Recent advances in the variational formulation of reduced Vlasov-Maxwell equations

Alain J. Brizard
Saint Michael's College

Plasma Theory Seminar
Princeton Plasma Physics Laboratory
Thursday, February 2, 2017

I. Vlasov-Maxwell Variational Principles

- **Variational formulations:** Lagrange, Euler, or Euler-Poincaré

$$\delta\mathcal{A} = \int \delta\mathcal{L}\, d^4x = 0 \;\;\rightarrow\;\; \delta\mathcal{L} = \delta\mathcal{L}_{\rm V} + \frac{1}{4\pi}\left(\mathbf{E}\cdot\delta\mathbf{E} \;-\; \mathbf{B}\cdot\delta\mathbf{B}\right)$$

- ○ Lagrange variational principle (Low, 1958)

$$\delta\mathcal{A}_{\rm V}^{\rm L} \;=\; \sum\int \delta{\sf L}\; f_0\; d^6z_0$$

- ○ Euler variational principle (Brizard, 2000)

$$\delta\mathcal{A}_{\rm V}^{\rm E} \;=\; -\sum\int \left(\delta\mathcal{F}\,\mathcal{H} \;+\; \mathcal{F}\,\delta\mathcal{H}\right) d^8\mathcal{Z}$$

- ○ Euler-Poincaré variational principle (Cendra *et al.*, 1998)

$$\delta\mathcal{A}_{\rm V}^{\rm EP} \;=\; \sum\int \left(\delta f\; {\sf L}_{\rm EP} \;+\; f\;\delta{\sf L}_{\rm EP}\right) d^6z$$

◦ Constrained variations on electromagnetic fields

$$\delta\mathbf{E} \;=\; -\nabla\delta\Phi - c^{-1}\partial_t\delta\mathbf{A} \;\;\text{and}\;\; \delta\mathbf{B} = \nabla\times\delta\mathbf{A}$$

- **Reduced polarization & magnetization**

$$\mathcal{L}_{\rm redV}(\cdots;\Phi,\mathbf{A};\mathbf{E},\mathbf{B}) \;\rightarrow\; \left\{ \begin{array}{l} \mathbf{P}_{\rm red} \equiv \delta\mathcal{L}_{\rm redV}/\delta\mathbf{E} \\ \\ \mathbf{M}_{\rm red} \equiv \delta\mathcal{L}_{\rm redV}/\delta\mathbf{B} \end{array} \right.$$

◦ Reduced polarization charge density

$$-\,\nabla\delta\Phi\cdot\frac{\delta\mathcal{L}_{\rm redV}}{\delta\mathbf{E}} \;\;\rightarrow\;\; \delta\Phi\,(\nabla\cdot\mathbf{P}_{\rm red})$$

◦ Reduced polarization & magnetization current densities

$$-\frac{1}{c}\frac{\partial\delta\mathbf{A}}{\partial t}\cdot\frac{\delta\mathcal{L}_{\rm redV}}{\delta\mathbf{E}} - \nabla\times\delta\mathbf{A}\cdot\frac{\delta\mathcal{L}_{\rm redV}}{\delta\mathbf{B}} \rightarrow \delta\mathbf{A}\cdot\left(\frac{1}{c}\frac{\partial\mathbf{P}_{\rm red}}{\partial t} + \nabla\times\mathbf{M}_{\rm red}\right)$$

OUTLINE

- **Part I. Guiding-center Vlasov-Maxwell Equations**
  - Pre-gyrokinetic theory: No background-fluctuation separation
  - Vlasov-Maxwell fields $(f, \mathbf{E}, \mathbf{B})$ satisfy standard guiding-center orderings $(\Omega^{-1}\partial_t \ll 1,\ E_\| \ll |\mathbf{E}_\perp|)$
  - Guiding-center magnetization (magnetic + moving-electric) plays a crucial role in momentum & angular-momentum conservation (Brizard & Tronci, 2016)

- **Part II. Parallel-symplectic Gyrokinetic Equations**
  - Parallel-symplectic representation: gyrocenter Poisson bracket contains terms due to perturbed magnetic field: $\langle A_{1\|\mathrm{gc}}\rangle$
  - The Parallel-symplectic representation is equivalent to the Hamiltonian representation since the gyrocenter magnetic moment μ is the same in all representations (Brizard, 2017)

II. Guiding-center Vlasov-Maxwell Equations

- **Guiding-center Lagrangian** $Z^{\alpha} = (\mathbf{X}, p_{\|}, w, t)$ & $\mu =$ label

$$L_{\rm gc} = \frac{e}{c}\,\mathbf{A}^{*}\cdot\dot{\mathbf{X}} - w\,(\dot{t}-1) - \left(\frac{p_{\|}^{2}}{2m} + e\,\Phi^{*}\right) \rightarrow \left\{ \begin{array}{rcl} \Phi^{*} & = & \Phi + \mu\,B/e \\ \mathbf{A}^{*} & = & \mathbf{A} + p_{\|}\,c\widehat{\rm b}/e \end{array} \right.$$

- ◦ Modified electromagnetic fields $(\mathbf{E}^{*}, \mathbf{B}^{*})$:

$$\left. \begin{array}{l} \mathbf{E}^{*} = -\,\nabla\Phi^{*} - c^{-1}\partial_{t}\mathbf{A}^{*} \\ \\ \mathbf{B}^{*} = \nabla\times\mathbf{A}^{*} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \nabla\times\mathbf{E}^{*} = -\,c^{-1}\partial_{t}\mathbf{B}^{*} \\ \\ \nabla\cdot\mathbf{B}^{*} = 0 \end{array} \right.$$

- ◦ Reduced guiding-center Euler-Lagrange equations $(B_{\|}^{*} \equiv \widehat{\rm b}\cdot\mathbf{B}^{*})$

$$\dot{\mathbf{X}} = \frac{p_{\|}}{m}\,\frac{\mathbf{B}^{*}}{B_{\|}^{*}} + \mathbf{E}^{*}\times\frac{c\widehat{\rm b}}{B_{\|}^{*}},\quad \dot{p}_{\|} = e\,\mathbf{E}^{*}\cdot\frac{\mathbf{B}^{*}}{B_{\|}^{*}} \;\rightarrow\; \frac{\partial B_{\|}^{*}}{\partial t} = -\,\frac{\partial}{\partial z^{a}}\left(\dot{z}^{a}\;B_{\|}^{*}\right)$$

Guiding-center Poisson bracket

$$\{F,\ G\}_{\rm gc} \;=\; \frac{\mathbf{B}^*}{B_\|^*}\cdot\left(\nabla^* F\,\frac{\partial G}{\partial p_\|} - \frac{\partial F}{\partial p_\|}\,\nabla^* G\right) - \frac{c\widehat{\mathsf{b}}}{eB_\|^*}\cdot\nabla^* F\times\nabla^* G$$
$$+\;\frac{\partial F}{\partial w}\,\frac{\partial G}{\partial t} \;-\; \frac{\partial F}{\partial t}\,\frac{\partial G}{\partial w}$$

- Notation: $\nabla^* \equiv \nabla - (e/c)\partial_t\mathbf{A}^*\,\partial_w$
- Guiding-center Hamilton equations

$$\mathcal{H}_{\rm gc} = \frac{p_\|^2}{2m} + e\,\Phi^* - w \;\;\rightarrow\;\; \dot{Z}^\alpha \;=\; \left\{Z^\alpha,\ \mathcal{H}_{\rm gc}\right\}_{\rm gc}$$

- Liouville property

$$\{F,\ G\}_{\rm gc} \;=\; \frac{1}{B_\|^*}\,\frac{\partial}{\partial Z^\alpha}\left(B_\|^*\,F\,\left\{Z^\alpha,\ G\right\}_{\rm gc}\right)$$

- **Guiding-center Vlasov equation** $z^a = (\mathbf{X}, p_\|)$

$$F_\mu \;\equiv\; (2\pi\, m\, B_\|^*)\, f_\mu \;\;\rightarrow\;\; \frac{\partial F_\mu}{\partial t} \;+\; \frac{\partial}{\partial z^a}\left(\dot{z}^a\, F_\mu\right) \;=\; 0$$

- **Guiding-center Maxwell equations** ($\sum^\mu \equiv \sum_{\rm species} \int d\mu$)

$$\begin{aligned} \nabla\cdot\mathbf{E} \;&=\; 4\pi\,\varrho_{\rm gc} \;\equiv\; -\,4\pi \sum^\mu \int \frac{\partial L_{\rm gc}}{\partial\Phi} F_\mu\, dp_\| \\ \nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} \;&=\; \frac{4\pi}{c}\left(\mathbf{J}_{\rm gc} + c\,\nabla\times\mathbf{M}_{\rm gc}\right) \\ &\equiv\; 4\pi \sum^\mu \int \left[\frac{\partial L_{\rm gc}}{\partial\mathbf{A}}\, F_\mu + \nabla\times\left(\frac{\partial L_{\rm gc}}{\partial\mathbf{B}}\, F_\mu\right)\right] dp_\| \end{aligned}$$

- ◦ Guiding-center magnetization: intrinsic & moving-electric dipole

$$\frac{\partial L_{\rm gc}}{\partial\mathbf{B}} \;=\; -\,\mu\,\frac{\partial B}{\partial\mathbf{B}} + p_\|\,\frac{\partial\widehat{\mathsf{b}}}{\partial\mathbf{B}}\cdot\dot{\mathbf{X}} \;=\; \boldsymbol{\mu}_{\rm gc} \;+\; \boldsymbol{\pi}_{\rm gc}\times\frac{p_\|\,\widehat{\mathsf{b}}}{mc}$$

Higher-order guiding-center theory

- **Higher-order guiding-center theory** (Tronko & Brizard, 2015)

$$\mathbf{A}^* = \mathbf{A} + \frac{c}{e}\left[p_\parallel \widehat{b} - \epsilon_B \frac{\mu B}{\Omega}\left(\mathbf{R} + \frac{1}{2}\nabla\times\widehat{b}\right)\right]$$

$$H_{gc} = \frac{p_\parallel^2}{2m} + \mu B + \epsilon_B^2 \Psi_2 = \frac{m}{2}\left\langle\left|\dot{\mathbf{X}} + \dot{\boldsymbol{\rho}}_{gc}\right|^2\right\rangle$$

  - Gyrogauge vector ($\widehat{b} \equiv \widehat{e}_1 \times \widehat{e}_2$): $\mathbf{R} \equiv \nabla\widehat{e}_1 \cdot \widehat{e}_2$
  - Guiding-center polarization correction: $(\widehat{b}\cdot\nabla\times\widehat{b})\,\widehat{b} \rightarrow \nabla\times\widehat{b}$

- Guiding-center polarization (Pfirsch, 1984; Kaufman, 1986)

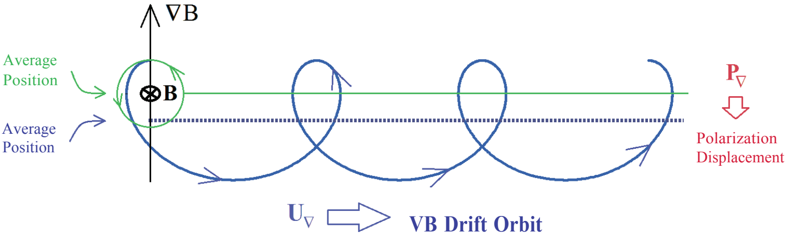
$$\boldsymbol{\pi}_{gc} = e\,\langle\boldsymbol{\rho}_{gc}\rangle - \nabla\cdot\left\langle\frac{e}{2}\,\boldsymbol{\rho}_{gc}\boldsymbol{\rho}_{gc}\right\rangle + \cdots = \frac{e\widehat{b}}{\Omega}\times\dot{\mathbf{X}}$$

Geometry of Reduced Polarization

- **Guiding-center Polarization driven by ∇B-drift**

  ○ Ion polarization driven by $\mathbf{U}_\nabla = (c\widehat{\mathsf{b}}/eB) \times \mu\,\nabla B$

$$\boldsymbol{\pi}_\nabla = \frac{e\,\widehat{\mathsf{b}}}{\Omega} \times \mathbf{U}_\nabla = -\,\frac{e\,\mu}{m\,\Omega^2}\,\nabla_\perp B \;\;\rightarrow\;\; \frac{|\boldsymbol{\pi}_\nabla|}{e\,\rho_\perp} = \rho_\perp\,|\nabla_\perp \ln B| \;\ll\; 1$$



III. Guiding-center Euler Variational Principle

- **Guiding-center Euler action** (Brizard & Tronci, 2016)

$$\mathcal{A}_{\rm gc} \;=\; -\sum^{\mu}\int \mathcal{F}_{\mu}\,\mathcal{H}_{\rm gc}\; d^{6}Z \;+\; \int \frac{d^{4}x}{8\pi}\;\Big(|\mathbf{E}|^{2} \;-\; |\mathbf{B}|^{2}\Big)$$

  - Extended Vlasov phase-space density (Brizard, 2000)

$$\mathcal{F}_{\mu} \;\equiv\; F_{\mu}\,\delta(w - H_{\rm gc}) \;\text{ and }\; \mathcal{H}_{\rm gc} \;\equiv\; H_{\rm gc} \;-\; w \;=\; 0$$

  - Eulerian Hamiltonian variation → intrinsic magnetization

$$\delta\mathcal{H}_{\rm gc} \;\equiv\; e\,\delta\Phi^{*} \;=\; e\,\delta\Phi \;+\; \mu\,\widehat{\mathsf{b}}\cdot\delta\mathbf{B}$$

- **Guiding-center Eulerian Vlasov variation**

$$\delta\mathcal{F}_\mu \;\equiv\; B_\|^* \left\{\delta\mathcal{S},\; \mathcal{F}_\mu/B_\|^*\right\}_{\rm gc} \;+\; \delta B_\|^*\; \mathcal{F}_\mu/B_\|^* \;+\; \frac{e}{c}\,\delta\mathbf{A}^*\cdot\left(B_\|^*\left\{\mathbf{X},\; \mathcal{F}_\mu/B_\|^*\right\}_{\rm gc}\right)$$

◦ Eulerian magnetic variations → moving electric dipole

$$\frac{e}{c}\,\delta\mathbf{A}^* = \frac{e}{c}\,\delta\mathbf{A} + p_\|\,\delta\mathbf{B}\cdot\frac{\partial\widehat{b}}{\partial\mathbf{B}} \quad \text{and} \quad \delta B_\|^* = \delta\mathbf{B}^*\cdot\widehat{b} + \left(\delta\mathbf{B}\cdot\frac{\partial\widehat{b}}{\partial\mathbf{B}}\right)\cdot\mathbf{B}^*$$

Guiding-center Vlasov constraint

$$\int \delta\mathcal{F}_\mu\, d^6Z \;=\; 0 \quad\rightarrow\quad \delta\mathcal{F}_\mu \;\equiv\; \frac{\partial}{\partial Z^\alpha}\left(\mathcal{F}_\mu\,\delta Z^\alpha\right)$$

◦ Guiding-center phase-space virtual displacement

$$\delta Z^\alpha \;\equiv\; \{\delta\mathcal{S}, Z^\alpha\}_{\rm gc} + (e/c)\,\delta\mathbf{A}^*\cdot\{\mathbf{X}, Z^\alpha\}_{\rm gc}$$

Guiding-center Lagrangian density

- Eulerian variation

$$\delta\mathcal{L}_{\rm gc} = \frac{1}{4\pi}\left(\delta\mathbf{E}\cdot\mathbf{E} - \delta\mathbf{B}\cdot\mathbf{B}\right) - \sum^{\mu}\int\left(\delta\mathcal{F}_{\mu}\,\mathcal{H}_{\rm gc} + \mathcal{F}_{\mu}\,e\,\delta\Phi^{*}\right)\,dp_{\|}\,dw$$

- Usefull expression

$$\delta\mathcal{F}_{\mu}\,\mathcal{H}_{\rm gc} = -\,\mathcal{F}_{\mu}\left(\frac{e}{c}\,\delta\mathbf{A}^{*}\cdot\frac{d_{\rm gc}\mathbf{X}}{dt}\right) + B_{\|}^{*}\,\delta\mathcal{S}\,\left\{\mathcal{F}_{\mu}/B_{\|}^{*},\,\mathcal{H}_{\rm gc}\right\}_{\rm gc} + \frac{\partial}{\partial Z^{\alpha}}\left[\mathcal{F}_{\mu}\;(\textbf{Noether terms})\right]$$

- Identity

$$e\,\delta\Phi^{*} - \frac{e}{c}\,\delta\mathbf{A}^{*}\cdot\frac{d_{\rm gc}\mathbf{X}}{dt} = e\,\delta\Phi - \frac{e}{c}\,\delta\mathbf{A}\cdot\frac{d_{\rm gc}\mathbf{X}}{dt} - \delta\mathbf{B}\cdot\left(\boldsymbol{\mu}_{\rm gc} + \boldsymbol{\pi}_{\rm gc}\times\frac{\mathrm{p}_{\|}\,\widehat{\mathrm{b}}}{mc}\right)$$

- Eulerian variation ($\mathbf{H} \equiv \mathbf{B} - 4\pi\,\mathbf{M}_{\rm gc}$)

$$
\begin{aligned}
\delta\mathcal{L}_{\rm gc} \;\;\equiv\;\; & -\sum^{\mu}\int B_{\|}^{*}\,\delta\mathcal{S}\,\{\mathcal{F}_{\mu}/B_{\|}^{*},\;\mathcal{H}\}_{\rm gc}\,dp_{\|}\,dw \\
& + \frac{\delta\Phi}{4\pi}\left(\nabla\cdot\mathbf{E} \;-\; 4\pi\,\varrho_{\rm gc}\right) \\
& + \frac{\delta\mathbf{A}}{4\pi}\cdot\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} - \nabla\times\mathbf{H} + \frac{4\pi}{c}\,\mathbf{J}_{\rm gc}\right) \\
& + \frac{\partial\delta\mathcal{J}}{\partial t} \;+\; \nabla\cdot\delta\boldsymbol{\Gamma}
\end{aligned}
$$

- Noether components

$$
\begin{aligned}
\delta\mathcal{J} \;\;&\equiv\;\; \sum^{\mu}\int\delta\mathcal{S}\;\mathcal{F}_{\mu}\;dp_{\|}\,dw \;-\; \frac{\mathbf{E}\cdot\delta\mathbf{A}}{4\pi\,c} \\
\delta\boldsymbol{\Gamma} \;\;&\equiv\;\; \sum^{\mu}\int\delta\mathcal{S}\;\mathcal{F}_{\mu}\,\frac{d_{\rm gc}\mathbf{X}}{dt}\;dp_{\|}\,dw - \frac{1}{4\pi}\left(\delta\Phi\,\mathbf{E} \;+\; \delta\mathbf{A}\times\mathbf{H}\right)
\end{aligned}
$$

Guiding-center Vlasov-Maxwell variational principle

$$
\begin{aligned}
0 \;=\; \delta\mathcal{A}^{\rm E}_{\rm gc} \;\; &= \;\; -\,\Sigma^{\mu}\int B_{\|}^{*}\,\delta\mathcal{S}\,\left\{\mathcal{F}_{\mu}/B_{\|}^{*},\;\mathcal{H}\right\}_{\rm gc}\,d^{6}Z \\
&+\;\int\frac{\delta\Phi}{4\pi}\left(\nabla\cdot\mathbf{E}\;-\;4\pi\,\varrho_{\rm gc}\right)d^{3}x\;dt \\
&+\;\int\frac{\delta\mathbf{A}}{4\pi}\cdot\left(\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}-\nabla\times\mathbf{H}+\frac{4\pi}{c}\,\mathbf{J}_{\rm gc}\right)d^{3}x\;dt
\end{aligned}
$$

- Stationarity with respect to the variations $(\delta\Phi, \delta\mathbf{A})$ yields the guiding-center Maxwell equations.

- Stationarity with respect to the variation $\delta\mathcal{S}$ yields the extended guiding-center Vlasov equation $B_{\|}^{*}\,\{\mathcal{F}_{\mu}/B_{\|}^{*},\,\mathcal{H}\}_{\rm gc} = 0$

$$
0 \;=\; \int dw\;B_{\|}^{*}\,\{\mathcal{F}_{\mu}/B_{\|}^{*},\,\mathcal{H}\}_{\rm gc} \;=\; \frac{\partial F_{\mu}}{\partial t}\;+\;\frac{\partial}{\partial z^{a}}\left(\dot{z}^{a}\,F_{\mu}\right)
$$

Guiding-center Noether Equation

- **Hamiltonian constraint:** $\mathcal{H}_{\rm gc} = H_{\rm gc} - w = 0 \;\;\Rightarrow\;\; \mathcal{L}_{\rm gc} \rightarrow \mathcal{L}_{\rm M}$

$$\delta\mathcal{L}_{\rm gc} \;\equiv\; -\,\left(\delta t\,\frac{\partial}{\partial t} + \delta\mathbf{x}\cdot\nabla\right)\mathcal{L}_{\rm M} \;=\; \frac{\partial\delta\mathcal{J}}{\partial t} \;+\; \nabla\cdot\delta\mathbf{\Gamma}$$

Energy-momentum conservation law

- Space-time translations generated by $\delta\mathcal{S}$:

$$\delta\mathcal{S} \;=\; \frac{e}{c}\,\mathbf{A}^{*}\cdot\delta\mathbf{x} \;-\; w\,\delta t \;\equiv\; \mathbf{P}\cdot\delta\mathbf{x} \;-\; w\,\delta t$$

- Eulerian potential variations (gauge: $\delta\chi \equiv \mathbf{A}\cdot\delta\mathbf{x} - \Phi\,c\,\delta t$)

$$\begin{aligned} \delta\Phi &\equiv \delta\mathbf{x}\cdot\mathbf{E} \;+\; c^{-1}\partial_t\delta\chi \\ \delta\mathbf{A} &\equiv c\,\delta t\,\mathbf{E} \;+\; \delta\mathbf{x}\times\mathbf{B} \;-\; \nabla\delta\chi \end{aligned}$$

Guiding-center energy conservation law

$$\frac{\partial \mathcal{E}_{\rm gc}}{\partial t} \;+\; \nabla\cdot\mathbf{S}_{\rm gc} \;=\; 0$$

- Guiding-center energy density ($K_{\rm gc} = \mu\, B + p_{\|}^2/2m$)

$$\mathcal{E}_{\rm gc} \;\equiv\; \sum{}^{\mu} \int F_{\mu}\; K_{\rm gc}\, d{\rm p}_{\|} \;+\; \frac{1}{8\pi}\left(|\mathbf{E}|^2 \;+\; |\mathbf{B}|^2\right)$$

- Guiding-center energy-density flux

$$\mathbf{S}_{\rm gc} \;\equiv\; \sum{}^{\mu} \int F_{\mu}\; K_{\rm gc}\, \dot{\mathbf{X}}\, d{\rm p}_{\|} \;+\; \frac{c}{4\pi}\,\mathbf{E}\times\mathbf{H}$$

Guiding-center momentum conservation law

$$\frac{\partial \mathbf{P}_{\rm gc}}{\partial t} + \nabla\cdot\mathsf{T}_{\rm gc} = 0$$

- Guiding-center momentum density

$$\mathbf{P}_{\rm gc} \equiv \sum^{\mu}\int p_{\|}\,\widehat{\mathsf{b}}\,F_{\mu}\,dp_{\|} + \frac{\mathbf{E}\times\mathbf{B}}{4\pi\,c}$$

- Symmetric guiding-center stress tensor $\mathsf{T}_{\rm gc} \equiv \mathsf{T}_{\rm M} + \mathsf{T}_{\rm gcV}$

$$\mathsf{T}_{\rm M} \equiv \left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)\frac{\mathbf{I}}{8\pi} - \frac{1}{4\pi}\left(\mathbf{EE} + \mathbf{BB}\right)$$

$$\mathsf{T}_{\rm gcV} \equiv \mathsf{P}_{\rm CGL} + \sum^{\mu}\int\left(\dot{\mathbf{X}}_{\perp}\;p_{\|}\,\widehat{\mathsf{b}} + p_{\|}\,\widehat{\mathsf{b}}\;\dot{\mathbf{X}}_{\perp}\right)F_{\mu}\,dp_{\|}$$

- CGL pressure tensor

$$\mathsf{P}_{\rm CGL} \equiv \sum^{\mu}\int\left[\frac{p_{\|}^2}{m}\,\widehat{\mathsf{b}}\,\widehat{\mathsf{b}} + \mu\,B\,\left(\mathbf{I} - \widehat{\mathsf{b}}\widehat{\mathsf{b}}\right)\right]F_{\mu}\,dp_{\|}$$

Guiding-center toroidal angular momentum conservation law

- Toroidal covariant component $P_{\mathrm{gc}\varphi} \equiv \mathbf{P}_{\mathrm{gc}} \cdot \partial\mathbf{x}/\partial\varphi$

$$\frac{\partial P_{\mathrm{gc}\varphi}}{\partial t} + \nabla\cdot\left(\mathsf{T}_{\mathrm{gc}}\cdot\frac{\partial\mathbf{x}}{\partial\varphi}\right) = \nabla\left(\frac{\partial\mathbf{x}}{\partial\varphi}\right) : \mathsf{T}_{\mathrm{gc}}^{\top} \equiv 0$$

- **Symmetric guiding-center stress tensor**

$$\mathsf{T}_{\mathrm{gc}}^{\top} \equiv \mathsf{T}_{\mathrm{gc}}$$

- Guiding-center stress tensor T_{gc} was previously only assumed to be symmetric (e.g., Similon 1985).

- Guiding-center polarization is crucial in establishing symmetry

$$\mathsf{T}_{\mathrm{gcV}} \equiv \mathsf{P}_{\mathrm{CGL}} + \sum\nolimits^{\mu} \int \left(\dot{\mathbf{X}}_{\perp}\; p_{\|}\,\widehat{\mathsf{b}} + p_{\|}\,\widehat{\mathsf{b}}\;\dot{\mathbf{X}}_{\perp}\right) F_{\mu}\, dp_{\|}$$

Summary of Part I

- Variational formulations (Lagrange, Euler, & Euler-Poincaré) of guiding-center Vlasov-Maxwell equations have been derived.

- Guiding-center Vlasov-Maxwell theory is a pre-gyrokinetic theory that does not separate background and perturbed Vlasov-Maxwell fields.

- Exact energy-momentum & angular-momentum conservation laws rely on the symmetry of the guiding-center Vlasov-Maxwell stress tensor.

- The symmetry of the guiding-center Vlasov-Maxwell stress tensor depends on the complete representation of the guiding-center magnetization as the sum of the intrinsic magnetic-dipole and the moving electric-dipole contributions.

IV. Parallel-symplectic Gyrokinetic Equations

- **Gyrocenter symplectic one-form** ($\mathbf{R}_0^* \equiv \mathbf{R} + \frac{1}{2}\nabla\times\widehat{\mathsf{b}}_0$)

$$\Gamma_{\rm gy} = \left[\frac{e}{c}\left(\mathbf{A}_0 + \epsilon\,\langle A_{1\|{\rm gc}}\rangle\,\widehat{\mathsf{b}}_0\right) + p_\|\,\widehat{\mathsf{b}}_0\right]\cdot d\mathbf{X} + \mu\,\frac{B}{\Omega}\left(d\zeta - \mathbf{R}_0^*\cdot d\mathbf{X}\right) - w\,dt$$

- Gyrocenter Poisson bracket ($\{\ ,\ \}_0 \equiv \{\ ,\ \}_{0{\rm gc}}$)

$$\{F,\ G\}_{\rm gy} = \frac{\Omega}{B}\left(\frac{\partial F}{\partial\zeta}\frac{\partial G}{\partial\mu} - \frac{\partial F}{\partial\mu}\frac{\partial G}{\partial\zeta}\right) + \left(\frac{\partial F}{\partial w}\frac{\partial G}{\partial t} - \frac{\partial F}{\partial t}\frac{\partial G}{\partial w}\right) + \frac{\mathbf{B}_\epsilon^*}{B_{\epsilon\|}^*}\cdot\left(\nabla_\epsilon^* F\,\frac{\partial G}{\partial p_\|} - \frac{\partial F}{\partial p_\|}\,\nabla_\epsilon^* G\right) - \frac{c\widehat{\mathsf{b}}_0}{eB_{\epsilon\|}^*}\cdot\nabla_\epsilon^* F\times\nabla_\epsilon^* G$$

where $\mathbf{B}_\epsilon^* \equiv \mathbf{B}_0^* + \epsilon\,\nabla\times\left(\langle A_{1\|{\rm gc}}\rangle\,\widehat{\mathsf{b}}_0\right)$, $B_{\epsilon\|}^* \equiv \widehat{\mathsf{b}}_0\cdot\mathbf{B}_\epsilon^*$, and

$$\nabla_\epsilon^* F \equiv \left(\nabla F + \mathbf{R}_0^*\frac{\partial F}{\partial\zeta}\right) - \frac{\epsilon e}{c}\,\widehat{\mathsf{b}}_0\left(\frac{\partial\langle A_{1\|{\rm gc}}\rangle}{\partial t}\frac{\partial F}{\partial w} + \frac{\Omega}{B}\frac{\partial\langle A_{1\|{\rm gc}}\rangle}{\partial\mu}\frac{\partial F}{\partial\zeta}\right)$$

- **Gyrocenter Hamiltonian**: $(\Phi_{1gc}, \mathbf{A}_{1gc}) \rightarrow (\mathbf{E}_{1gc}, \mathbf{B}_{1gc})$

$$
\begin{aligned}
H_{gy} \;=\; & \left(\mu B + p_{\|}^{2}/2m\right) \;+\; \epsilon\, e \left(\langle \Phi_{1gc} \rangle \;-\; \left\langle \mathbf{A}_{1\perp gc} \cdot \frac{\Omega}{c} \frac{\partial \boldsymbol{\rho}_0}{\partial \zeta} \right\rangle \right) \\
& -\frac{\epsilon^2}{2} \left[\left\langle e\, \boldsymbol{\rho}_{1gc} \cdot \left(\mathbf{E}_{1gc} + \frac{1}{c} \{ \mathbf{X} + \boldsymbol{\rho}_0,\; H_{0gc} \}_0 \times \mathbf{B}_{1gc} \right) \right\rangle \right. \\
& \left. + \left\langle \frac{e}{c} \mathbf{A}_{1gc} \cdot \left(\{ \mathbf{X} + \boldsymbol{\rho}_0,\; e \langle \psi_{1gc} \rangle \}_0 \;+\; \frac{e}{mc} \langle A_{1\| gc} \rangle\, \widehat{\mathsf{b}}_0 \right) \right\rangle \right]
\end{aligned}
$$

- First-order gyrocenter displacement: Polarization/Magnetization

$$
\boldsymbol{\rho}_{1gc} \;\equiv\; \left(\frac{d}{d\epsilon} \mathsf{T}_{gy}^{-1} (\mathbf{X} + \boldsymbol{\rho}_0) \right)_{\epsilon = 0} \;=\; \{ \mathbf{X} + \boldsymbol{\rho}_0,\; S_1 \}_0
$$

- First-order effective potential & gyrocenter gauge function:

$$
\psi_{1gc} \equiv \Phi_{1gc} - \mathbf{A}_{1gc} \cdot \frac{1}{c} \{ \mathbf{X} + \boldsymbol{\rho}_0,\; H_{0gc} \}_0
$$

$$
\frac{\partial S_1}{\partial t} + \{ S_1,\; H_{0gc} \}_0 \;=\; e \left(\psi_{1gc} \;-\; \langle \psi_{1gc} \rangle \right)
$$

- **Gyrokinetic Vlasov equation**

  - Gyrocenter Vlasov distribution $F_\mu(\mathbf{X}, p_\|, t)$

$$\frac{\partial F_\mu}{\partial t} \;+\; \dot{\mathbf{X}}\cdot\nabla F_\mu \;+\; \dot{p}_\| \,\frac{\partial F_\mu}{\partial p_\|} \;=\; 0$$

  - Gyrocenter Hamilton equations

$$\dot{\mathbf{X}} = \frac{\partial H_{\rm gy}}{\partial p_\|}\,\frac{\mathbf{B}^*_\epsilon}{B^*_{\epsilon\|}} + \frac{c\widehat{\mathrm{b}}_0}{B^*_{\epsilon\|}}\times\nabla H_{\rm gy}, \quad \dot{p}_\| = -\,\frac{\mathbf{B}^*_\epsilon}{B^*_{\epsilon\|}}\cdot\nabla H_{\rm gy} \;-\; \epsilon\,\frac{e}{c}\frac{\partial\langle A_{1\|{\rm gc}}\rangle}{\partial t}$$

  - Gyrocenter Liouville theorem ($\mathcal{J}_{\rm gy} \equiv 2\pi\, m\, B^*_{\epsilon\|}$)

$$\frac{\partial B^*_{\epsilon\|}}{\partial t} \;=\; -\,\frac{\partial}{\partial z^a}\left(\dot{z}^a\; B^*_{\epsilon\|}\right) \;\rightarrow\; \frac{\partial (B^*_{\epsilon\|}\,F_\mu)}{\partial t} \;=\; -\,\frac{\partial}{\partial z^a}\left(\dot{z}^a\; B^*_{\epsilon\|}\,F_\mu\right)$$

- **Gyrokinetic Maxwell equations**

◦ Gyrokinetic Poisson equation $[\delta^3_{\rm gc} \equiv \delta^3(\mathbf{X} + \boldsymbol{\rho}_{\rm gc} - \mathbf{r})]$

$$\epsilon\,\nabla^2\Phi_1(\mathbf{r}) \;=\; -\,4\pi\int \mathcal{J}_{\rm gy}\,F_\mu\left\langle \mathsf{T}^{-1}_{\rm gy}\,e\,\delta^3_{\rm gc}\right\rangle d^6Z$$
$$=\; -\,4\pi\left(\varrho_{\rm gy} \;-\; \nabla\cdot\mathbf{P}_{\rm gy}\right)$$

◦ Gyrokinetic Ampère equation $(\mathbf{B} = \mathbf{B}_0 + \epsilon\,\mathbf{B}_1)$

$$\nabla\times\mathbf{B}(\mathbf{r}) \;=\; \frac{4\pi}{c}\int \mathcal{J}_{\rm gy}\,F_\mu\left\langle \mathsf{T}^{-1}_{\rm gy}\left(e\,\delta^3_{\rm gc}\,\{\mathbf{X} + \boldsymbol{\rho}_{\rm gc},\; H_{\rm gc}\}_0\right)\right\rangle d^6Z$$
$$=\; \frac{4\pi}{c}\left(\mathbf{J}_{\rm gy} \;+\; \frac{\partial\mathbf{P}_{\rm gy}}{\partial t} \;+\; c\,\nabla\times\mathbf{M}_{\rm gy}\right)$$

◦ Note: Because of variational derivation, $\mathsf{T}^{-1}_{\rm gy}f = f \;-\; \epsilon\,\mathcal{L}_{1\rm gy}f$

Parallel-symplectic Gyrokinetic Euler Variational Principle

- **Gyrokinetic Euler action**

$$\mathcal{A}_{\rm gy} \;=\; -\int \mathcal{F}_{\rm gy}\,\mathcal{H}_{\rm gy}\; d^8\mathcal{Z} + \int \frac{d^4x}{8\pi}\left(\epsilon^2\,|\nabla\Phi_1|^2 \;-\; |\mathbf{B}_0 + \epsilon\,\nabla\times\mathbf{A}_1|^2\right)$$

◦ Extended gyrocenter Vlasov density

$$\mathcal{F}_{\rm gy} \;\equiv\; B^*_{\epsilon\|}\; F_\mu\,\delta(w - H_{\rm gy}) \;\equiv\; B^*_{\epsilon\|}\;\mathcal{F}_\mu$$

◦ Eulerian gyrocenter Hamiltonian variation

$$\delta H_{\rm gy} \;=\; \epsilon\left\langle \mathsf{T}_{\rm gy}^{-1}\left(e\,\delta\psi_{1\rm gc}\right)\right\rangle \;+\; \epsilon\,\frac{ep_\|}{mc}\,\langle\delta A_{1\|\rm gc}\rangle$$

◦ Note: $\partial H_{\rm gy}/\partial p_\| \;=\; p_\|/m + \mathcal{O}(\epsilon^2)$

- **Gyrocenter Vlasov Eulerian variation**

$$
\begin{aligned}
\delta\mathcal{F}_{\rm gy} &= \delta B^{*}_{\epsilon\|}\,\mathcal{F}_{\mu} \;+\; B^{*}_{\epsilon\|}\,\delta\mathcal{F}_{\mu} \\
&= \left(\epsilon\,\langle\delta A_{1\|{\rm gc}}\rangle\,\widehat{\sf b}_0\cdot\nabla\times\widehat{\sf b}_0\right)\,\mathcal{F}_{\mu} \\
&\quad +\; B^{*}_{\epsilon\|}\left(\{\delta\mathcal{S},\;\mathcal{F}_{\mu}\}_{\rm gy} \;+\; \epsilon\,\frac{e}{c}\,\langle\delta A_{1\|{\rm gc}}\rangle\,\frac{\partial\mathcal{F}_{\mu}}{\partial p_{\|}}\right) \\
&= \frac{\partial}{\partial\mathcal{Z}^{\alpha}}\left(\mathcal{F}_{\rm gy}\,\delta\mathcal{Z}^{\alpha}\right)
\end{aligned}
$$

- Gyrocenter phase-space virtual displacement

$$
\delta\mathcal{Z}^{\alpha} \;\equiv\; \{\delta\mathcal{S},\,Z^{\alpha}\}_{\rm gy} + \epsilon\,\frac{e}{c}\langle\delta A_{1\|{\rm gc}}\rangle\;\widehat{\sf b}_0\cdot\{\mathbf{X},\;\mathcal{Z}^{\alpha}\}_{\rm gy}
$$

- Extended gyrokinetic Vlasov equation: $0 \;=\; \{\mathcal{F},\;\mathcal{H}_{\rm gy}\}_{\rm gy}$

$$
0 \;=\; \int\{\mathcal{F}_{\mu},\;\mathcal{H}_{\rm gy}\}_{\rm gy}\,dw \;=\; \frac{\partial(B^{*}_{\epsilon\|}\,F_{\mu})}{\partial t} + \frac{\partial}{\partial z^{a}}\left(\dot{z}^{a}\,B^{*}_{\epsilon\|}\,F_{\mu}\right)
$$

- **Gyrokinetic variational principle**

$$
\begin{aligned}
\delta\mathcal{A}_{\rm gy} &= -\int d^8\mathcal{Z}\left[\mathcal{H}_{\rm gy}\,\frac{\partial}{\partial\mathcal{Z}^\alpha}\left(\mathcal{F}_{\rm gy}\,\delta\mathcal{Z}^\alpha\right)\right. \\
&\quad \left. +\,\mathcal{F}_{\rm gy}\left(\epsilon\left\langle \mathsf{T}_{\rm gy}^{-1}\left(e\,\delta\psi_{1\rm gc}\right)\right\rangle \,+\,\epsilon\,\frac{ep_\|}{mc}\,\langle\delta A_{1\|\rm gc}\rangle\right)\right] \\
&\quad -\int\frac{d^4x}{4\pi}\left(\epsilon^2\,\delta\Phi_1\,\nabla^2\Phi_1\,+\,\epsilon\,\delta\mathbf{A}_1\cdot\nabla\times\mathbf{B}\right) \\
&= -\int d^8\mathcal{Z}\,B^*_{\epsilon\|}\left[\delta\mathcal{S}\,\left\{\mathcal{F}_\mu,\,\mathcal{H}_{\rm gy}\right\}_{\rm gy}\right. \\
&\qquad\qquad \left. -\,\epsilon\,\frac{e}{c}\langle\delta A_{1\|\rm gc}\rangle\,\mathcal{F}_\mu\left(\frac{\partial H_{\rm gy}}{\partial p_\|}-\frac{p_\|}{m}\right)\right] \\
&\quad -\left[\int\frac{d^4x}{4\pi}\left(\epsilon^2\,\delta\Phi_1\,\nabla^2\Phi_1\,+\,\epsilon\,\delta\mathbf{A}_1\cdot\nabla\times\mathbf{B}\right)\right. \\
&\qquad \left. +\int d^8\mathcal{Z}\,\mathcal{F}_{\rm gy}\left(\epsilon\,\left\langle\mathsf{T}_{\rm gy}^{-1}\left(e\,\delta\psi_{1\rm gc}\right)\right\rangle\right)\right] \;=\; \mathcal{O}(\epsilon^3)
\end{aligned}
$$

Summary of Part II

- Equivalent representations of guiding-center and gyrokinetic Vlasov-Maxwell equations are available.

- Equivalent gyrokinetic Vlasov-Maxwell equations can be derived by variational principle.

- Future work will look at truncated parallel-symplectic gyrokinetic Vlasov-Maxwell equations and derive its energy-momentum conservation laws by Noether method.

Lectures Notes on Gyrokinetic Theory
(graduate-level textbook)
to be completed by Fall of 2017