Temperature Gradients are supported by Cantori in Chaotic Magnetic Fields

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Motivation

→ Error fields, 3D effects, . . create chaotic fields.

Method

→ Heat transport is solved numerically:

\[ \nabla \cdot \left( \kappa_\parallel \mathbf{b} \cdot \nabla T + \kappa_\perp \nabla_\perp T \right) = 0 \text{ with } \kappa_\parallel / \kappa_\perp = 10^{10}. \]

We found

→ Isotherms coincide with cantori,
→ chaotic coordinates, based on ghost - surfaces,
    solves for the temperature profile in a chaotic field.

eg. M3D simulation of CDX-U
Field line transport is restricted by irrational field-lines

→ the irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict field line transport even after the onset of chaos.

→ KAM surfaces stop radial field line transport

→ broken KAM surfaces ≡ cantori do not stop, but do slow down radial field line transport

Poincaré plot (model field → next slide)
Cantori are approximated by high-order periodic orbits;

→ high-order (minimizing) periodic orbits are located using variational methods;

- Magnetic field lines, $B = \nabla \times A$, are stationary curves $C$ of the action integral $S = \int_C A \cdot dl$, where $A = \psi \nabla \theta - \chi \nabla \phi$ and $\chi(\psi, \theta, \phi) = \psi^2 / 2 + \sum k_{mn}(\psi) \cos(m\theta - n\phi)$.

- Setting $\delta S = 0$ gives $\dot{\theta} = B^\theta / B^\phi = \dot{\theta}(\psi, \theta, \phi)$ and $\dot{\psi} = B^\psi / B^\phi$.

- A piecewise linear, $\theta(\phi) = \theta_i + (\theta_{i+1} - \theta_i) / \Delta \phi$, trial curve allows analytic evaluation of the action integral, $S = S(\theta_0, \theta_1, \theta_2 \ldots) \rightarrow \text{fast!}$

- To find $(p, q)$ periodic curves, use Newton's method to find $\partial S / \partial \theta_i = 0 \rightarrow \text{robust!}$ with constraint $\phi_N = 2\pi q$, $\theta_N = \theta_0 + 2\pi p$.

- Two types of periodic orbit: $O :$ stable, action-minimax
  $X :$ unstable, action-minimizing $\rightarrow \text{cantori as } p/q \rightarrow \text{irrational}$
**Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.**


- At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\partial^2 S / \partial \theta_{ij}$, with negative eigenvalue indicates the direction in which the action integral decreases.

- Pushing trial curve from minimax (stable) $p/q$ orbit down action-gradient flow to minimizing (unstable) $p/q$ orbit defines *ghost-surfaces*.

- Ghost-surfaces may be thought of as rational coordinate surfaces that pass through island chains.

\[
\text{Action Gradient Flow } \frac{\partial \theta_i}{\partial \alpha} = -\frac{\partial S}{\partial \theta_i}
\]
Steady state temperature is solved numerically; isotherms coincide with ghost-surfaces.

→ ghost-surface for high order periodic orbits “fill in the gaps” in the irrational cantori;

→ ghost-surfaces and isotherms are almost indistinguishable;

NUMERICS

• heat flux $\nabla \cdot q = 0$, where $q = \kappa || \mathbf{b} \cdot \nabla T + \kappa_\perp \nabla T$;

  strongly anisotropic $\kappa_|| / \kappa_\perp = 10^{10}$;

• parallel relaxation, use field-aligned coordinates

  $B = \nabla \alpha \times \nabla \beta$, so $\nabla_||^2 T = B^\phi \frac{\partial}{\partial \phi} \left( \frac{B^\phi}{B^2} \frac{\partial T}{\partial \phi} \right)$

• perpendicular relaxation, use symmetric finite-diff.

  $\nabla_\perp^2 T = \partial_{xx}^2 T + \partial_{yy}^2 T$

• solve sparse linear system iteratively

  on numerical grid $2^{12} \times 2^{12}$
Chaotic-coordinates simplifies temperature profile

→ ghost-surfaces can be used as radial coordinate surfaces → chaotic-coordinates \((s, \theta, \phi)\)

- From \(0 = \frac{\partial}{\partial s} \int V \nabla \cdot \mathbf{q} \, dV = \frac{\partial}{\partial s} \int V \mathbf{q} \cdot \mathbf{n} \, d\sigma\) assume \(T = T(s)\) to derive \(T' = \frac{\text{const.}}{|\kappa_\parallel \Omega + \kappa_\perp G|}\)
  
  for quadratic-flux \(\Omega = \int d\sigma g^{ss} (B_n / B)^2\), and metric \(G = \int d\sigma g^{ss}\), where \(g^{ss} = \nabla s \cdot \nabla s\), \(B_n = B \cdot \nabla s / |\nabla s|\)
  
- in the "ideal limit" \(\kappa_\perp \to 0\), \(T' \to \infty\) on irrational KAM surfaces where \(\varphi = 0\);
  
- non-zero \(\kappa_\perp\) ensures \(T(s)\) is smooth, \(T'\) peaks on minimal-\(\varphi\) surfaces (noble cantori).

Temperature Profile

\((\kappa_\parallel / \kappa_\perp = 10^{10})\)
Summary

→ in chaotic fields, anisotropic heat transport is restricted by irrational field lines ≡ cantori
→ interpolating a suitable selection of ghost-surfaces allows chaotic-magnetic-coordinates to be constructed
→ the temperature takes the form $T=T(s)$, where $s$ labels the chaotic coordinate surfaces, and an expression for the temperature gradient is derived.

Future Work

→ For a practical implementation of this theory, eg. in MHD codes, the following points must be addressed:
   → what is the best selection of rational $p/q$ ghost-surfaces for a given chaotic field?
   → how does the best selection of ghost-surfaces depend on $\kappa$?
   → how should the ghost-surfaces be interpolated?