Modeling efforts in hybrid kinetic-MHD and fully kinetic theories

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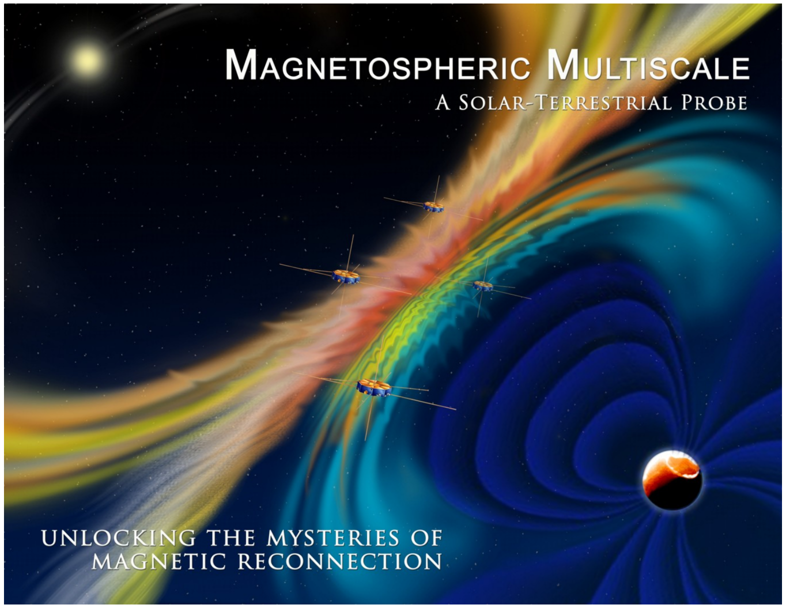
Hybrid kinetic-fluid models for plasma physics

- MHD simulations are invalidated by the presence of energetic particles
- **Hybrid philosophy: a fluid interacts with a hot particle gas**
- Many linear hybrid models exist – here, we focus on **nonlinear models**.



Energetic solar wind interacts with Earth's magnetosphere

- In fusion, two coupling options arose by *inserting assumptions in the equations* [Park & al. (1992); Kim & al. (1994); Todo & al. (1995)]

Formulating hybrid models require powerful and general methods

. . . **we shall use symmetry methods!**

Different hybrid models [Park *et al.*(1992)]

$$\rho_b \frac{d\mathbf{v}_b}{dt} + \frac{\partial \rho_h \mathbf{v}_{h,\|}}{\partial t} = -\nabla p_b - \nabla \cdot \mathsf{P}_h + \mathbf{J} \times \mathbf{B}, \tag{1}$$

where, $\partial(\rho_h \mathbf{v}_{h,\perp})/\partial t$ is neglected compared to the perpendicular momentum change of the bulk plasma, and $\mathsf{P}_h \equiv \int \mathbf{v}\mathbf{v} f_h \, d^3 v$ without the usual velocity shift. For the hot particles alone, we have

$$\frac{\partial \rho_h \mathbf{v}_h}{\partial t} = -\nabla \cdot \mathsf{P}_h + \mathbf{J}_h \times \mathbf{B} + q_h \mathbf{E}_\perp. \tag{2}$$

By subtracting the parallel component of Eq. (2) from Eq. (1), we obtain the pressure coupling equation:

$$\boxed{\rho_b \frac{d\mathbf{v}_b}{dt} = -\nabla p_b - (\nabla \cdot \mathsf{P}_h)_\perp + \mathbf{J} \times \mathbf{B}.} \tag{3}$$

Alternatively, by subtracting all components of Eq. (2) from Eq. (1), we obtain the current coupling equation:

$$\boxed{\rho_b \frac{d\mathbf{v}_b}{dt} = -\nabla p_b + (\nabla \times \mathbf{B} - \mathbf{J}_h) \times \mathbf{B} + q_h \mathbf{v}_b \times \mathbf{B}.} \tag{4}$$

Two couplings are possible: pressure coupling vs. current coupling.

Let's derive them...

Starting point: Vlasov-multifluid system

Two fluid species (electrons + fluid ions) interact with energetic ions:

$$\rho_s \frac{\partial \boldsymbol{u}_s}{\partial t} + \rho_s \left(\boldsymbol{u}_s \cdot \nabla \right) \boldsymbol{u}_s = a_s \rho_s \left(\mathbf{E} + \boldsymbol{u}_s \times \mathbf{B} \right) - \nabla \mathsf{p}_s$$

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot \left(\rho_s \boldsymbol{u}_s \right) = 0$$

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m_h} \cdot \frac{\partial f}{\partial \mathbf{x}} + q_h \left(\mathbf{E} + \frac{\mathbf{p}}{m_h} \times \mathbf{B} \right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \sum_s a_s \rho_s \boldsymbol{u}_s - \mu_0 \, a_h \int \mathbf{p} \, f \, \mathrm{d}^3\mathbf{p}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\epsilon_0 \nabla \cdot \mathbf{E} = \sum_s a_s \rho_s + q_h \int f \, \mathrm{d}^3\mathbf{p} \,, \qquad \nabla \cdot \mathbf{B} = 0$$

<u>Notice</u>: *Vlasov eqn is used here; see later for drift-kinetic approximation*

Current-coupling scheme for hybrid MHD

- Take the sum $\rho_i \boldsymbol{u}_i + \rho_e \boldsymbol{u}_e$ and neglect electron inertia. Neutrality $\epsilon_0 \to 0$ and ideal Ohm's law $\mathbf{E}+\boldsymbol{u}\times\mathbf{B}=0$ (*neglects hot density*) yield

$$\rho\frac{\partial \boldsymbol{u}}{\partial t}+\rho\left(\boldsymbol{u}\cdot\nabla\right)\boldsymbol{u}=\left(q_h\,\boldsymbol{u}\int f\,\mathrm{d}^3\mathbf{p}-\,a_h\int\mathbf{p}\,f\,\mathrm{d}^3\mathbf{p}+\frac{1}{\mu_0}\nabla\times\mathbf{B}\right)\times\mathbf{B}-\nabla\mathsf{p}$$

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m_h}\cdot\frac{\partial f}{\partial \mathbf{x}}+q_h\left(\frac{\mathbf{p}}{m_h}-\boldsymbol{u}\right)\times\mathbf{B}\cdot\frac{\partial f}{\partial \mathbf{p}}=0$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0\,,\qquad\qquad \frac{\partial \mathbf{B}}{\partial t}=\nabla\times\left(\boldsymbol{u}\times\mathbf{B}\right)\,.$$

Current-coupling scheme (CCS) used in [Belova et al.(1997), Chen et al.(1999)].

- At this point, one would like to insert the assumptions

$$\frac{1}{\rho}\int f\,\mathrm{d}^3\mathbf{p}\ll 1\,,\qquad \frac{1}{\rho}\int\mathbf{p}\,f\,\mathrm{d}^3\mathbf{p}\ll 1\,,\qquad T_h\gg T_c$$

where T_h and T_c are the hot and cold temperatures, respectively.

Pressure-coupling MHD scheme (PCS)

- Dynamics of total momentum $\mathbf{M} = \rho\boldsymbol{u} + \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ yields

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \frac{\partial}{\partial t}\int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p} \;=\; -\nabla\cdot\mathbb{P} - \nabla \mathsf{p} + \frac{1}{\mu_0}\,\mathrm{curl}\,\mathbf{B}\times\mathbf{B}\,.$$

  where $m_h\mathbb{P} = \int \mathbf{p}\mathbf{p}\, f\, \mathrm{d}^3\mathbf{p}$ is the kinetic stress tensor (absolute pressure)

- In the literature, the PCS is obtained from above by assuming

$$\frac{\partial}{\partial t}\int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p} \simeq 0\,,$$

  and leaving all other equations unchanged (including Vlasov).

- [Park & al.(1992)] claimed essential equivalence of CCS and PCS

PCS doesn't conserve the CCS energy exactly: how are they equivalent?

These points could be approached by hard analytical methods

. . . *we shall use geometry instead!*

Geometry & symmetry in Hamiltonian plasma dynamics

Poisson brackets and symmetry

- Particles carry *canonical PB*, not applicable to Eulerian continuum theories
- Special noncanonical PBs arise from **geometric symmetry arguments**
- *Symmetric Hamiltonian systems* $\dot{\mu} = \{\mu, H\}$ carry the **Lie-Poisson bracket** (LPB)

$$\{F, G\}(\mu) = \left\langle \mu, \left[\frac{\mathrm{d}F}{\mathrm{d}\mu}, \frac{\mathrm{d}G}{\mathrm{d}\mu}\right]\right\rangle$$

where $\langle\cdot,\cdot\rangle$ is a scalar product and $[\cdot,\cdot]$ *is a symmetry commutator*.

Commutator arises from symmetry underlying dynamics!

- Rotational symmetry for vectors (*rigid body motion*):

$$[\mathbf{g},\mathbf{k}]=\mathbf{g}\times\mathbf{k} \;\rightarrow\; \{F,G\}=\boldsymbol{\mu}\cdot\frac{\mathrm{d}F}{\mathrm{d}\boldsymbol{\mu}}\times\frac{\mathrm{d}G}{\mathrm{d}\boldsymbol{\mu}}$$

- Relabeling symmetry for velocities (*Euler fluid dynamics*):

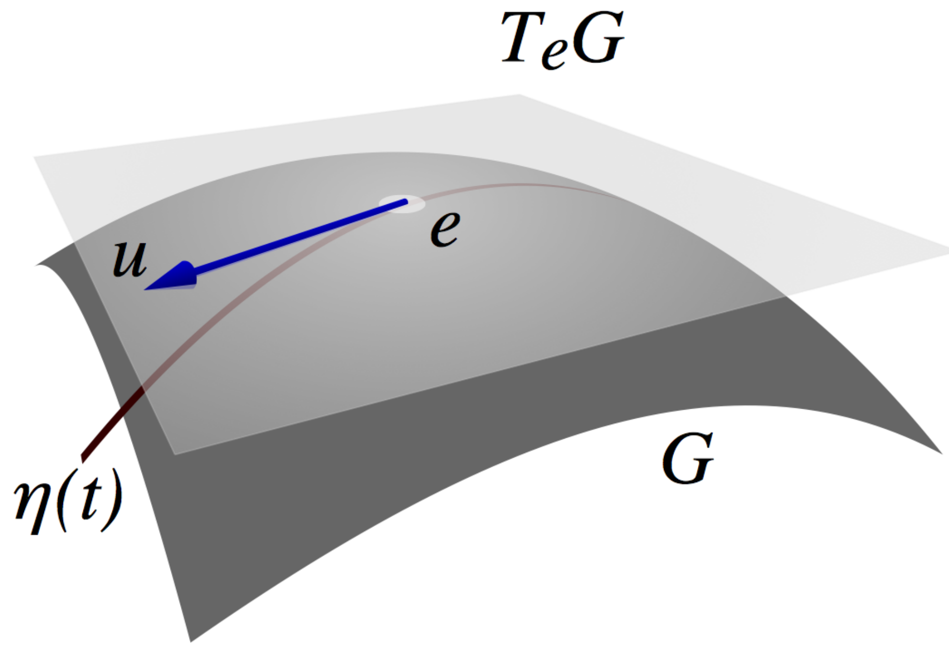
$$[\mathbf{v},\mathbf{u}]=(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v} \;\rightarrow\; \{F,G\}=\int\boldsymbol{\mu}(\mathbf{x})\cdot\left[\frac{\delta F}{\delta\boldsymbol{\mu}},\frac{\delta G}{\delta\boldsymbol{\mu}}\right]\mathrm{d}^3\mathbf{x}$$

- Unitary symmetry for matrix operators (*quantum dynamics*):

$$[A,B]=AB-BA \;\rightarrow\; \{F,G\}=\hbar\,\mathrm{Tr}\left(i\rho\left[\frac{\delta F}{\delta\rho},\frac{\delta G}{\delta\rho}\right]\right)$$

- Canonical symmetry for phase-space functions (**Vlasov equation**):

$$[h,k]=\frac{\partial h}{\partial\mathbf{x}}\cdot\frac{\partial k}{\partial\mathbf{p}}-\frac{\partial h}{\partial\mathbf{p}}\cdot\frac{\partial k}{\partial\mathbf{x}} \;\rightarrow\; \{F,G\}=\int f(\mathbf{x},\mathbf{p})\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right]\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$$



While Lagrangian dynamics of $\boldsymbol{\eta}(\mathbf{a},t)$ on G possesses the canonical PB

$$\{F,G\} = \int \left(\frac{\delta F}{\delta \boldsymbol{\eta}} \cdot \frac{\delta G}{\delta \boldsymbol{\psi}} - \frac{\delta F}{\delta \boldsymbol{\psi}} \cdot \frac{\delta G}{\delta \boldsymbol{\eta}} \right) \mathrm{d}^3\mathbf{a}\,,$$

Eulerian dynamics on the tangent space (at the identity) possesses the LPB

$$\{F,G\}(\boldsymbol{\mu}) = \left\langle \boldsymbol{\mu}, \left[\frac{\delta F}{\delta \boldsymbol{\mu}}, \frac{\delta G}{\delta \boldsymbol{\mu}} \right] \right\rangle$$

Fluids*: (γ, ψ) are Lagrangian coordinates, while $\boldsymbol{\mu} =$ fluid momentum $\mathbf{m}$.*

Vlasov*: (γ, ψ) are Lagrangian coordinates, while $\mu =$ distribution function f.*

The Maxwell-Vlasov (MV) system

$$
\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{x}} + q\left(\mathbf{E} + \frac{\mathbf{p}}{m} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{p}} = 0 \\
&\epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \frac{q}{m} \int \mathbf{p}\, f\, \mathrm{d}^3\mathbf{p} \\
&\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \\
&\epsilon_0 \nabla \cdot \mathbf{E} = q \int f\, \mathrm{d}^3\mathbf{p}\,, \qquad \nabla \cdot \mathbf{B} = 0
\end{aligned}
$$

Maxwell fields possess the canonical Poisson bracket

$$
\begin{aligned}
\{F, G\}_{Max} &= \frac{1}{\epsilon_0} \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \frac{\delta G}{\delta \mathbf{A}} - \frac{\delta F}{\delta \mathbf{A}} \cdot \frac{\delta G}{\delta \mathbf{E}} \right) \mathrm{d}^3\mathbf{x}\,, \\
H &= \frac{\epsilon_0}{2} \int |\mathbf{E}|^2\, \mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0} \int |\nabla \times \mathbf{A}|^2\, \mathrm{d}^3\mathbf{x}
\end{aligned}
$$

MV also enjoys a geometric Hamiltonian structure! [Marsden & Weinstein ('82)]

Kinetic approaches are expensive!

Better forget details? fluid approach. . .

From Vlasov to fluids: ideal MHD

- The moment fluid closure

$$f \mapsto \left(\int \mathbf{p}\, f \,\mathsf{d}^3\mathbf{p},\ \int f \,\mathsf{d}^3\mathbf{p} \right)$$

  leads to the LPB structure for ideal barotropic fluids.

- Another LPB $\{F,G\}_{MHD}(\mathbf{m},\rho,\mathbf{A})$ was also found for ideal MHD

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} = -{\mu_0}^{-1}\,\mathbf{B}\times\nabla\times\mathbf{B} - \nabla \mathsf{p}$$

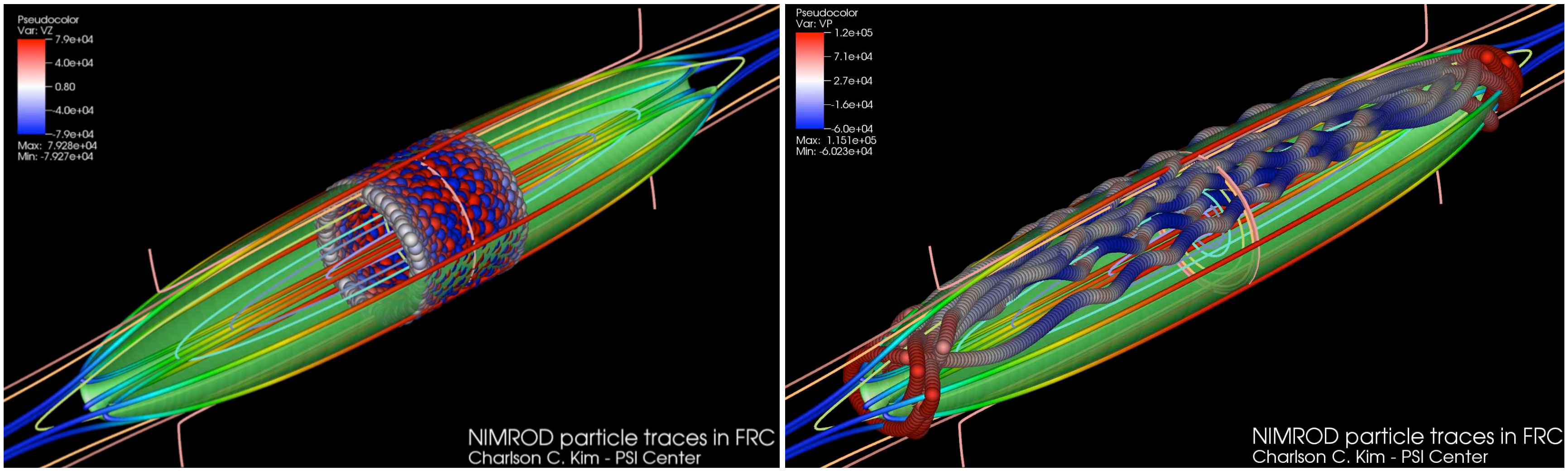
$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\mathbf{B})\ , \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0\,,$$

  [Morrison & Greene (1980), Holm & Kupershmidt (1983)].

- In terms of the fluid momentum $\mathbf{m} = \rho\,\boldsymbol{u}$, the MHD *Hamiltonian* is

$$H = \frac{1}{2}\int \frac{|\mathbf{m}|^2}{\rho}\,\mathsf{d}^3\mathbf{x} + \int \rho\,\mathcal{U}(\rho)\,\mathsf{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int |\nabla\times\mathbf{A}|^2\ ,$$

Still, energetic particles require kinetic theory!



Hot particle dynamics in PCS hybrid simulations for Field Reversed Configuration experiments (FRCs). *Right*: low energy particles colored by poloidal velocity. *Left*: high energy particles colored by axial velocity. Hot particles confine to the outboard region (higher magnetic gradients) and never cross the origin. *(Figure by the Plasma Science and Innovation Center, University of Washington).*

Let's apply geometric mechanics to formulate hybrid models!

Current-coupling scheme for hybrid MHD

Theorem [CT(2010)]
The CCS is Hamiltonian with the Lie-Poisson bracket (up to details)

$$\{F,G\}_{MHD}\,(\mathbf{M},\rho,\mathbf{A})+\{F,G\}_V\,(\hat{f}\,)$$

and the Hamiltonian (with fluid momentum $\mathbf{m}=\rho\boldsymbol{u}$)

$$H=\frac{1}{2}\int\frac{|\mathbf{m}|^2}{\rho}\,\mathrm{d}^3\mathbf{x}+\frac{1}{2\mathsf{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}+\int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x}+\frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}\,.$$

Things get worse for PCS: no Hamiltonian structure?

A geometric hybrid PCS model: assumptions

- Consider a plasma of a fluid (MHD) bulk and an energetic component
- Express the dynamics in terms of the total momentum $\mathbf{M} = \mathbf{m} + \mathbf{K}$, where $\mathbf{K} = \int \mathbf{p} f \, \mathrm{d}^3\mathbf{p}$. Then one wants to assume a *rarefied energetic component* so that $\mathbf{K}$-contributions can be neglected.
- In plasma literature, one replaces $\partial_t \mathbf{K} \simeq 0$ in the equation for the total momentum $\mathbf{M}$. *This breaks Hamiltonian structure: no energy balance!*
- The geometric Hamiltonian approach neglects $\mathbf{K}$-contributions by replacing $\mathbf{m} \simeq \mathbf{M}$ in the Hamiltonian, which is then given by [CT(2010)]

$$H = \frac{1}{2}\int \frac{|\mathbf{m}|^2}{\rho}\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mathrm{m}_h}\int f\,|\mathbf{p}|^2\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} + \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} + \frac{1}{2\mu_0}\int |\mathbf{B}|^2\,\mathrm{d}^3\mathbf{x}\,,$$

A geometric hybrid model: equations

- This process returns the same fluid equation as in the literature while inserting new transport term and inertial forces in the kinetic equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla \mathsf{p} - \frac{1}{\mathsf{m}_h\,\rho}\,\nabla\cdot\int \mathbf{p}\mathbf{p} f\,\mathrm{d}^3\mathbf{p} - \frac{1}{\mu_0\rho}\mathbf{B}\times\nabla\times\mathbf{B}$$

$$\frac{\partial f}{\partial t} + \left(\boldsymbol{u} + \frac{\mathbf{p}}{m_h}\right)\cdot\frac{\partial f}{\partial \mathbf{x}} - (\mathbf{p}\cdot\nabla\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}} + a_h\,\mathbf{p}\times(\mathbf{B} - \nabla\times\boldsymbol{u})\cdot\frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\,\boldsymbol{u}) = 0\,, \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\mathbf{B})\ ,$$

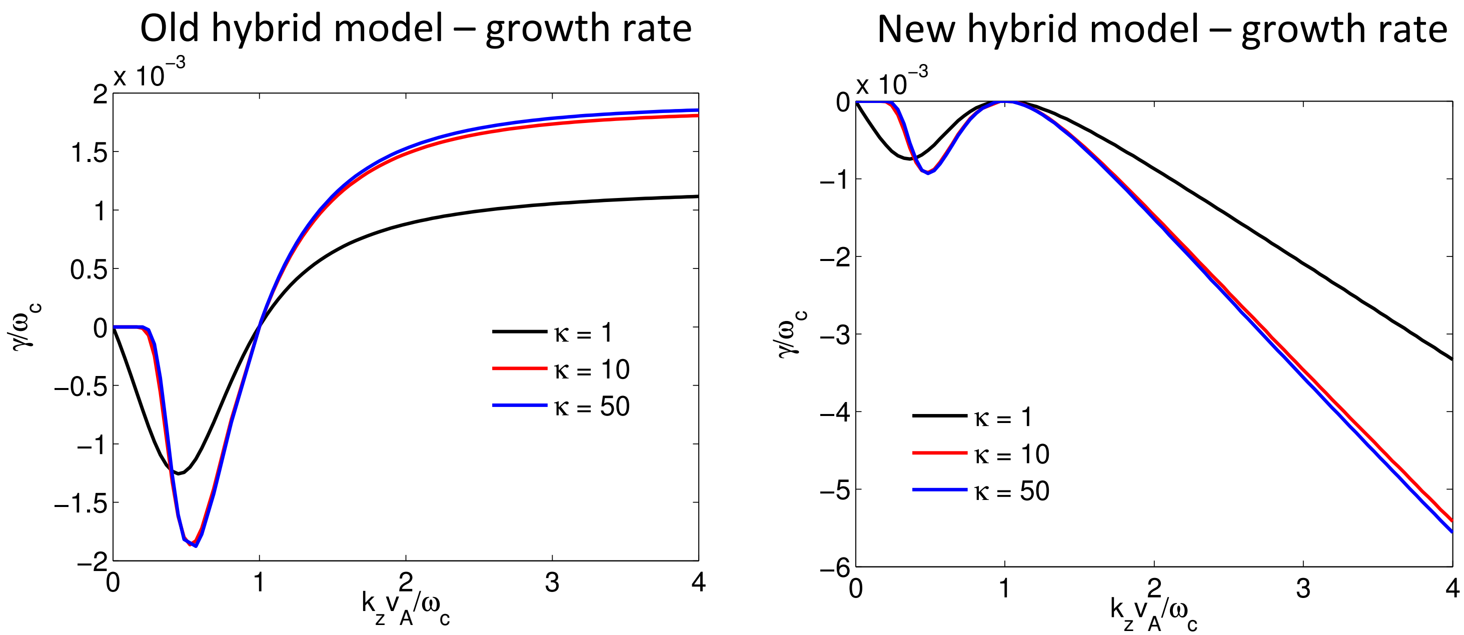
- Inertial force terms emerge since hot particle trajectories are now computed in the (Lagrangian) fluid frame.

- Dropping all $\boldsymbol{u}$-terms in the second equation and replacing $\mathbf{p}\times\mathbf{B}$ by $(\mathbf{p}-m_h\boldsymbol{u})\times\mathbf{B}$ yields the (non-Hamiltonian) model from the literature

Dispersion relation for κ-distributions

Linearize around static equilibria with $f_0 = f_0(p^2/2)$ and define $F = \int f_0 \, \mathrm{d}^2\mathbf{p}_\perp$. For longitudinal propagation, one obtains (with $v_A = b/\sqrt{\mu_0}$)

$$\omega^2 - k_z^2 v_A^2 + \omega\,(\alpha\omega \mp \omega_c)\left(n_0 + (\omega \mp \omega_c)\int_{-\infty}^{+\infty} \frac{F\,\mathrm{d}p_z}{k_z p_z - \omega \pm \omega_c}\right) = 0.$$



New model ($\alpha = 1$) gives magnetized 'Landau damping'

Spurious instability in the non-Hamiltonian model! ($\alpha = 0$)

[C.T., Tassi, Camporeale & Morrison (2014)]

Variational approach to the (low-frequency) CCS

Euler-Poincaré formulation of the (Vlasov) CCS

- The CCS possesses the EP (Eulerian) Lagrangian [C.T. & Holm (2010)]

$$\ell = \underbrace{\frac{1}{2}\int \rho\,|\boldsymbol{U}|^2\,\mathrm{d}^3\mathbf{x} - \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} - \frac{1}{2\mu_0}\int |\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}}_{\textit{Newcomb's MHD Lagrangian}}$$

$$+ \underbrace{\int f\left[(m_h\mathbf{v} + q_h\mathbf{A})\cdot\mathbf{u} - \frac{m_h}{2}|\mathbf{v}|^2 - q_h\Phi\right]\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{v}}_{\textit{Phase-space Lagrangian for hot particles}}$$

  where $\boldsymbol{\mathcal{X}} = (\mathbf{u}, \mathbf{a})$ is the 6D vector field governing $\partial_t f + \nabla_{\mathbf{z}}\cdot(f\boldsymbol{\mathcal{X}}) = 0$.

- Here, $\boldsymbol{\eta}(\mathbf{x}_0, t)$ and $\zeta(\mathbf{x}_0, \mathbf{v}_0, t)$ are Lagrangian paths for the fluid and the hot particles. Also we have

$$\dot{\boldsymbol{\eta}} = \boldsymbol{U}(\boldsymbol{\eta}, t)\,, \qquad \dot{\zeta} = \boldsymbol{\mathcal{X}}(\zeta, t)\,, \qquad \boldsymbol{\Phi} = \boldsymbol{U}\cdot\mathbf{A} \quad \textit{(gauge choice ensuring frozen-in condition)}$$

$$\rho(\boldsymbol{\eta}, t)\,\mathrm{d}^3\boldsymbol{\eta} = \rho_0(\mathbf{x}_0)\,\mathrm{d}^3\mathbf{x}_0\,, \quad \mathbf{A}(\boldsymbol{\eta}, t)\cdot\mathrm{d}\boldsymbol{\eta} = \mathbf{A}_0(\mathbf{x}_0)\cdot\mathrm{d}\mathbf{x}_0\,, \quad f(\boldsymbol{\zeta}, t)\,\mathrm{d}^6\boldsymbol{\zeta} = f_0(\zeta_0)\,\mathrm{d}^3\zeta_0$$

Low-frequency CCS: consistency issues

- Most of common hybrid models involve *drift-kinetic equations*, whose natural formulation is in terms of Hamilton's variational principle

- Although the CCS is both mathematically and physically consistent when hot particles follow full-orbit trajectories (Vlasov), *problems arise with the guiding-centre (GC) approximation:* **no energy conservation!**

- Indeed, hybrid codes are based on the usual magnetization expression

$$\mathbf{M}(\mathbf{X},t) = -\int \mu\, \mathbf{b}(\mathbf{X},t)\, f(\mathbf{X},v_\|,\mu,t)\, \mathrm{d}\mu\, \mathrm{d}v_\| \,,$$

  which is inconsistent in the presence of self-evolving EM fields!
  (Standard GC notation: $\mathbf{B} = B\,\mathbf{b}$)

Magnetization in guiding-center theory

- The variational structure of **GC dynamics in a self-evolving EM field** was first approached by in [Pfirsch (1983), Kaufman (1986)]

- The GC magnetization has **components perpendicular to** $\mathbf{B}$, whose explicit expression has been unfolded in [Brizard & Tronci (2016)]

- Variations $\delta\mathbf{A}$ in the action principle lead to

$$\mathbf{M} = -\int \left[\mu\mathbf{b} - \frac{mv_\|}{BB_\|^*}\left(v_\|\mathbf{B}_\perp^* - \mathbf{b}\times\mathbf{E}^*\right)\right] f\,\mathrm{d}\mu\,\mathrm{d}v_\|$$

('moving-dipole correction') with the effective EM field

$$\mathbf{B}^* := \mathbf{B} + \frac{mv_\|}{q}\nabla\times\mathbf{b}\,, \qquad \mathbf{E}^* := -\mathbf{E} - \frac{mv_\|}{q}\frac{\partial\mathbf{b}}{\partial t} - \mu\nabla B$$

Variational approach to low-frequency CCS

- The GC approximation for the particles lead to [Burby & C.T. (2016)]

$$\ell = \underbrace{\frac{1}{2}\int \rho\,|\boldsymbol{U}|^2\,\mathrm{d}^3\mathbf{x} - \int \rho\,\mathcal{U}(\rho)\,\mathrm{d}^3\mathbf{x} - \frac{1}{2\mu_0}\int |\nabla\times\mathbf{A}|^2\,\mathrm{d}^3\mathbf{x}}_{\textit{Newcomb's MHD Lagrangian}}$$

$$+ \underbrace{\int f\left[(m_h v_\parallel \mathsf{b} + q_h\mathbf{A})\cdot\mathbf{u}_{\mathrm{gc}} - \frac{m_h}{2}v_\parallel^2 - \mu B - q_h\Phi\right]\mathrm{d}\mu\,\mathrm{d}^4 z}_{\textit{GC Lagrangian for hot particles}}$$

where $\boldsymbol{\mathcal{X}}_{\mathrm{gc}} = (\mathbf{u}_{\mathrm{gc}}, a_\parallel)$ is the vector field governing $\partial_t f + \nabla_{\mathbf{z}}\cdot(f\boldsymbol{\mathcal{X}}_{\mathrm{gc}}) = 0$.

- This leads to the hybrid MHD momentum equation (new terms emerge!)

$$\boxed{\rho\,\partial_t\boldsymbol{U} + \rho\,(\boldsymbol{U}\cdot\nabla)\,\boldsymbol{U} = \Big(\mathbf{J} + q_h n_h\boldsymbol{U} - \mathbf{J}_{\mathrm{gc}} - \nabla\times\mathbf{M}\Big)\times\mathbf{B} - \nabla\mathsf{p}\,,}$$

where $\mathbf{M}$ was given previously and the hot density and GC current are

$$n_h = \int f\,\mathrm{d}\mu\,\mathrm{d}v_\parallel\,,\qquad \mathbf{J}_{\mathrm{gc}} = q_h\int\Big(v_\parallel\mathbf{B}^* - \mathsf{b}\times\mathbf{E}^*\Big)B_\parallel^{*\,-1} f\,\mathrm{d}\mu\,\mathrm{d}v_\parallel\,.$$

In addition, we recall ideal Ohm's law $\mathbf{E} = -\boldsymbol{U}\times\mathbf{B}$.

Comparison with previous versions [Todo & al. (1995)]

The new equations differ from previous models by three main features:

- **Standard guiding-center theory:** the parallel component of the effective magnetic field $B_\parallel^*$ is nowhere approximated by B.

- $\mathbf{E}\times\mathbf{B}$**−drift current:** had we assumed $B_\parallel^* \simeq B$, the Lorentz force $q_h n_h \boldsymbol{U}\times\mathbf{B}$ would cancel with the $E\times B-$drift current contribution $\mathbf{J}_{E\times B} = q_h(\int f/B_\parallel^*\,\mathrm{d}\mu\,\mathrm{d}v_\parallel)\mathbf{E}\times\mathbf{b}$ in the fluid equation, since

$$(q_h n_h \boldsymbol{U} - \mathbf{J}_{E\times B})\times\mathbf{B} = q_h\left[\int\left(1-\frac{B}{B_\parallel^*}\right) f\,\mathrm{d}\mu\,\mathrm{d}v_\parallel\right]\boldsymbol{U}\times\mathbf{B}\,.$$

Notice that this term does not affect the energy balance.

- **Energy conservation:** retaining the moving-dipole magnetization term ensures energy and momentum balance. Omitting this term yields

$$\dot{E} = -\int\left[(\boldsymbol{U}\times\mathbf{B})\cdot\nabla\times\int\frac{m_h v_\|}{BB_\|^*}\left(v_\|\mathbf{B}_\perp^* - \mathbf{b}\times\mathbf{E}^*\right) f\,\mathrm{d}\mu\,\mathrm{d}v_\|\right]\mathrm{d}^3x\,,$$

where

$$E = \frac{1}{2}\int\rho|\boldsymbol{U}|^2\,\mathrm{d}^3x + \int\left(\frac{m_h}{2}v_\|^2 + \mu B\right) f\,\mathrm{d}\mu\,\mathrm{d}^4z + \int\rho\,\mathcal{U}(\rho)\,\mathrm{d}^3x + \frac{1}{2\mu_0}\int|\mathbf{B}|^2\,\mathrm{d}^3x\,.$$

This contradicts previous results in the literature and it is unaffected by the presence of the $\mathbf{J}_{E\times B}$−terms in the fluid equation.

- It is likely that **spurious instabilities** are produced in previous models, although these may be filtered by the insertion of dissipation terms.

- Similar issues were found also when adopting the **gyrokinetic approximation** [Burby & C.T. (2016)]

- What about **PCS**? Requires Poisson brackets (ongoing)

Modeling efforts in fully kinetic systems

Fluid models, hybrids and kinetic theories

Although the Maxwell-Vlasov (MV) system is the most complete description, it is accompanied by several computational difficulties.

Various reduced models have been formulated over the decades:

- **Kinetic models** (retain all moments for all particle species)
  1. *Gyro-averaged kinetics*: low-frequencies (below cyclotron)
  2. *Darwin-Vlasov theory*: radiationless limit (keeps Langmuir waves)

- **Fluid and hybrid models** (hybrids retain all moments for one species)
  1. *Two-fluid and MHD*: quasi-neutral limit (no Langmuirs)
  2. *Hybrid kinetic-fluid models*: retain some kinetic effects

Question: *Can we use quasi-neutrality to* **neglect both radiation and Langmuirs** *in full Vlasov theory? Can we avoid solving for the plasma frequency?*

Previous attempts

- C. Z. Cheng and J. R. Johnson [JGR (1999)] formulated a full-orbit neutral kinetic theory that neglects $O(m_e/m_i)-$terms in Ohm's Law.

  $\rightarrow$ This does not preserve Ampère's balance $en(Z\mathbf{V}_i - \mathbf{V}_e) = \mu_0^{-1}\nabla\times\mathbf{B}$

- Motivated by previous works by Hesse, Kuznetsova and Winske, I used **variational methods** to formulate a self-consistent kinetic model *neglecting electron inertia* [C.T.(2013)].

  $\rightarrow$ Ampère is OK, but model goes wrong as lenghtscales approach δ_e.

Need for self-consistent kinetic model incorporating quasi-neutrality!

(Quasi-)Neutral Vlasov equations [CT & Camporeale (2015)]

- The neutral Vlasov theory is obtained by coupling the Vlasov equation to the **low-frequency Maxwell equations**:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}\,, \qquad \mu_0^{-1}\nabla \times \mathbf{B} = \mathbf{J}\,, \qquad \rho = 0\,,$$

  as they are obtained in the formal limit $\varepsilon_0 \to 0$.

- The closure for $\mathbf{E}$ arises from the **first moment of the Vlasov equation**

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \boldsymbol{x}} + \frac{q_s}{m_s}\Big(\mathbf{E} + \mathbf{v}\times\mathbf{B}\Big)\cdot\frac{\partial f_s}{\partial \mathbf{v}} = 0\,,$$

  so that (upon choosing $s = e$)

$$\mathbf{E} = -\mathbf{V}_e \times \mathbf{B} + \frac{1}{q_e n_e}\nabla\cdot\mathbb{P}_e + \frac{m_e}{q_e}\left(\frac{\partial \mathbf{V}_e}{\partial t} + \mathbf{V}_e\cdot\nabla\,\mathbf{V}_e\right)$$

  where $\mathbf{V}_e$ is given by Ampère's law $q_e n_e \mathbf{V}_e = \mu_0^{-1}\nabla\times\mathbf{B} - q_i\int \mathbf{v}\,f_i\,\mathrm{d}^3\mathbf{v}$

- **No electrostatic limit**: this model restricts MV to frequencies much smaller than the plasma frequency and lengths much larger than Debye.

Specializations

1. Neglecting $O(m_e/m_i)-$terms in Ohm's Law yields the **CJ model**.

2. Neglecting electron inertia and adopting a non-gyrotropic closure for electrons recovers the **hybrid reconnection model** in [Winske & Hesse ('94)]

3. Replacing electron kinetics by its standard fluid closure and retaining inertia yields the **hybrid model** proposed in [Valentini et al. (2007)].

4. Replacing electron kinetics by its standard fluid closure and discarding inertia yields a class of widely studied **hybrid models** [Freidberg (1972)].

5. Analogously, complete fluid closures recover the **two fluid model**, as well as **Hall-MHD** and (by neglecting the Hall term) **ideal MHD**.

Variational formulation for phase-space paths

- The action for the neutral Vlasov theory is obtained from that corresponding to Maxwell-Vlasov upon neglecting the electric energy

$$\frac{\epsilon_0}{2}\int |\mathbf{E}|^2 \, \mathrm{d}^3x = \frac{\epsilon_0}{2}\int \left|\frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi\right|^2 \mathrm{d}^3x \,,$$

- The model possesses the following **gauge-invariant Lagrangian**:

$$L = \sum_s \int f_{0s}(\mathbf{z}_{0s}) \Big[\Big(m_s \boldsymbol{v}_s(\mathbf{z}_{0s}, t) + q_s \mathbf{A}(\boldsymbol{x}_s(\mathbf{z}_{0s}, t), t) \Big) \cdot \dot{\boldsymbol{x}}_s(\mathbf{z}_{0s}, t)$$
$$- \frac{m_s}{2} |\boldsymbol{v}_s(\mathbf{z}_{0s}, t)|^2 - q_s \varphi(\boldsymbol{x}_s(\mathbf{z}_{0s}, t), t) \Big] \, \mathrm{d}^6 z_{0s}$$
$$- \frac{1}{2\mu_0} \int |\nabla \times \mathbf{A}(\mathbf{x}, t)|^2 \, \mathrm{d}^3x \,,$$

where $\mathbf{z}_{0s} = (\boldsymbol{x}_{0s}, \mathbf{v}_{0s})$ and $\boldsymbol{\zeta}_s(\mathbf{z}_{0s}, t) = (\boldsymbol{x}_s(\mathbf{z}_{0s}, t),\, \boldsymbol{v}_s(\mathbf{z}_{0s}, t))$ are **Lagrangian trajectories** on phase-space.

Main features

- Poisson's equation has been removed
- No electrostatic limit $\rightarrow$ absence of Langmuir waves
- Fast gyromotion effects are completely retained
- Gauge-invariant model (unlike Darwin-Vlasov)
- Mathematical consistency is ensured by action principle

Neutral Vlasov vs. hybrids: linear stability results

Linearized neutral Vlasov model

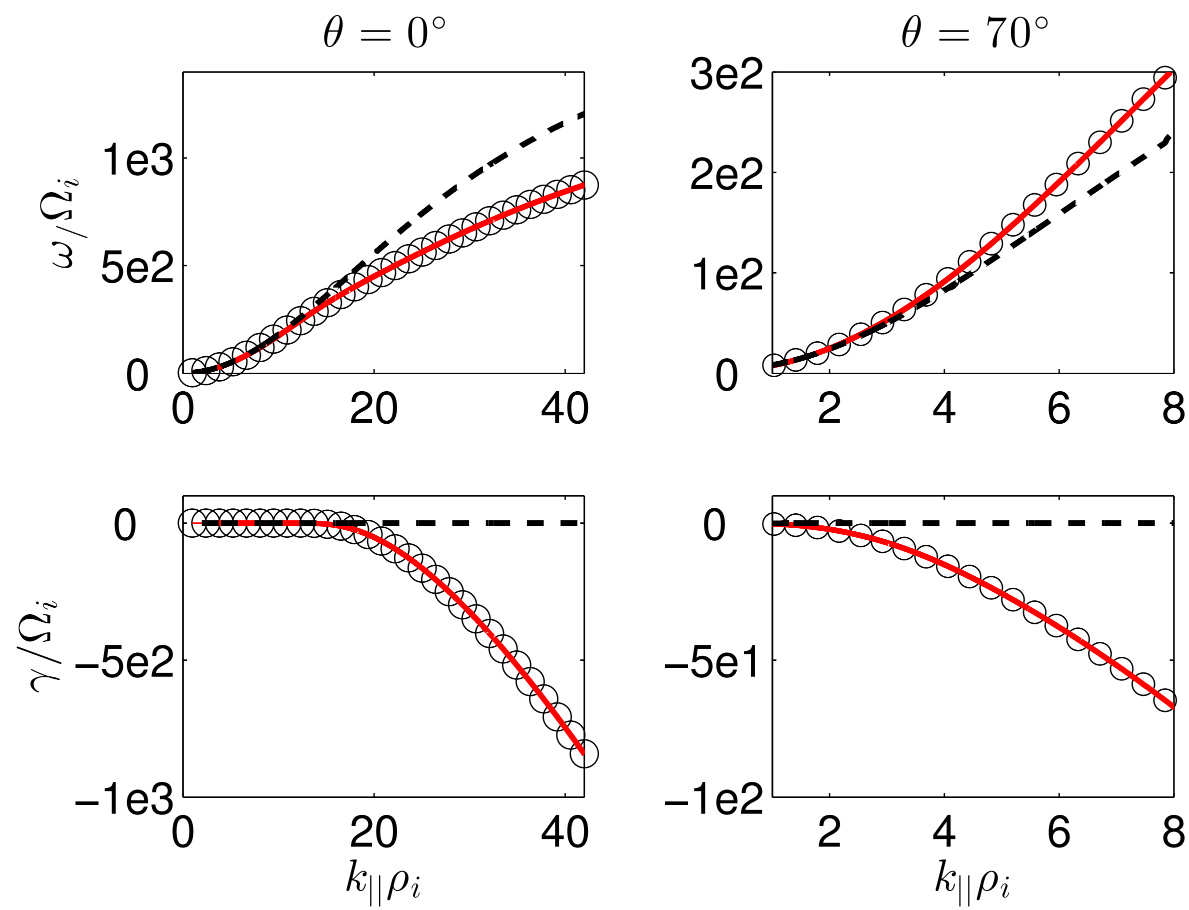
- We linearize around **isotropic Maxwellian equilibria** for both electrons and ions. Also, we consider a vertical equilibrium magnetic field.

- The **dispersion relation** is easily found by letting $\varepsilon_0 \to 0$ in the Maxwell-Vlasov dispersion relation

- We study **Alfvén and whistler waves**, upon comparing Maxwell-Vlasov with the neutral Vlasov model and a hybrid model [Valentini et al. (2007)] with kinetic ions and fluid electrons (with inertia).

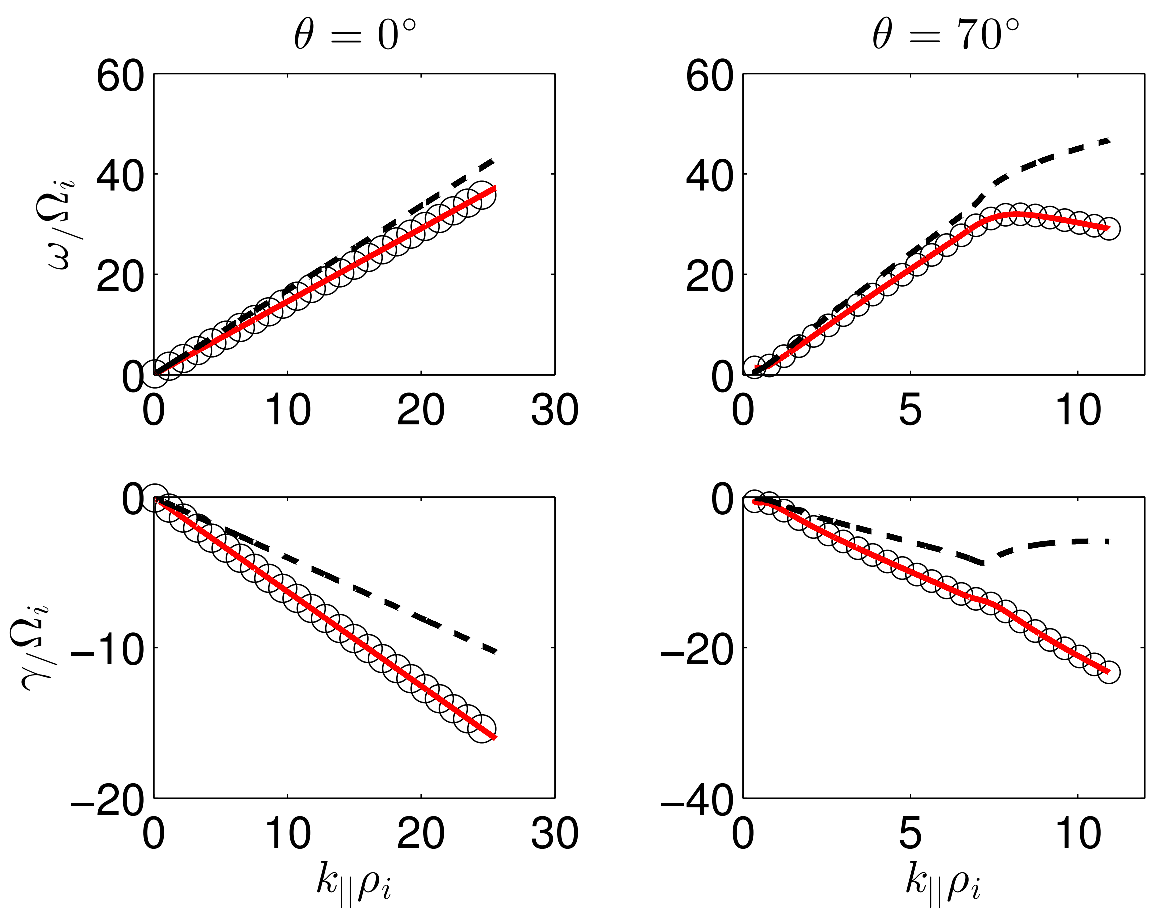
- The plasma has $\beta = 0.5$ (the ratio between thermal and magnetic energy) and $\omega_{pi}/\omega_{ci} \sim 7 \times 10^3$, which are typical for the solar wind.

Whistler wave propagation



Real frequency (top) and damping rate (bottom) for Whistler wave propagation at $\theta = 0^\circ$ (left) and $\theta = 70^\circ$ (right). Red line refers to neutral Vlasov, while the dashed line and the circles are used for the hybrid model and Maxwell-Vlasov, respectively.

Alfvén wave propagation



Real frequency (top) and damping rate (bottom) for Alfvén wave propagation at $\theta = 0^\circ$ (left) and $\theta = 70^\circ$ (right). Legend is as in previous figure.

Main message:

when it comes to deriving reduced models, better **insert the approximations in the Hamiltonian or the Lagrangian** than in the eqns of motion!

THANK YOU!

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