The simplest model of approximating global, macroscopic force-balance in toroidal plasma confinement with arbitrary geometry is magnetohydrodynamics (MHD).

Non-axisymmetric magnetic fields generally do not have a nested family of smooth flux surfaces, unless ideal surface currents are allowed at the rational surfaces.

If the field is non-integrable (chaotic, fractal phase space), then any continuous pressure that satisfies $B \cdot \nabla p = 0$ must have an infinitely discontinuous gradient, $\nabla p$.

Instead, solutions with stepped-pressure profiles are guaranteed to exist. A partially-relaxed, topologically-constrained, MHD energy principle is described.

Equilibrium solutions are calculated numerically. Results demonstrating convergence tests, benchmarks, and non-trivial solutions are presented.

The constraints of ideal MHD may be applied at the rational surfaces, in which case surface currents prevent the formation of islands. Or, these constraints may be relaxed in the vicinity of the rational surfaces, in which case magnetic islands will open if resonant perturbations are applied.
An ideal equilibrium with non-integrable (chaotic) field and continuous pressure, is infinitely discontinuous.

**Ideal MHD theory** = \( \nabla p = j \times B \), gives \( B \cdot \nabla p = 0 \)

**Chaos theory** = nowhere are flux surfaces continuously nested

*for non-symmetric systems, nested family of flux surfaces is destroyed

*islands & irregular field lines appear where transform is rational \((n/m)\); rationals are dense in space

Poincare-Birkhoff theorem \( \rightarrow \) periodic orbits, (e.g. stable and unstable) guaranteed to survive into chaos

*some irrational surfaces survive if there exists an \( r, k \in \mathbb{R} \) s.t. for all rationals, \( |t - n/m| > rm^{-k} \)

i.e. rotational-transform, \( \imath \), is poorly approximated by rationals,

**Ideal MHD + Chaos \( \rightarrow \) infinitely discontinuous equilibrium**

*iterative method for calculating equilibria is ill-posed;

1) \( B_n \cdot \nabla p = 0 \) \( \nabla p \) is everywhere discontinuous, or zero;

2) \( j_\perp = B_n \times \nabla p / B_n^2 \) \( j_\perp \) everywhere discontinuous or zero;

3) \( B_n \cdot \nabla \sigma = -\nabla \cdot j_\perp \) \( B \cdot \nabla \) is densely and irregularly singular;

\( \sigma \) is single valued if and only if \( \delta \sigma = -\oint_C \nabla \cdot j_\perp dl / B = 0 \)

pressure must be flat across every closed field line, or parallel current is not single-valued;

4) \( \nabla \times B_{n+1} = j \equiv \sigma B_n + j_\perp \) solution only if \( \nabla \cdot (\sigma B + j_\perp) = 0 \)

To have a well-posed equilibrium with chaotic \( B \) need to

\( \rightarrow \) introduce non-ideal terms, such as resistivity, \( \eta \), perpendicular diffusion, \( \kappa_\perp \), \([HINT, M3D, NIMROD,..] \),

\( \rightarrow \) or return to an energy principle, but relax infinity of ideal MHD constraints

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**Diophantine Condition**

Kolmogorov, Arnold and Moser
Instead, a multi-region, relaxed energy principle for MHD equilibria with non-trivial pressure and chaotic fields

Energy, helicity and mass integrals (defined in nested annular volumes)

\[ W_l = \int_{V_l} \left( \frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv, \quad H_l = \int_{V_l} (A \cdot B) dv, \quad M_l = \int_{V_l} p^{1/\gamma} dv \]

Seek constrained, minimum-energy state

\[ F = \sum_{l=1}^{N} \left( W_l - \mu_l H_l / 2 - \nu_l M_l \right) \]

1st variation due to unconstrained variations \( \delta p, \delta A, \) and interface geometry, \( \xi, \)

except ideal "topological" constraint \( \delta B = \nabla \times (\xi \times B) \) imposed discretely at interfaces

\[ \delta F = \sum_{l=1}^{N} \left\{ \int_{V_l} \left( \frac{1}{\gamma - 1} - \frac{\nu_l p^{1/\gamma - 1}}{\gamma} \right) \delta p \ dv + \int_{V_l} \delta A \cdot (\nabla \times B - \mu_l B) dv - \int_{\partial V_l} \left[ \left[ p + B^2 / 2 \right] \right] \xi \cdot dS \right\} \]

Equilibrium solutions when \( \nabla \times B = \mu_l B \) in annuli, \( [[p+B^2/2]]=0 \) across interfaces

→ partial Taylor relaxation allowed in each annulus; allows for topological variations/islands/chaos;
→ global relaxation prevented by ideal constraints; → non-trivial stepped – pressure solutions;
→ \( \nabla \times B = \mu_l B \) is a linear equation for \( B; \) depends on interface geometry; solved in parallel in each annulus;
→ solving force balance ≡ adjusting interface geometry to satisfy \( [[p+B^2/2]]=0; \)

ideal interfaces that support pressure generally have irrational rotational-transform;
standard numerical problem finding zero of multi-dimensional function; call NAG routine;
Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

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We establish an existence result for the three-dimensional MHD equations

\[(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p\]
\[\nabla \cdot \mathbf{B} = 0\]
\[\mathbf{B} \cdot n|_{\partial T} = 0\]

with \(p \neq \text{const in tori } T\) without symmetry. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps. © 1996 John Wiley & Sons, Inc.


→ this was a strong motivation for pursuing the stepped-pressure equilibrium model

→ how large the “sufficiently small” departure from axisymmetry can be needs to be explored numerically
By definition, an equilibrium code must constrain topology;

Definition: Equilibrium Code (fixed boundary)
given (1) boundary (2) pressure (3) rotational-transform \( \equiv \) inverse q-profile (or current profile)
\( \rightarrow \) calculate \( B \) that is consistent with force-balance; pressure profile is not changed!
c.f. "coupled equilibrium-transport" approach, that evolves pressure while evolving field

Cannot apriori specify pressure without apriori constraining topology of the field
\( \rightarrow \) the constraint \( B \cdot \nabla p = 0 \) means the structure of \( B \) and \( p \) are intimately connected;
if \( p \) is given and \( B \) that satisfies force balance is to be constructed,
then flux surfaces must coincide with pressure gradients; (e.g. if \( p \) is smooth, \( B \) must have nested surfaces).

\( \rightarrow \) specifying the profiles discretely is a practical means of retaining some control
over the profiles, whilst making minimal assumptions regarding the topology;
\( \rightarrow \) pressure gradients are assumed to coincide with a set of strongly-irrational \( \equiv \) "noble" flux surfaces

noble irrational
\( \equiv \) limit of alternating path down Farey-tree
\( \equiv \) Fibonacci sequence

\[ \frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_1 + p_2}{q_1 + q_2}, \ldots \rightarrow \frac{p_1 + \gamma p_2}{q_1 + \gamma q_2}, \quad \gamma = \text{golden mean} \]
Extrema of energy functional obtained numerically; introducing the Stepped Pressure Equilibrium Code (SPEC)

The vector-potential is discretized

* toroidal coordinates \((s, \vartheta, \zeta)\), *interface geometry \(R_l = \sum_{m,n} R_{l,m,n} \cos(m\vartheta - n\zeta), Z_l = \sum_{m,n} Z_{l,m,n} \sin(m\vartheta - n\zeta)\)

* exploit gauge freedom \(A = A_\vartheta(s, \vartheta, \zeta) \nabla \vartheta + A_\zeta(s, \vartheta, \zeta) \nabla \zeta\)

* Fourier \(A_\vartheta = \sum_{m,n} a_{\vartheta}(s) \cos(m\vartheta - n\zeta)\)

* Finite-element \(a_{\vartheta}(s) = \sum_i a_{\vartheta, i}(s) \varphi(s)\) piecewise cubic or quintic basis polynomials

and inserted into constrained-energy functional \(F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 - \nu_l M_l \right)\)

* derivatives w.r.t. vector-potential → linear equation for Beltrami field \(\nabla \times \mathbf{B} = \mu \mathbf{B}\) solved using sparse linear solver

* field in each annulus computed independently, distributed across multiple cpus

* field in each annulus depends on enclosed toroidal flux (boundary condition) and

  → poloidal flux, \(\psi_p\), and helicity-multiplier, \(\mu\) adjusted so interface transform is strongly irrational

  → geometry of interfaces, \(\xi \equiv \{R_{m,n}, Z_{m,n}\}\)


* interface geometry is adjusted to satisfy force \(\mathbf{F}[\xi] = \{[[p+ B^2/2 ]]_{m,n}\} = 0\)

* angle freedom constrained by spectral-condensation, adjust angle freedom to minimize \(\sum (m^2 + n^2) \left(R_{mn}^2 + Z_{mn}^2 \right)\) minimal spectral width [Hirshman, VMEC]

* derivative matrix, \(\nabla \mathbf{F}[\xi]\), computed in parallel using finite-differences

* call NAG routine: quadratic-convergence w.r.t. Newton iterations; robust convex-gradient method;
Numerical error in Beltrami field scales as expected

Scaling of numerical error with radial resolution depends on finite-element basis

\[ A = A_\vartheta \nabla \vartheta + A_\zeta \nabla \zeta, \quad B = \nabla \times A, \quad j = \nabla \times B, \]

need to quantify error \( \text{error} = j - \mu B \)

\[ A_\vartheta, A_\zeta \sim O(h^n) \]

\[ \sqrt{g} B^s = \partial_\vartheta A_\zeta - \partial_\zeta A_\vartheta \sim O(h^n) \]

\[ \sqrt{g} B^\vartheta = -\partial_s A_\zeta \sim O(h^{n-1}) \]

\[ \sqrt{g} B^\zeta = \partial_s A_\vartheta \sim O(h^{n-1}) \]

\[ \sqrt{g} j^s \sim O(h^{n-1}) \]

\[ \sqrt{g} j^\vartheta \sim O(h^{n-2}) \]

\[ \sqrt{g} j^\zeta \sim O(h^{n-2}) \]

Example of chaotic Beltrami field in single given annulus;

\[ R = 1.0 + r(\vartheta, \zeta) \cos \vartheta, \]

\[ Z = r(\vartheta, \zeta) \sin \vartheta, \]

inner surface \( r = 0.1 \)

outer interface \( r = 0.2 + \delta [\cos(2\vartheta - \zeta) + \cos(3\vartheta - \zeta)] \)

Poincaré plot, \( \zeta = 0 \)

sub-radial grid, \( N=16 \)
Stepped-pressure equilibria accurately approximate smooth-pressure *axisymmetric* equilibria

in axisymmetric geometry . . .

→ magnetic fields have family of nested flux surfaces
→ equilibria with smooth profiles exist,
→ may perform benchmarks (e.g. with VMEC)
  (arbitrarily approximate smooth-profile with stepped-profile)
→ approximation improves as number of interfaces increases
→ location of magnetic axis converges w.r.t radial resolution

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**Increasing pressure resolution ≡ number of interfaces**

step-profile approximation to smooth profile

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Pressure, $p$

**toroidal flux $\psi$**

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**transformation**

---

magnetic axis vs. radial resolution
using quintic-radial finite-element basis
(for high pressure equilibrium)
(dotted line indicates VMEC result)

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$N$ ≡ finite-element resolution
Equilibria with (i) perturbed boundary & chaotic fields, and (ii) pressure are computed.

Boundary deformation induces islands

\[ R = 1.0 + r \cos \vartheta, \quad Z = r \sin \vartheta \]

\[ r = 0.3 + \delta \cos(2\vartheta - \phi) + \delta \cos(3\vartheta - \phi) \]

\[ \delta = 10^{-4} \]

**Demonstrated Convergence**

of high-pressure equilibrium with islands, with Fourier Resolution,

Convergence of (2,1) & (3,1) island widths ..

with Fourier resolution, \( \beta \approx 4\% \) case
poloidal resolution \( 0 \leq m \leq M \)
toroidal resolution \( -N \leq n \leq N \)
Sequence of equilibria with increasing pressure shows plasma *can* have significant response to external perturbation.

axisymmetric

\[ R = 1.00 + 0.30 \cos(\varphi) + 0.05 \cos(2\varphi) \]

plus small perturbation

\[ Z = 1.00 + 0.40 \sin(\varphi) + \left[ \delta_{21} \cos(2\varphi - \zeta) + \delta_{31} \cos(3\varphi - \zeta) \right] \cos(\varphi) \]

\[ \left[ \delta_{21} \cos(2\varphi - \zeta) + \delta_{31} \cos(3\varphi - \zeta) \right] \sin(\varphi) \]

**\( \beta_{tot} \approx 0.000 \)**

**\( \beta_{tot} \approx 0.018 \)**

**\( T_{VMEC} \approx 100s \)**

**\( T_F \approx 30s \)**

**\( T_{VF} \approx 1.5h \)**

M = 6 N = 3 RESOLUTION

PRESSURE

TRANSFORM

resonant normal error field

\( B_{2,1}^n \)

\( B_{3,1}^n \)
If ideal constraint applied at rational surfaces, then shielding currents prevent island formation.

axisymmetric boundary, \( R = 1.0 + 0.3 \cos \theta + 0.05 \cos 2\theta, \)
\( Z = 0.4 \sin \theta \)

plus perturbation (\( \delta = 10^{-4} \))

\( \delta R = \delta \cos(2\theta - \phi) \cos \theta \)
\( \delta Z = \delta \cos(2\theta - \phi) \sin \theta \)

pressure gradients coincide with irrational interfaces

with rational ideal interface \( \approx \) non-linear IPEC

without rational ideal interface \( \rightarrow q=1/2 \) island opens
Summary

→ A partially-relaxed, topologically-constrained energy principle has been described and the equilibrium solutions constructed numerically

* using a high-order (piecewise quintic) radial discretization, and a spectrally condensed Fourier representation
* workload distributed across multiple cpus,
* extrema located using standard numerical methods (NAG): modified Newton’s method, with quadratic-convergence
* non-axisymmetric solutions with chaotic fields and non-trivial pressure guaranteed to exist (under certain conditions)

→ Specifying the profiles discretely is a practical means of retaining some control over the profiles, while making minimal assumptions regarding the topology of the field

* it is only assumed that some flux surfaces exist
* pressure gradients coincide with strongly irrational flux surfaces

→ Convergence studies have been performed

* expected error scaling with radial resolution confirmed
* detailed benchmark with axisymmetric equilibria (with smooth profiles)
* demonstrated convergence of island widths with Fourier resolution

→ By enforcing the ideal constraint at the rational surfaces, the formation of magnetic islands is prohibited by the formation of surface “shielding” currents

* similar to non-linear generalization of IPEC
* relaxing ideal constraint at rational surfaces allows islands to open
Force balance condition at interfaces gives rise to auxiliary pressure-jump Hamiltonian system.

→ Beltrami condition, $\nabla \times \mathbf{B} = \mu \mathbf{B}$, and interface constraint, $\mathbf{B} \cdot \mathbf{n} = 0$, gives $\nabla \times \mathbf{B} \cdot \nabla s = 0$,
suggests surface potential, $B_\varphi = \partial_\varphi f$, $B_\zeta = \partial_\zeta f$, so that $\partial_\varphi B_\zeta - \partial_\zeta B_\varphi = 0$,

$$B^2 = \left( g_{\varphi\varphi} f_\varphi f_\varphi - 2 g_{\varphi\zeta} f_\varphi f_\zeta + g_{\zeta\zeta} f_\zeta f_\zeta \right) / \left( g_{\varphi\varphi} g_{\zeta\zeta} - g_{\varphi\zeta} g_{\zeta\varphi} \right),$$

metric elements $g_{\alpha\beta} \equiv \partial_\alpha \mathbf{x} \cdot \partial_\beta \mathbf{x}$

→ Force balance condition, $[[p + B^2 / 2]] = 0$, introduce $H \equiv 2(p_1 - p_2) = B_2^2 - B_1^2 = \text{const}$.

→ Let tangential field on "inner-side" of interface be given, $B_{1\varphi} = \partial_\varphi f$, $B_{1\zeta} = \partial_\zeta f$,
tangential field on "outer-side", $B_{2\varphi} = p_\varphi$, $B_{2\zeta} = p_\zeta$, determined by characteristics

$$\dot{\varphi} = \left. \frac{\partial H(\varphi, \zeta, p_\varphi, p_\zeta)}{\partial p_\varphi} \right|_{\zeta, p_\varphi, p_\zeta}, \quad \dot{p}_\varphi = - \left. \frac{\partial H}{\partial \varphi} \right|_{\zeta, p_\varphi, p_\zeta}, \quad \dot{\zeta} = \left. \frac{\partial H}{\partial p_\zeta} \right|_{\varphi, p_\varphi, p_\zeta}, \quad \dot{p}_\zeta = - \left. \frac{\partial H}{\partial \zeta} \right|_{\varphi, p_\varphi, p_\zeta}

→ 2 d.o.f. Hamiltonian system, and invariant surfaces only exist if "frequency" is irrational

⇒ ideal interfaces that support pressure must have irrational transform

Hamilton-Jacobi theory for continuation of magnetic field across a toroidal surface supporting a plasma pressure discontinuity

Sequence of equilibria with increasing pressure shows plasma *can* have significant response to external perturbation.

axisymmetric plus small perturbation

\[ R = 1.00 + 0.30 \cos(\vartheta) + 0.05 \cos(2\vartheta) + [\delta_{21} \cos(2\vartheta - \zeta) + \delta_{31} \cos(3\vartheta - \zeta)] \cos(\vartheta) \]

\[ Z = 1.00 + 0.40 \sin(\vartheta) + [\delta_{21} \cos(2\vartheta - \zeta) + \delta_{31} \cos(3\vartheta - \zeta)] \sin(\vartheta) \]
Sequence of equilibria with increasing pressure shows plasma can have significant response to external perturbation.

axisymmetric plus perturbation \( \delta_{21} = \delta_{31} = 10^{-4} \)

\[
R = 1.00 + [0.30 + \delta_{21} \cos(2 \varphi - \zeta) + \delta_{31} \cos(3 \varphi - \zeta)] \cos(\varphi)
\]

\[
Z = 1.00 + [0.30 + \delta_{21} \cos(2 \varphi - \zeta) + \delta_{31} \cos(3 \varphi - \zeta)] \sin(\varphi)
\]

Resonant radial field at rational surface; \( n=1,2,3 \) stability from PEST;
Sequence of equilibria with slowly increasing pressure

\( \beta_{\text{tot}} \approx 0.000 \)

axisymmetric : \( R = 1.00 + 0.30 \cos(\vartheta) + 0.05 \cos(2\vartheta) \)

plus \( Z = 1.00 + 0.40 \sin(\vartheta) \)

perturbation : \( \delta R = [\delta_{21} \cos(2\vartheta - \zeta) + \delta_{31} \cos(3\vartheta - \zeta)] \cos(\vartheta) \)

\( \delta Z = [\delta_{21} \cos(2\vartheta - \zeta) + \delta_{31} \cos(3\vartheta - \zeta)] \sin(\vartheta) \)

\( T_F \approx 20 \text{s} \)

\( T_{VF} \approx 60 \text{ m} \)
Toroidal magnetic confinement depends on flux surfaces

Transport in magnetized plasma dominately parallel to $\mathbf{B}$

→ if the field lines are not confined (e.g. by flux surfaces), then the plasma is poorly confined

Axisymmetric magnetic fields possess a continuously nested family of flux surfaces

→ nested family of flux surfaces is guaranteed if the system has an ignorable coordinate

→ rational field-line ≡ periodic trajectory

→ irrational field-lines cover *irrational* flux surface

$m$ magnetic field is called *integrable*

$rational field-line \ \mathcal{G} = 0.3333… \ \xi$

$irrational field-line \ \mathcal{G} = 0.3819… \ \xi$

straight-field-line flux coordinates,

$\mathbf{B} \cdot \nabla \psi = 0$

$\mathbf{B} = \nabla \psi \times \nabla \mathcal{G} + i (\psi) \nabla \zeta \times \nabla \psi$

$\sqrt{g} \mathbf{B} \cdot \nabla \equiv \partial_{\zeta} + i \partial_{\mathcal{G}}$

magnetic differential equation, $\mathbf{B} \cdot \nabla \sigma = s$,

is singular at rational surfaces, $(m \ i - n) \sigma_{m,n} = i(\sqrt{g} s)_{m,n}$
Ideal MHD equilibria are extrema of energy functional

The energy functional is

$$W = \int_V (p + B^2 / 2) \, dv$$

*ideal variations*

mass conservation

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

state equation

$$d_t (p \rho^{-\gamma}) = 0$$

Faraday's law, ideal Ohm's law

$$\delta \mathbf{B} = \nabla \times (\delta \mathbf{\xi} \times \mathbf{B})$$

→ ideal variations don’t allow field topology to change “frozen-flux”

the first variation in plasma energy is

$$\delta W = \int_V (\nabla p - j \times \mathbf{B}) \cdot \delta \mathbf{\xi} \, dv$$

→ two surface functions, e.g. the pressure, \( p(s) \), and rotational-transform \( \equiv \) inverse-safety-factor, \( \psi(s) \), and → a boundary surface (… for fixed boundary equilibria…)

constitute “boundary-conditions” that must be provided to uniquely define an equilibrium solution

...... The computational task is to compute the magnetic field that is consistent with the given boundary conditions…

nested flux surface topology maintained by singular currents at rational surfaces

from \( \nabla \cdot (\sigma \mathbf{B} + j_\perp) = 0 \), parallel current must satisfy \( \mathbf{B} \cdot \nabla \sigma = -\nabla \cdot j_\perp \), where \( j_\perp = \mathbf{B} \times \nabla p / B^2 \)

→ magnetic differential equations are singular at rational surfaces (periodic orbits)

→ pressure-driven “Pfirsch-Schlüter currents” have \( 1/x \) type singularity

→ \( \delta \) - function singular currents shield out islands

\[
\sigma_{m,n} = \frac{i (\sqrt{g} \cdot \nabla \cdot j_\perp)_{m,n}}{(mt - n)} + \delta(mt - n)
\]
Topological constraints: pressure gradients coincide with flux surfaces

The ideal interfaces are chosen to coincide with pressure gradients

→ parallel transport dominates perpendicular transport,
→ simplest approximation is $\mathbf{B} \cdot \nabla p = 0$
→ pressure gradients must coincide with KAM surfaces $\equiv$ ideal interfaces

The next order of approximation is $\partial_t p = \kappa_\parallel \nabla_\parallel^2 p + \kappa_\perp \nabla_\perp^2 p = 0$, with $\kappa_\parallel \gg \kappa_\perp$, e.g. $\kappa_\perp/\kappa_\parallel \sim 10^{-10}$

*pressure gradients coincide with KAM surfaces, cantori...
*pressure flattened across islands, chaos with width $\Delta w_c \sim (\kappa_\perp/\kappa_\parallel)^{1/4}$
*anisotropic diffusion equation solved analytically, $p' \propto 1/(\kappa_\parallel \delta^2 + \kappa_\perp G)$, $\delta_2$ is quadratic-flux across cantori, $G$ is metric term

A fixed boundary equilibrium is defined by:
(i) given pressure, $p(\psi)$, and rotational-transform profile, $\tau(\psi)$
(ii) geometry of boundary;

(a) only stepped pressure profiles are consistent (numerically tractable) with chaos and $\mathbf{B} \cdot \nabla p = 0$
(b) the computed equilibrium magnetic field must be consistent with the input profiles
(a) + (b) = where the pressure has gradients, the magnetic field must have flux surfaces.
→ non-trivial stepped pressure equilibrium solutions are guaranteed to exist
Taylor relaxation: a weakly resistive plasma will relax, subject to single constraint of conserved helicity.

Taylor relaxation, [Taylor, 1974]

\[
W = \int_V \left( p + B^2 / 2 \right) dv, \quad H = \int_V (A \cdot B) dv
\]

Constrained energy functional \( F = W - \mu H / 2, \) \( \mu \equiv \) Lagrange multiplier.

Euler-Lagrange equation, for unconstrained variations in magnetic field, \( \nabla \times B = \mu B \)

But, . . . Taylor relaxed fields have no pressure gradients

Ideal MHD equilibria and Taylor-relaxed equilibria are at opposite extremes . . . .

Ideal-MHD → imposition of infinity of ideal MHD constraints

non-trivial pressure profiles, but structure of field is over-constrained.

Taylor relaxation → imposition of single constraint of conserved global helicity

structure of field is not-constrained, but pressure profile is trivial, i.e. under-constrained.

We need something in between . . .

. . . perhaps an equilibrium model with finitely many ideal constraints, and partial Taylor relaxation?
Introducing the multi-volume, partially-relaxed model of MHD equilibria with topological constraints

Energy, helicity and mass integrals

\[ W_l = \int_{V_l} \left( \frac{p}{\gamma-1} + \frac{B^2}{2} \right) dv, \quad H_l = \int_{V_l} (A \cdot B) dv, \quad M_l = \int_{V_l} p^{1/\gamma} dv \]

Multi-volume, partially-relaxed energy principle

* A set of \( N \) nested toroidal surfaces enclose \( N \) annular volumes
  → the interfaces are assumed to be ideal, \( \delta B = \nabla \times (\delta \xi \times B) \)
* The multi-volume energy functional is

\[ F = \sum_{l=1}^{N} \left( W_l - \mu_l H_l / 2 - \nu_l M_l \right) \]

Euler-Lagrange equation for unconstrained variations in \( A \)

In each annulus, the magnetic field satisfies \( \nabla \times B_l = \mu_l B_l \)

Euler-Lagrange equation for variations in interface geometry

Across each interface, pressure jumps allowed, but total pressure is continuous \([ [p + B^2/2] ] = 0\)
→ an analysis of the force-balance condition is that the interfaces must have strongly irrational transform

→ field remains tangential to interfaces,
→ a finite number of ideal constraints, imposed topologically!

ideal interfaces coincide with KAM surfaces