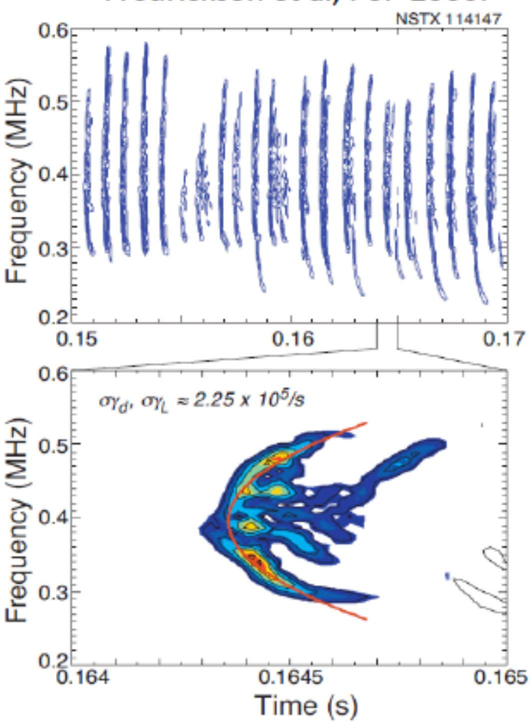


Closing remarks and future work

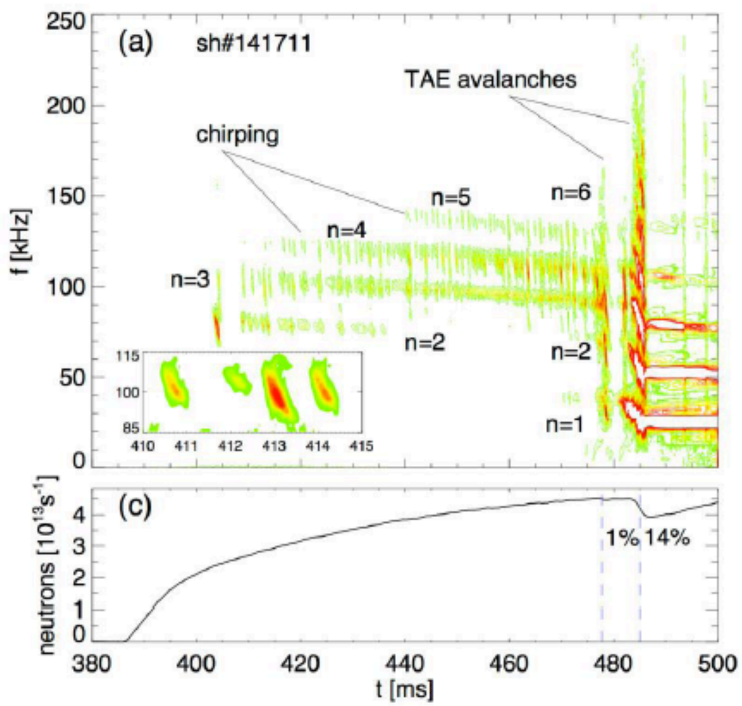
- During the explosion, the cubic equation eventually breaks down. However,
  - The solutions found are the precursors of the full nonlinear evolution that has been studied via a code that solves Liouville-type equations.
  - The chirping effect of the explosion continues beyond the applicability of the cubic equation with the formation of a hole and/or clump in phase space.

Chirping modes can degrade the confinement of energetic particles

GAE/CAE
Fredrickson et al, PoP 2006.



TAE Podestà et al, NF 2012



Up to 40% of injected beam is observed to be lost in DIII-D and NSTX

Chirping is ubiquitous in NSTX but rare in DIII-D. Why??

This presentation focuses on the conditions for chirping onset rather than their long-term evolution

Summary

- By direct simulations and analytic work, new explosive solutions to the Hickernell/Berk-Breizman equation have been found.
- Over most of the ϕ domain, we found a family of analytical solutions that agrees well with the simulations. We find that when this agreement happens, the analytical solutions are linearly stable. Out of all the explosive solutions that are possible, the system is attracted to a specific solution that depends on the parameter ϕ.
- There are some discrepancies in the region $\phi \lessapprox 0.05$, but the overall structure of the theoretical prediction is not far off from the simulations. The possibility of two attractors in this region is being considered.

Thank You

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