Adjoint method and runaway electron dynamics in momentum space

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Outline

• Introduction to runaway electron dynamics in momentum space

• Adjoint Method
  • Runaway Probability function
  • Expected Loss time

• Large angle scattering in runaway electron energy loss.

• Summary
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Motivation: develop a theoretical tool to help understand RE momentum space structure

- Due to the decrease of the Coulomb collision force with $p$, electrons with momentum larger than $p_{\text{crit}}$ can be continuously accelerated by the toroidal electric field to very high energy.
- Runaway electron (RE) beam is considered to be causing severe damage in ITER disruption.
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  • Runaway-loss separatrix formed by $E$ force and collisional drag
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RE Kinetic Equation in Momentum Space

\[ \frac{\partial f}{\partial t} + E\{f\} + C\{f\} + R_s\{f\} + R_B\{f\} = S\{f\} \]

*E*: Parallel electric field acceleration  
*C*: Relativistic collision operator (slowing-down and pitch angle scattering)  
*R_s*: Synchrotron radiation reaction force (SRRF)  
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**Notes**:
- The kinetic equation is a 2-D Fokker-Planck equation (ignoring the source term).
- Diffusion term mainly comes from pitch angle scattering term.

\[
\frac{\partial f}{\partial t} = \hat{L}[f] = -\nabla \cdot [af] + \nabla \nabla \cdot [Df]
\]
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- Kinetic equation can be written as a conservation form
  \[ \frac{\partial f}{\partial t} + \nabla \cdot J = 0 \quad J = a(p) f - \nabla \cdot [D(p) f] \]
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    \frac{\partial f}{\partial t} + \frac{\partial J}{\partial p} = 0 \quad U = a(p) f - \frac{\partial}{\partial p} [D(p)f]
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Adjoint method I: Runaway Probability Function

$P$ is solution of adjoint Fokker-Planck equation.

\[ \hat{L}^\dagger [P] = a(p) \frac{\partial P}{\partial p} + D(p) \frac{\partial^2 P}{\partial p^2} = 0 \]

$P(p_1) = 0, P(p_2) = 1$

Adjoint method I: Runaway Probability Function

\( F \) is the Green’s function of the Fokker-Planck operator \( L \).

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} - \hat{L}[F] = \delta(p - p_0)
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\[
\int_{p_1}^{p_2} \hat{L}[F]P\,dp = \left[ PU + D \frac{\partial P}{\partial p} F \right]_{p_1}^{p_2} + \int_{p_1}^{p_2} F \hat{L}^*[P]\,dp
\]

\( P(p = p_0) = J \bigg|_{p = p_2} \) 

\( P \) characterize the probability for electron to eventually reach boundary \( p=p_2 \).

Runaway Probability Function for $Z=1$

\[
a(p) \frac{dP(p)}{dx} + D(p) \frac{d^2P(p)}{dp^2} = 0 \quad P|_{p_1} = 0 \quad P|_{p_2} = 1
\]

- $P$ gives probability for electron to reach high momentum boundary
- Result of $P$ shows smooth transition near separatrix
  - The test-particle method (relying on truncation of pitch angle scattering) only gives a line of separatrix, equivalent to a Heaviside $P$ function.

\[
\frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial f}{\partial \xi} = \frac{\partial}{\partial \xi} (\xi f) + \frac{\partial^2}{\partial \xi^2} \left( \frac{1 - \xi^2}{2} f \right)
\]

Runaway probability function

$P$ at $\theta=0$ near separatrix

$E/E_{CH}=6$
$Z=1$
$\tau_r/\tau=100$
($B=3T$, $n_e=10^{21}m^{-3}$)
Runaway Probability Function for $Z=1$

\[ a(p) \frac{dP(p)}{dx} + D(p) \frac{d^2P(p)}{dp^2} = 0 \quad P|_{p_1} = 0 \quad P|_{p_2} = 1 \]

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- Result of $P$ shows smooth transition near separatrix
  - The test-particle method (relying on truncation of pitch angle scattering) only gives a line of separatrix, equivalent to a Heaviside $P$ function.
- Results agree well with Monte-Carlo Simulation
Runaway Probability Function for $Z=7$

\[
a(p) \frac{dP(p)}{dx} + D(p) \frac{d^2P(p)}{dp^2} = 0 \quad P|_{p_1} = 0 \quad P|_{p_2} = 1
\]

- Separatrix location and width of transition region both increase with pitch angle scattering ($Z$).
- Transition region is asymmetric at two sides of separatrix.
Use Runaway Probability to Calculate the Avalanche Growth Rate

\[ \frac{df}{dt} + E\{f\} + C\{f\} + R_S\{f\} + R_B\{f\} = S\{f\} \]

\[ \gamma_A = \int dp \frac{S\{f\} \cdot P}{n_{RE}} \]

- Calculated \( \gamma_A \) agrees well with CODE simulation result.
Use Runaway Probability to Calculate the Avalanche Growth Rate

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- For tokamak disruption, \( P \) can be used to estimate the number of seed RE in thermal quench.
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Adjjoint Method II: Expected Loss Time (ELT)

$T$ is solution of nonhomogeneous adjoint Fokker-Planck equation.

\[ \hat{L}^*[T] = a(p) \frac{\partial T}{\partial p} + D(p) \frac{\partial^2 T}{\partial p^2} = -1 \]

$T(p_1) = 0, T(p_2) = 0$

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$F$ is the Green’s function of the Fokker-Planck operator $L$.

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$S = \delta(p - p_0)$

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\int_{p_1}^{p_2} \hat{L}[F]T \, dp = \left[ TU + D \frac{\partial T}{\partial p} F \right]_{p_1}^{p_2} + \int_{p_1}^{p_2} F\hat{L}^\dagger[T] \, dp
$$

$T$ characterize the expected loss time, which is the expected time for an electron to reach the boundary.


Expected Loss Time for Runaway Electron Decay

\[ a(p) \frac{dT(p)}{dp} + D(p) \frac{d^2T(p)}{dp^2} = -1 \quad T\bigg|_{p_1,p_2} = 0 \]

- \( 1/T = 1/T_S + 1/T_R \), \( T_S \) (slowing-down time) and \( T_R \) (runaway time) are expected time to reach low/high energy boundary.
- For \( E<E_0 \), all electrons will end up in low energy boundary. \( T_S \) represents the timescale for runaway electron energy decay.
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- For \(E < E_0\), all electrons will end up in low energy boundary. \(T_S\) represents the timescale for runaway electron energy decay.
- In the marginal case (\(E\) is close to \(E_0\)), \(T\) has a big jump near the separatrix.
  - \(E\) field force can form a potential barrier near the separatrix that hinder particle losing energy.

\[ E = 1.5E_{CH} \quad Z=1 \]
Expected Loss Time for Runaway Electron Decay

- \(1/T = 1/T_S + 1/T_R\), \(T_S\) and \(T_R\) (the subscript represents the timescale for runaway electron energy decay).

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\[
\int \left( \frac{1}{T} \right) \, dp = -1, \quad p_1 = 0, \quad p_2 = 1.5E_{CH}Z = 1
\]
Expected Loss Time including Secondary RE Generation

\[ \frac{dF}{dt} = \frac{\partial F}{\partial t} - \hat{L}[f] = \delta(p - p_0) + S\{F\} \]

\[ S\{F\} = \int dq \sigma(p,q)F(q) \]

\[ \hat{L}^*[T] = -1 + \int dq \sigma(q,p)T(q) \]
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- \( T \to \infty \) when RE growth rate (with avalanche) is positive.

Expected loss time for \( \tau / \tau = 2 \) \text{ and } \text{Z} = 1
Expected Loss Time including Secondary RE Generation

\[
\frac{dF}{dt} = \frac{\partial F}{\partial t} - \hat{\mathcal{L}}[f] = \delta(p - p_0) + S\{F\}
\]

\[
S\{F\} = \int dq \sigma(p, q) F(q)
\]

\[
\hat{\mathcal{L}}^+[T] = -1 + \int dq \sigma(q, p) T(q)
\]

- \( T \to \infty \) when RE growth rate (with avalanche) is positive.

Expected loss time for \( \tau_r/\tau = 2 \) \( Z=1 \)

Critical electric field for avalanche

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Large Angle Scattering in Nonhomogeneous Momentum Space

- Like runaway electron avalanche where electrons gain a large amount of energy through large angle scattering (LAS), electrons can also lose a large fraction of energy through LAS.
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- For collisional energy loss, the contribution of LAS is $1/\ln \Lambda$ of the accumulation of small angle scattering.

\[
\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p}\left[\frac{\langle \Delta p \rangle}{\Delta t}\right]_c f + \frac{\partial^2}{\partial p^2}\left[\frac{\langle \Delta p \Delta p \rangle}{\Delta t}\right]_c f + C_L[f]
\]

No electric field

\[
\begin{align*}
1 & \quad 1/\ln \Lambda + v_{th}^2 / v_{test}^2 & \quad 1/\ln \Lambda
\end{align*}
\]

Large angle scattering
Large Angle Scattering in Nonhomogeneous Momentum Space

- Like runaway electron avalanche where electrons gain a large amount of energy through large angle scattering (LAS), electrons can also lose a large fraction of energy through LAS.
- For collisional energy loss, the contribution of LAS is $1/\ln \Lambda$ of the accumulation of small angle scattering.
- In nonhomogeneous momentum space, $E$ field balances collisional drag near the separatrix, thus LAS can be more important.

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p} \left[ \left\langle \frac{\Delta p}{\Delta t} \right\rangle_c f \right] + \frac{\partial^2}{\partial p^2} \left[ \left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle_c f \right] + C_L[f]$$

$E$ field balance collisional drag

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial p} \left[ eE + \left\langle \frac{\Delta p}{\Delta t} \right\rangle_c f \right] + \frac{\partial^2}{\partial p^2} \left[ \left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle_c f \right] + C_L[f]$$

No electric field

$1$ $1/\ln \Lambda + v_{th}^2/v_{test}^2$ $1/\ln \Lambda$ Large angle scattering
Expected Loss Time in Nonhomogeneous Momentum Space

- Large angle collision is important for electron energy loss when $E$ is close to $E_0$ (marginal case).

![Graph showing Expected Loss Time in Nonhomogeneous Momentum Space](image)
Expected Loss Time in Nonhomogeneous Momentum Space

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- Results of expected loss time shows that large angle collisions help electrons overpass the potential barrier, therefore significantly reduce the jump of $T$ at marginal case.
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Summary

• Adjoint method gives a new angle to study the nonhomogeneous momentum space of runaway electrons.

• Both runaway probability ($P$) and expected loss time ($T$) are derived from the adjoint method.

• For marginal case ($E$ close to $E_0$), large angle scattering (LAS) plays an important role in energy decaying of existing RE population.

• The adjoint method can also be applied to other dynamical systems.
Thanks!