Hot Particle Equilibrium code (HPE) with plasma anisotropy and toroidal rotation*

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Abstract

The development of HPE *code addresses the needs for accurate evaluation of the effects of plasma toroidal rotation and anisotropic pressure generated by NBI on tokamak equilibrium. These effects are becoming increasingly important after the upgrades of DIII-D and NSTX machines in the US where the hot particle pressure can become comparable with the core plasma pressure and toroidal rotation can routinely exceed the poloidal Mach number 1.*

The HPE *code represents the upgrade of the Equilibrium and Stability Code (*ESC*). One of specifics of* HPE *is its prescribed safety factor $q(a)$ regime, where a is a normalized radial flux coordinate. The isotropic pressure $p(a)$ profile is another input function to* HPE*. Other combinations of input profiles are also possible, similarly to* ESC*.*

The hot particle pressure is specified by the parallel pressure $p_\parallel(\bar{\Psi}, B)$ as function of poloidal flux and module of the magnetic field. In the present version of HPE *$p_\parallel$ is generated using a mono-energetic distribution function of hot particles with multiple pitch angles. The extension to the general case would only modify the input of the code. Its equilibrium solver is already designed for the general case. The kinetic parallel pressure $p_\parallel$ contributes to both isotropic pressure $p_\parallel(a)$ and to oscillatory part $\tilde{p}_\parallel(a,\theta)$, where θ is poloidal angle.*

The important novel elements of HPE *is the inclusion to hot particle pressure of the finite width of the drift orbits of fast ions, both passing and trapped.*

The plasma toroidal rotation is specified by the poloidal Mach number profile $\mathcal{M}^2(a)$ under assumption that the plasma temperature (energy) is constant along the magnetic surface.

For the case of powerful NBI injection in tokamaks the HPE *code is capable to provide the equilibrium calculations for theory needs as well as for interpretation of experiments with the use of experimental (or kinetic simulations) data.*

Also the free-boundary version of HPE *was created for non-tokamak geometry as a separate development.*

Contents

Scalar pressure equilibrium (r, ϕ, z are cylindrical coordinates)

$$
\begin{aligned}
&\mu_0 \mathbf{j} \equiv (\nabla \times \mathbf{B}),\\
&\mathbf{B} = \frac{1}{r}(\nabla \bar{\Psi} \times \mathbf{e}_\phi) + \frac{1}{r}\bar{F}\mathbf{e}_\phi, \quad \bar{F} \equiv rB_\phi,\\
&\nabla \bar{p}_{pl} - \bar{\rho}\Omega^2 r\mathbf{e}_r = (\mu_0 \mathbf{j} \times \mathbf{B}),\\
&\bar{p}_{pl} \equiv \mu_0 p_{pl}, \quad \bar{p}_{\|} \equiv \mu_0 p_{\|}, \quad \bar{\rho} \equiv \mu_0 \rho.
\end{aligned}
\tag{1.1}
$$

Here, p_{pl} is the core plasma pressure, ρ is the plasma density, $\Omega = \Omega(\bar{\Psi})$ is toroidal rotation frequency.

Following Grad (the 1950s), the equilibrium with hot particles is described by

$$
\begin{aligned}
&\mathbf{K} \equiv (\nabla \times \sigma \mathbf{B}),\\
&\mathbf{B} = \frac{1}{r}(\nabla \bar{\Psi} \times \mathbf{e}_\phi) + \frac{1}{r}\bar{F}\mathbf{e}_\phi, \quad \bar{F} \equiv rB_\phi,\\
&\sigma \equiv 1 - \tau, \quad \tau \equiv \frac{\bar{p}_{\|} - \bar{p}_{\perp}}{B^2}, \quad B \equiv |\mathbf{B}|,\\
&\nabla \bar{p}_{pl} + \nabla \bar{p}_{\|} - \bar{\rho}\Omega^2 r\mathbf{e}_r = B\tau\nabla B + (\mathbf{K} \times \mathbf{B}),\\
&\bar{p}_{pl} \equiv \mu_0 p_{pl}, \quad \bar{p}_{\|} \equiv \mu_0 p_{\|}, \quad \bar{\rho} \equiv \mu_0 \rho,
\end{aligned}
\tag{1.2}
$$

where r $p_{\|}$ is the hot particle parallel pressure.

The perpendicular pressure $\bar{p}_\perp$ does not enter the equilibrium equations (1.2). Instead, it can be determined from anisotropy function τ

$$\bar{p}_\perp = \bar{p}_\parallel - B^2\tau. \tag{1.3}$$

In* HPE *the parallel pressure of hot particles is split in two terms

$$\bar{p}_\parallel(a,\theta) \equiv \bar{p}_\parallel^{hot}(a) + \tilde{p}_\parallel(a,\theta), \quad \int_{-\pi}^{\pi} \tilde{p}_\parallel(a,\theta)d\theta = 0, \tag{1.4}$$

***thus, composing the input function $\bar{p}(a)$ of isotropic pressure for* HPE**

$$\bar{p}(a) \equiv \bar{p}^{pl}(a) + \bar{p}_\parallel^{hot}(a), \tag{1.5}$$

which is considered as the isotropic part of the pressure.

Here a,θ are flux coordinates, $\bar{\Psi} = \bar{\Psi}(a)$. In* HPE *there are 6 choices of radial coordinate a.

1.2 Toroidal rotation in HPE

Toroidal rotation is considered under assumption that plasma temperatures and rotation frequency are uniform along the magnetic surface and

$$\begin{aligned} &T_{e,i} = T_{e,i}(a), \quad \Omega = \Omega(a), \\ &n_e = n_e(a, r), \\ &\bar{p} = \mu_0 n_e(a, r)[T_e(a) + T_i(a)]. \end{aligned} \tag{1.6}$$

For $\mathrm{B} \cdot \nabla T_{e,i} = 0$ ***the rotation term can be written as***

$$\bar{\rho}\Omega^2 r = \bar{p}\frac{m_i\Omega^2 R^2}{T_e(a) + T_i(a)}\frac{r}{R^2} = \bar{p}\mathcal{M}^2\frac{r}{R^2}, \tag{1.7}$$

where R ***is the major radius of the magnetic axis, and***

$$\mathcal{M}^2(a) \equiv \frac{m_i\Omega^2 R^2}{T_e(a) + T_i(a)}, \tag{1.8}$$

the square of the poloidal Mach number.

Rearrangement of the centrifugal force

The isotropic plasma pressure $\bar{p}(a,r)$ is determined by compensation of the centrifugal force

$$\begin{aligned} &\frac{\partial \bar{p}(a,r)}{\partial r}\mathrm{e}_r = \bar{\rho}\Omega^2 r \mathrm{e}_r = \bar{p}\mathcal{M}^2\frac{r}{R^2}\mathrm{e}_r, \\ &\bar{p}(a,r) = \widehat{p}(a) e^{\mathcal{M}^2\frac{r^2-R^2}{2R^2}}. \end{aligned} \tag{1.9}$$

With plasma rotation $\widehat{p}(a)$ serves as the input function. It has the meaning of the isotropic plasma pressure along the vertical line $r = R$,

$$\widehat{p}(a) \equiv \bar{p}(a,R). \tag{1.10}$$

With $\bar{p}$ in the form of Eq. (1.9) the force along e_r is eliminated from equilibrium equation

$$\nabla\bar{p} - \bar{\rho}\Omega^2 r \mathrm{e}_r = \left(\widehat{p}(a) e^{\mathcal{M}^2\frac{r^2-R^2}{2R^2}}\right)'_a \nabla a. \tag{1.11}$$

In* HPE *the contribution of asymmetric part hot particles pressure $\tilde{p}_{\|}$ to rotation force is neglected.

In the GSh form, the equilibrium equations with rotation and hot particles can be written as

$$\Delta^{*\sigma}\bar{\Psi} \equiv r\nabla\cdot\left(\frac{\sigma\nabla\bar{\Psi}}{r}\right) = -r^2\left(\widehat{p}(\bar{\Psi})e^{\mathcal{M}^2\frac{r^2-R^2}{2R^2}} + \tilde{p}_\parallel\right)'_{\bar{\Psi}} - \frac{1}{\sigma}T,$$
$$T(a) \equiv \sigma\bar{F}\frac{d(\sigma\bar{F})}{d\bar{\Psi}}, \quad (\mathrm{B}\cdot\nabla(\sigma\bar{F})) = 0, \tag{2.1}$$
$$B\tau = \frac{\partial\tilde{p}_\parallel}{\partial B}.$$

Here the first term $\propto \widehat{p}$ represents the total isotropic pressure, $\tilde{p}_\parallel(\bar{\Psi}, B)$ is the oscillatory part of the parallel pressure. The plasma density is consumed by the poloidal Mach number (1.8).

The differential operator $\Delta^{*\sigma}$ represents the modification by the presence of the hot particles of the conventional GSh operator Δ^* and implemented in** HPE **in the self-conjugate form.

The input profiles for the** HPE **code are $\widehat{p}(a)$, $\bar{p}_\parallel(a, B)$, $\mathcal{M}^2(a)$ and $q(a)$. By convention, $\widehat{p}(a)$ takes into account the contributions from both core plasma and hot particles.

Note, that the "q-solver" regime of** HPE **is taken from the** ESC **code and is described later.

2.1 The algorithm of q-solver in HPE

In flux coordinates a, ϕ, θ (with $a = 1$ at the plasma boundary), the GSh equation for standard input profiles has the form

$$\begin{aligned}&(K\bar\Psi'_a - N\bar\Psi'_\theta)_a + (M\bar\Psi'_\theta - N\bar\Psi'_a)'_\theta = -V\left(\widehat{p}(\bar\Psi)e^{\mathcal{M}^2(\bar\Psi)\frac{r^2-R^2}{2R^2}} + \tilde{p}_\parallel\right)'_{\bar\Psi} - LT,\\ &K \equiv \sigma\frac{g_{\theta\theta}}{rD}, \quad N \equiv \sigma\frac{g_{a\theta}}{rD}, \quad M \equiv \sigma\frac{g_{aa}}{rD}, \quad D \equiv \frac{D(z,r)}{D(a,\theta)}, \quad V \equiv rD, \quad L \equiv \frac{D}{\sigma r}.\end{aligned} \tag{2.2}$$

In HPE *as in* ESC *the second order differential equation is written as a system of the first order equations*

$$Y' = -(M\bar\Psi'_\theta - N\bar\Psi'_a)'_\theta - V\left(\ldots\right)'_{\bar\Psi} - LT, \quad K\bar\Psi'_a = Y + N\bar\Psi'_\theta. \tag{2.3}$$

The zeroth harmonics of the first equation gives T in terms of Y'_0

$$Y'_0 = -\left(V\left(\ldots\right)'_{\bar\Psi}\right)_0 - L_0T, \tag{2.4}$$

and allows for elimination of unknown T

$$\left(Y' - \frac{L}{L_0}Y_0\right)' = -(M\bar\Psi'_\theta - N\bar\Psi'_a)'_\theta - V\left(\ldots\right)'_{\bar\Psi} + \frac{L}{L_0}\left(V\left(\ldots\right)'_{\bar\Psi}\right)_0 - \left(\frac{L}{L_0}\right)'_a Y_0. \tag{2.5}$$

The second equation (2.3), determines Y_0 for Eq. (2.5) in terms of $\bar\Psi'_0$, which is determined by q-profile.

For given $q(a)$-profile this algorithm results in fast convergence with the same, 3-5 iteration speed, as for given current density profile.

3 Mono-energetic model of hot particles

At present, the mono-energetic $E = E_h$ distribution function of hot particles is used in **HPE**

$$f = f(\bar\Psi, E, \mu) = f(a)\delta(E - E_h)s(\tau),$$
$$\mu \equiv \frac{mv_\perp^2}{2B}, \quad \lambda \equiv \frac{\mu B_{min}}{E} = \left.\frac{v_\perp^2}{v^2}\right|_{B=B_{min}}, \quad \tau = \left.\frac{v_\parallel}{v}\right|_{B=B_{min}}. \tag{3.1}$$

The pitch angle dependence is specified by factor $s(\tau)$ as a function of the pitch angle τ.

$$s(\tau)d\tau = \frac{sd\lambda}{2\sqrt{1-\lambda}}. \tag{3.2}$$

This specification leads to the space distribution of the hot particle density

$$n(a, B) = \int f d^3v = n(a)\frac{B}{B_{min}} \cdot \frac{1}{2}\int_0^t \frac{sd\lambda}{\sqrt{1 - \lambda\frac{B}{B_{min}}}},$$
$$n(a) \equiv n(a, B_{min}), \tag{3.3}$$
$$t \equiv \frac{B_{min}}{B}, \quad \epsilon \equiv \frac{1}{R_m} = \frac{B_{min}}{B_{max}}.$$

The integral over λ is taken within the accessibility region $0 \le \lambda \le t$ at each magnetic surface.

3.1 Hot particle pressure with finite orbit corrections

Parallel pressure

$$p_{\|}(a,B) = p_E(a)\frac{B}{B_{min}}\int_0^t \sqrt{1-\lambda\frac{B}{B_{min}}}\,s d\lambda + \delta p^{pass} + \delta p^{tr}, \qquad (3.4)$$
$$p_E(a) \equiv n(a, B_{min})E.$$

Two 1-D functions $p_E(a)$ and $s(\tau)$ specify the hot particle parallel pressure $p_{\|}(a,B)$.

Finite width corrections $\delta p^{pass}, \delta p^{tr}$ are given in the linear approximation as

$$\delta p_{\|}^{pass}(a,B) = \pm\frac{2}{3}\rho_L \bar{F}\frac{dp_E}{d\bar{\Psi}}\frac{B}{B_{min}}\left(1-\frac{B}{B_{min}}\right)\int_0^{\epsilon}\sqrt{1-\lambda\frac{B}{B_{min}}}\,s d\lambda,$$
$$\delta p_{\|}^{tr}(a,B) = -\frac{2}{3}\rho_L \bar{F}\frac{dp_E}{d\bar{\Psi}}\int_{\epsilon}^{t}\left(1-\lambda\frac{B}{B_{min}}\right) s d\lambda, \qquad (3.5)$$

***where $\rho_\perp$ is the perpendicular Larmor radius of hot particles. In* HPE**

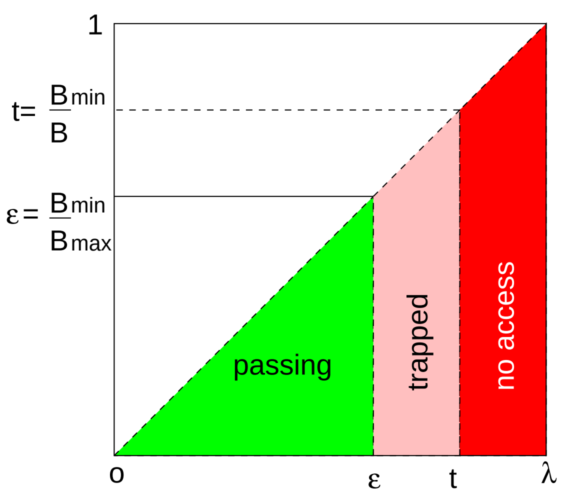
$$\rho_\perp = \frac{v_\perp mc}{eB_{a=0}}. \qquad (3.6)$$

The integration is taken within the regions of passing $0 \le \lambda \le \epsilon$ and trapped particles $\epsilon \le \lambda \le t$.

3.2 Numerical evaluation of parallel pressure

Six following integral functions are sufficient for calculation of $p_{\|}(a, B)$ and its derivative $\frac{\partial p_{\|}}{\partial B}$

$$
\begin{aligned}
&I_{-1}(t) \equiv \int_0^t \frac{s(\tau)d\lambda}{\sqrt{1-\frac{\lambda}{t}}}, \quad I_1(t) \equiv \int_0^t \sqrt{1-\frac{\lambda}{t}}s(\tau)d\lambda, \\
&I_{-1}^{pass}(t) \equiv \int_0^{\frac{1}{R_m}} \frac{\lambda s(\tau)d\lambda}{\sqrt{1-\frac{\lambda}{t}}}, \quad I_1^{pass}(t) \equiv \int_0^{\frac{1}{R_m}} \sqrt{1-\frac{\lambda}{t}}s(\tau)d\lambda, \\
&J_0(t) = \int_0^t s(\tau)d\lambda, \quad J_1(t) = \int_0^t \lambda s(\tau)d\lambda.
\end{aligned}
\tag{3.7}
$$



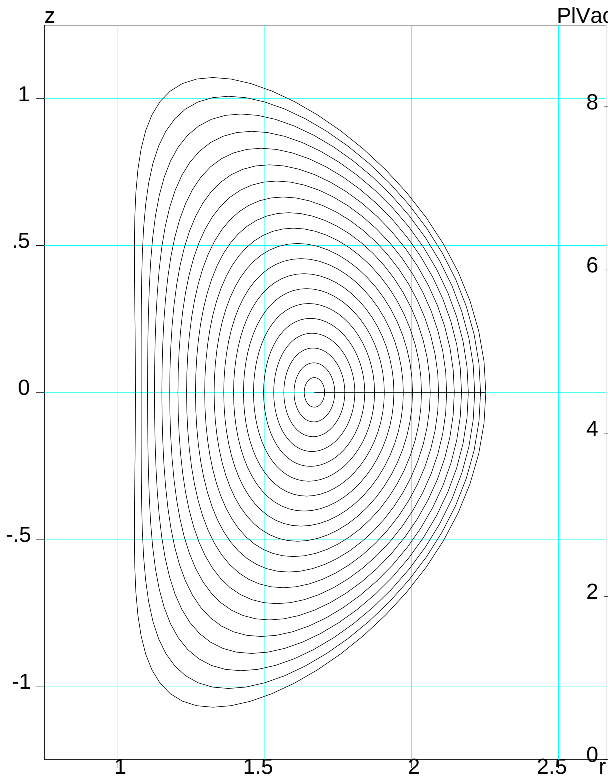
The argument space for I_1, I_{-1}

$$
\begin{aligned}
&\tau = \sqrt{1-\lambda}, \\
&t \equiv \frac{B_{min}}{B}, \quad \epsilon \le t \le 1
\end{aligned}
\tag{3.8}
$$

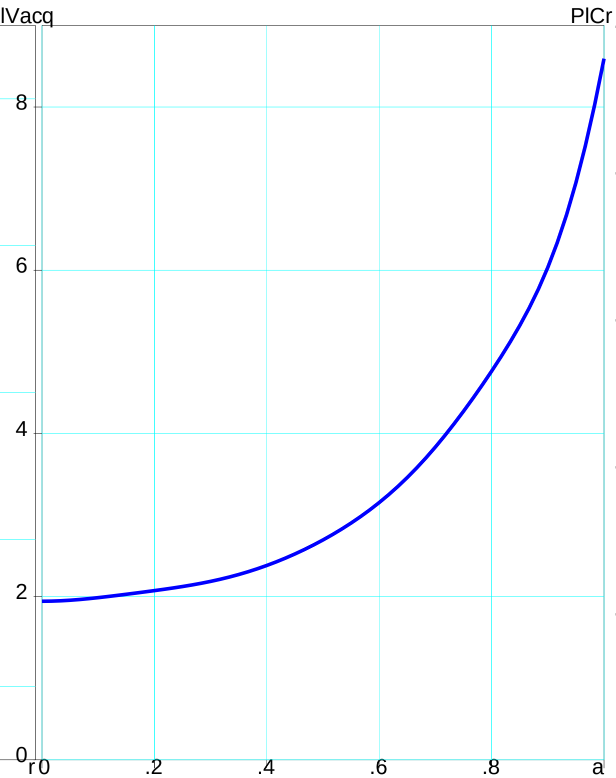
Given the $s(\tau)$ as a spline function, the integrals are evaluated numerically.

4 Simulation examples

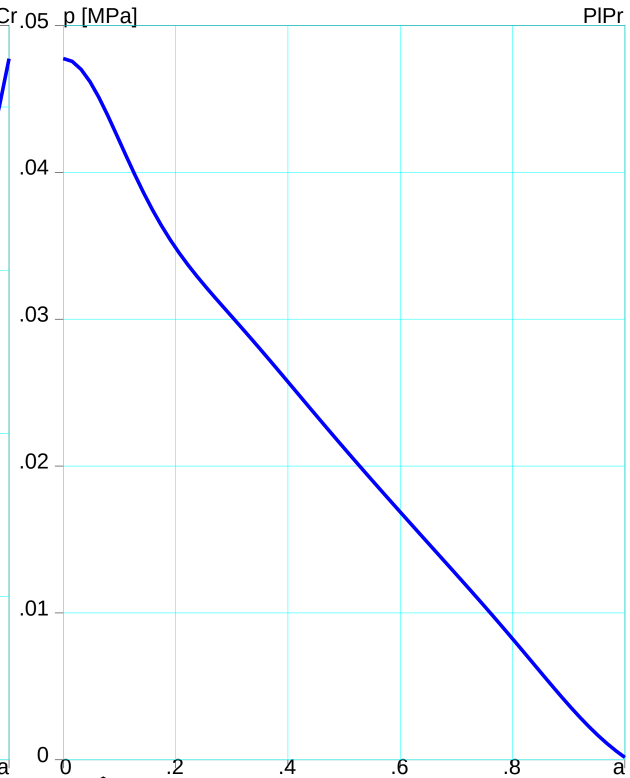
DIII-D experimental $\bar{p}(a),\ q(a)$ profiles are taken for illustration of* HPE *functionlity



DIII-D plasma cross section

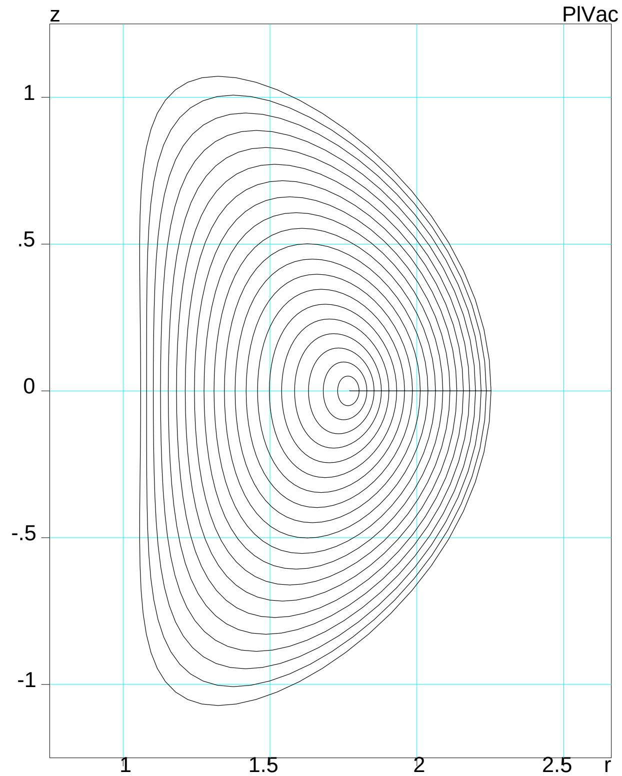


q-profile

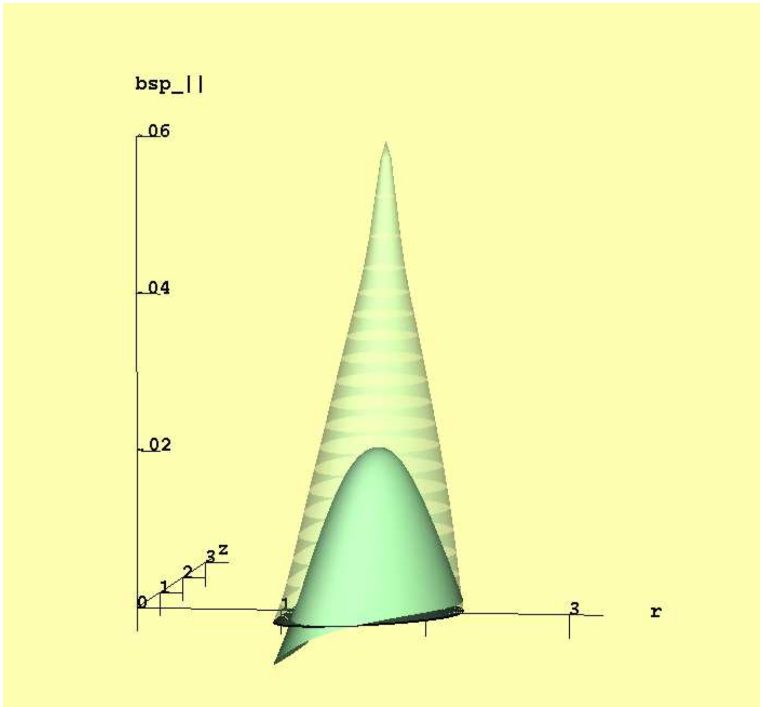


***$\widehat{p}$-profile,** $\beta = 1.4\%$*

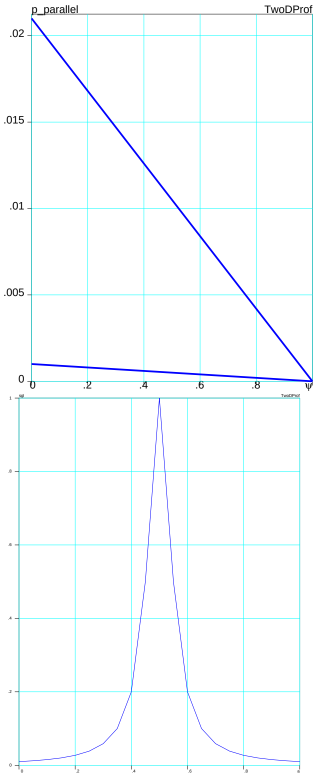
Equilibrim with anisotropic pressure



Plasma cross section



$\mathbf{p}_{\|}(r, z)$ ***2-D profile (green). Shadowed surface is*** $\widehat{\mathbf{p}}(a)$



Input $\bar{\mathbf{p}}_{\|}(\psi),\ s(\lambda)$ ***profiles***

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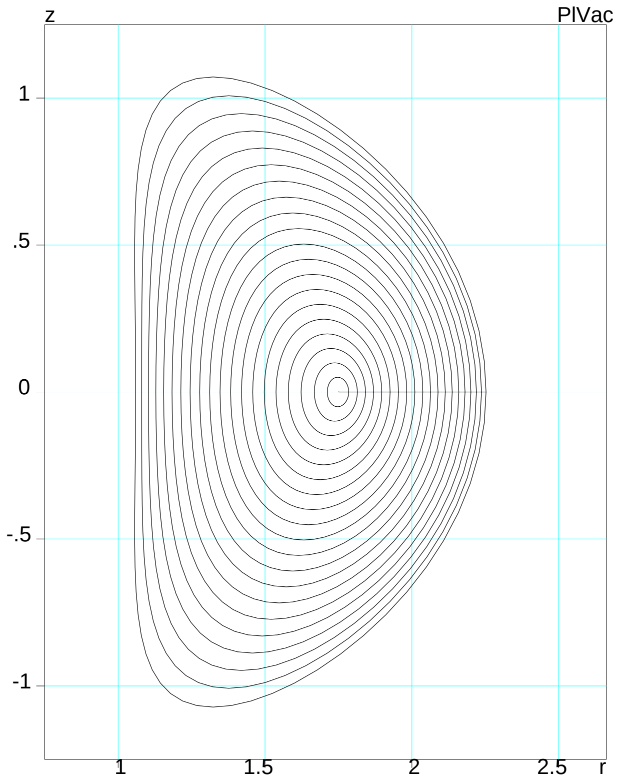
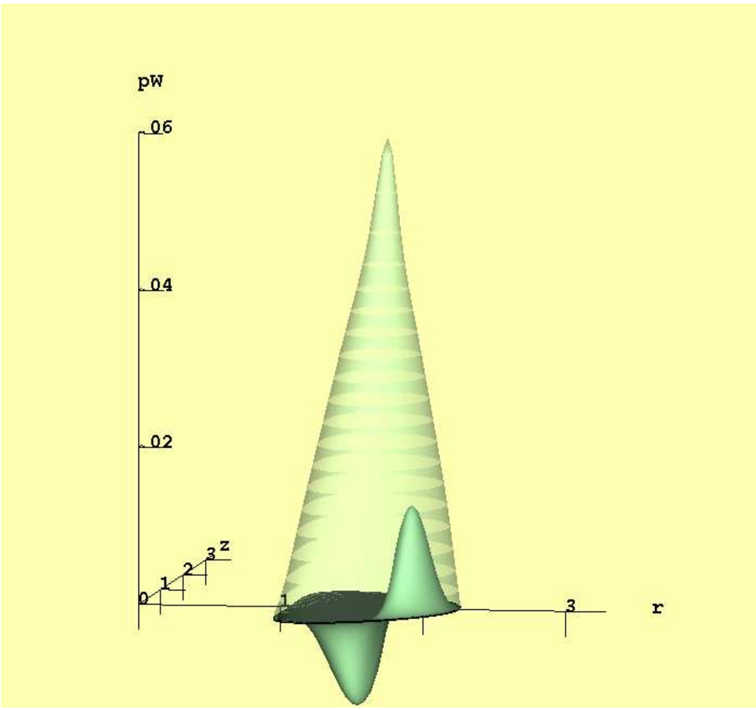
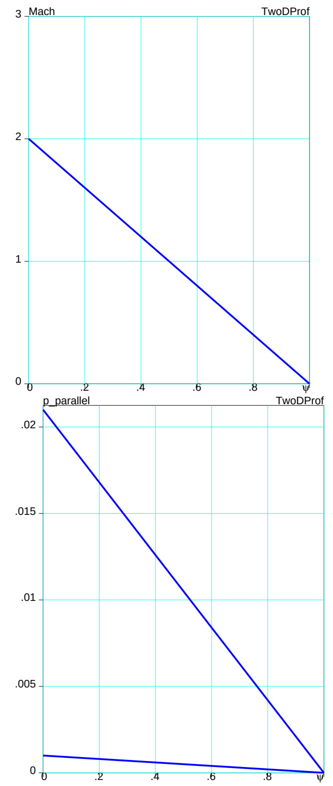
Equilibrim with anisotropic pressure and toroidal rotation



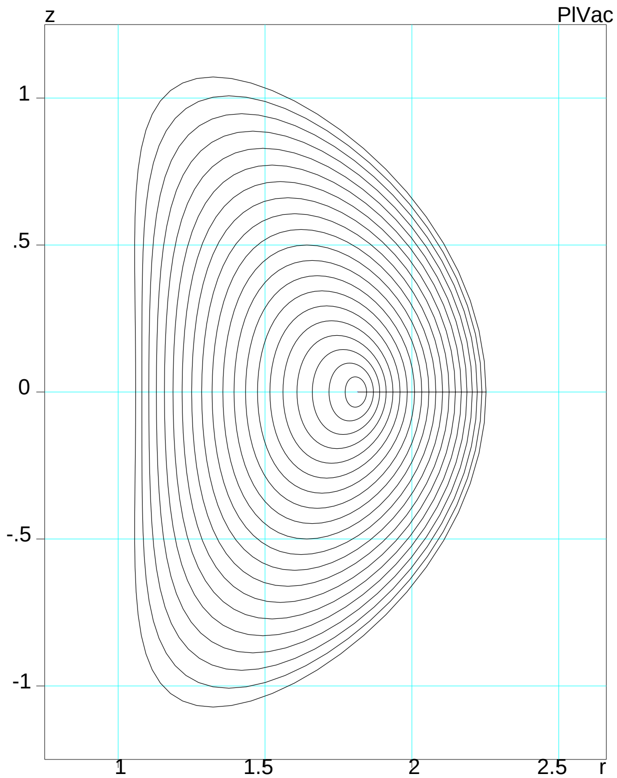
Plasma cross section



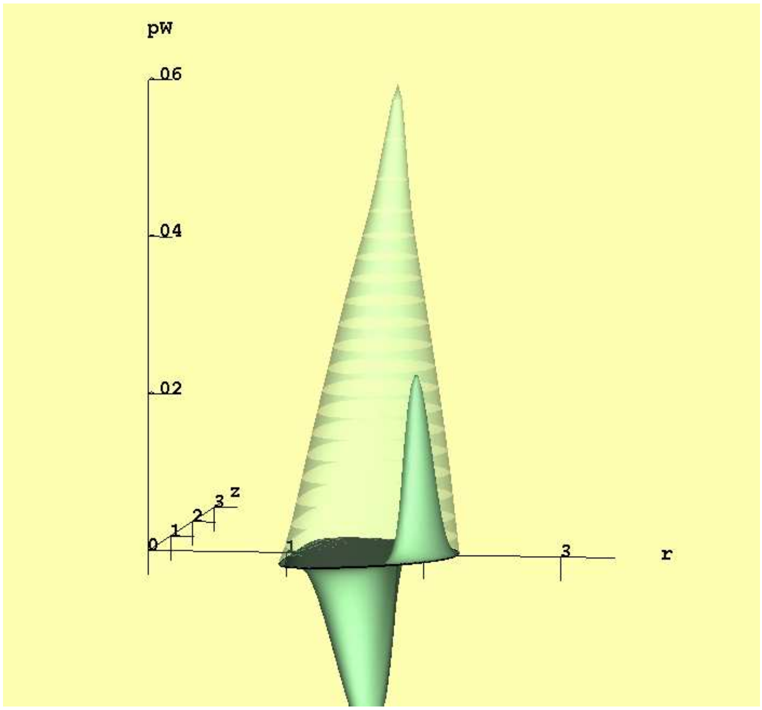
Rotation contribution to plasma pressure



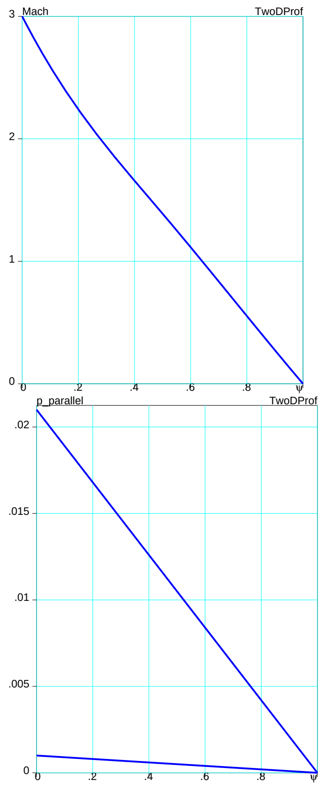
Input profiles $\mathcal{M}(\psi)$, $\bar{p}_{\|}(\psi)$



Plasma cross section



Rotation contribution to plasma pressure



Input profiles $\mathcal{M}(\psi)$, $\bar{p}_{||}(\psi)$

5 Summary

The fast equilibrium code* HPE *for plasma equilibrium with hot particle and toroidal rotation is created and functional.

Working in both modes with

(a) specified current density, and
(b) given q-profile,

the* HPE *provides for fast (3-5 iterations) convergence in flux coordinates, suitable for transport and stability analysis.

For the first time, the corrections due to finite width of hot ion orbits are included in the parallel hot ion pressure as the input for the flux coordinate equilibrium q-solver.

The present version of the parallel pressure specification corresponds to mono-energetic distribution function of hot ions, which in turn is specified by the 1-D radial profile and the 1-D distribution of pitch angle.

At the same time the* HPE *equilibrium solver itself is general and can use any $\bar{p}_{\|}(\bar{\Psi}, B)$ as the input.

In particular, the realistic $\bar{p}_{\|}(\bar{\Psi}, B)$ profiles can be provided by kinetic NBI codes working together with* HPE*. Because of coupling through the magnetic geometry, such an interface might be a challenge.