General Theory of the Plasmoid Instability

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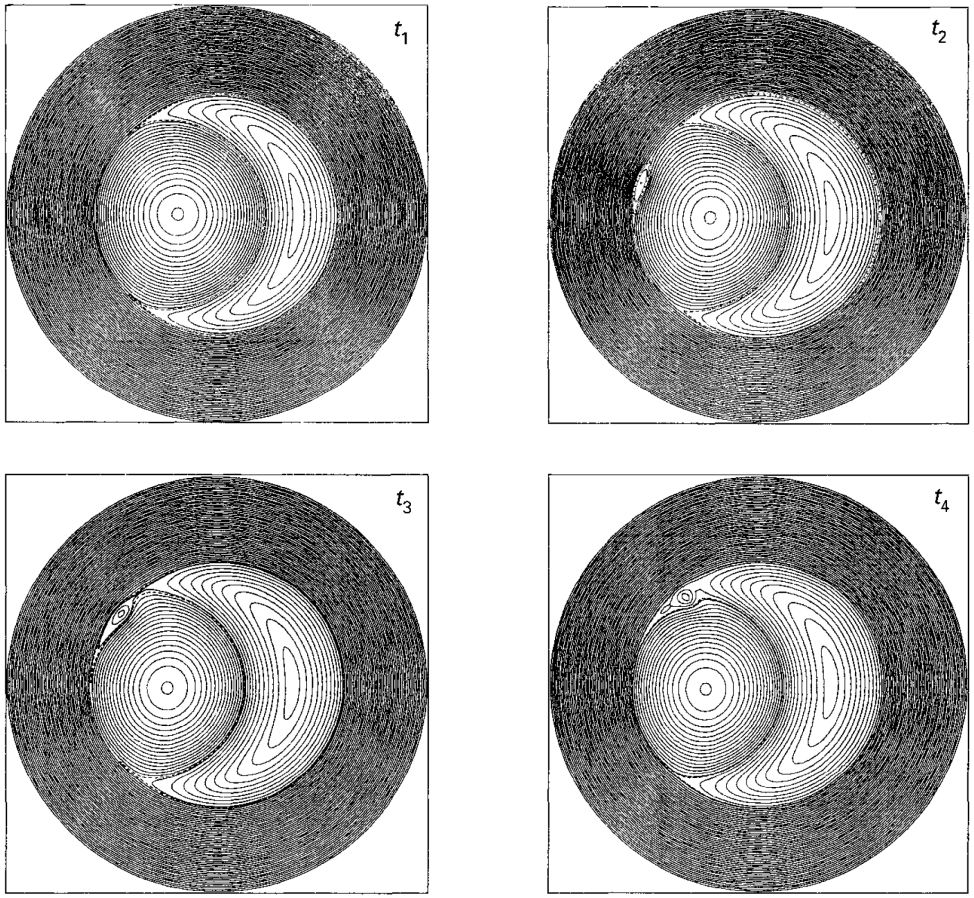
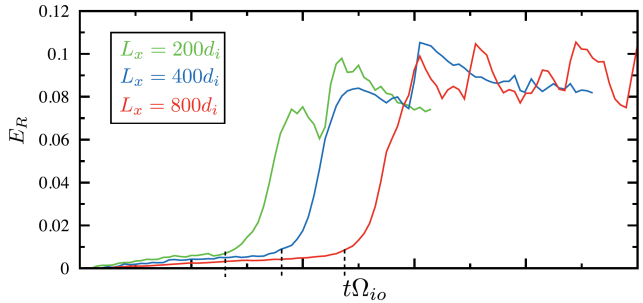
Plasmoid Instability and its importance in reconnection

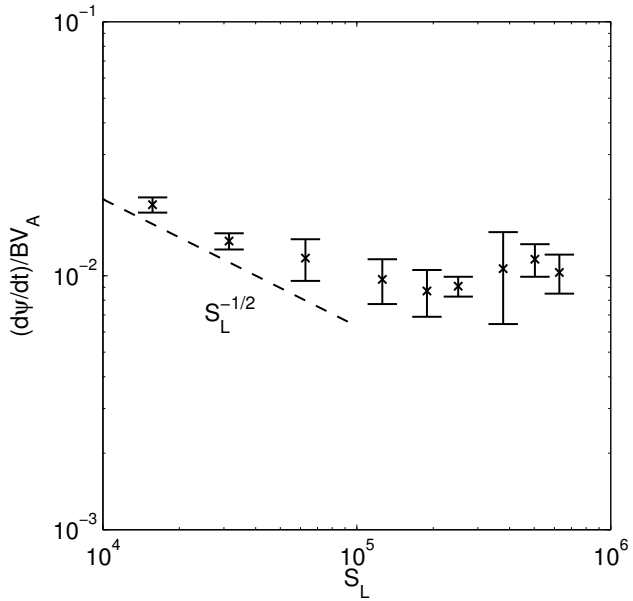
- Elongated current sheets are subject to plasmoid instability
- The plasmoid instability leads to fast reconnection!



Biskamp, PoF 1986



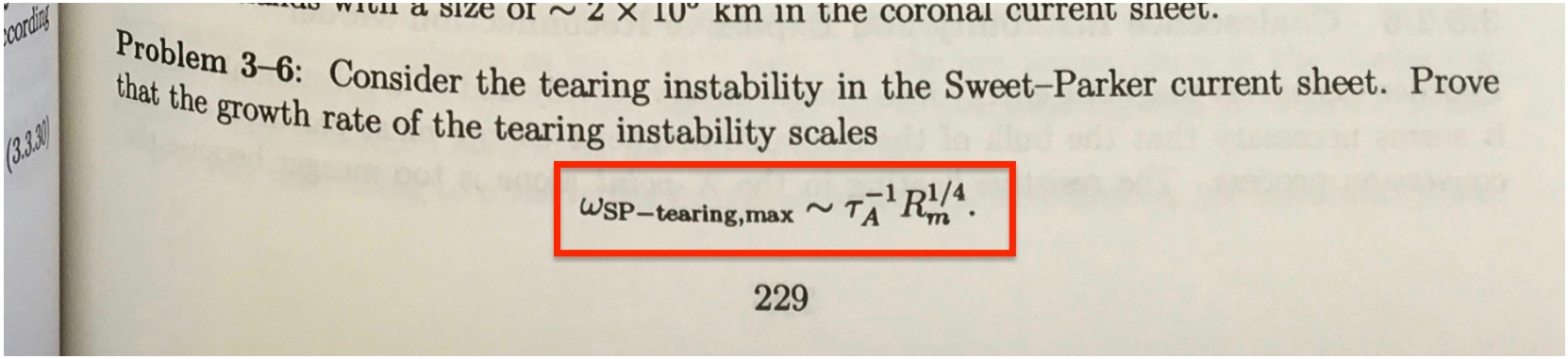
Daughton *et al.*, PRL 2009



Bhattacharjee *et al.*, PoP 2009

Assuming a Sweet-Parker aspect ratio...

- Tajima & Shibata, *Plasma Astrophysics* (1997)



with a size of $\sim 2 \times 10^5$ km in the coronal current sheet.

Problem 3–6: Consider the tearing instability in the Sweet–Parker current sheet. Prove that the growth rate of the tearing instability scales

$$\omega_{\text{SP-tearing,max}} \sim \tau_A^{-1} R_m^{1/4}.$$

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Here $\tau_A = L/V_A$ and $R_m = LV_A/\eta$. L is the length of the diffusion region (current sheet). (Hint: Note the relation $R_{m,*} = (a/L)R_m$ and $a/L = R_m^{-1/2}$ from Sweet–Parker theory.) Note that the growth rate *increases* with increasing the magnetic Reynolds number. Prove also that

$$\frac{\lambda}{L} = 2\pi R_m^{-3/8}.$$

Thus when $R_m \sim 10^{14}$, we find $\omega_{\text{SP-tearing,max}} \tau_A \sim 10^{-3.5}$. If we apply this result to the solar corona $L \sim 10^4$ km and $V_A \sim 10000$ km/s, then we find $\tau_{\text{SP-tearing,max}} \sim 3 \times 10^{-4}$ sec, and $\lambda \sim 0.5$ km. That is, we have many small magnetic islands in the long current sheet.

(Note that the same result has been re-obtained 10 years later by Loureiro *et al.*, PoP 2007)

Assuming a Sweet-Parker aspect ratio...

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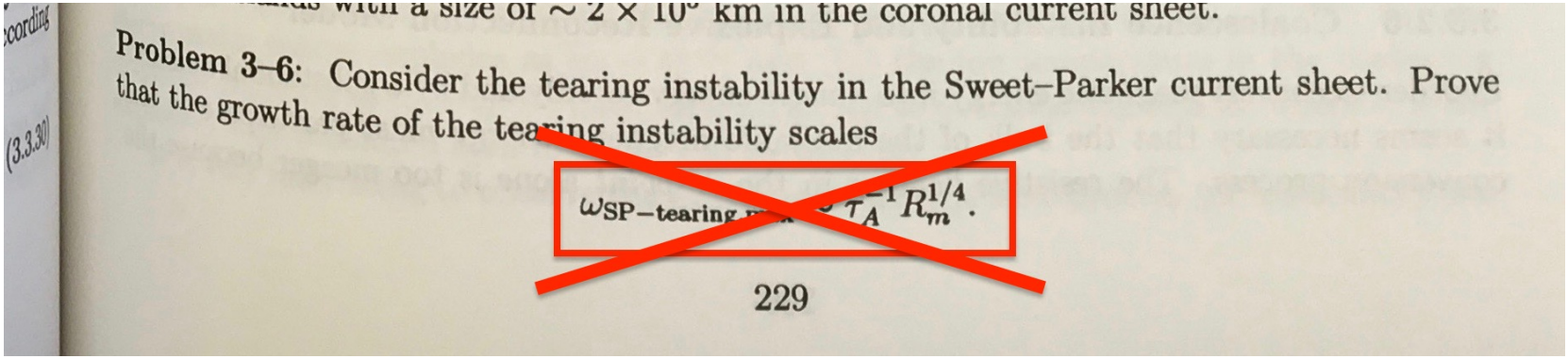

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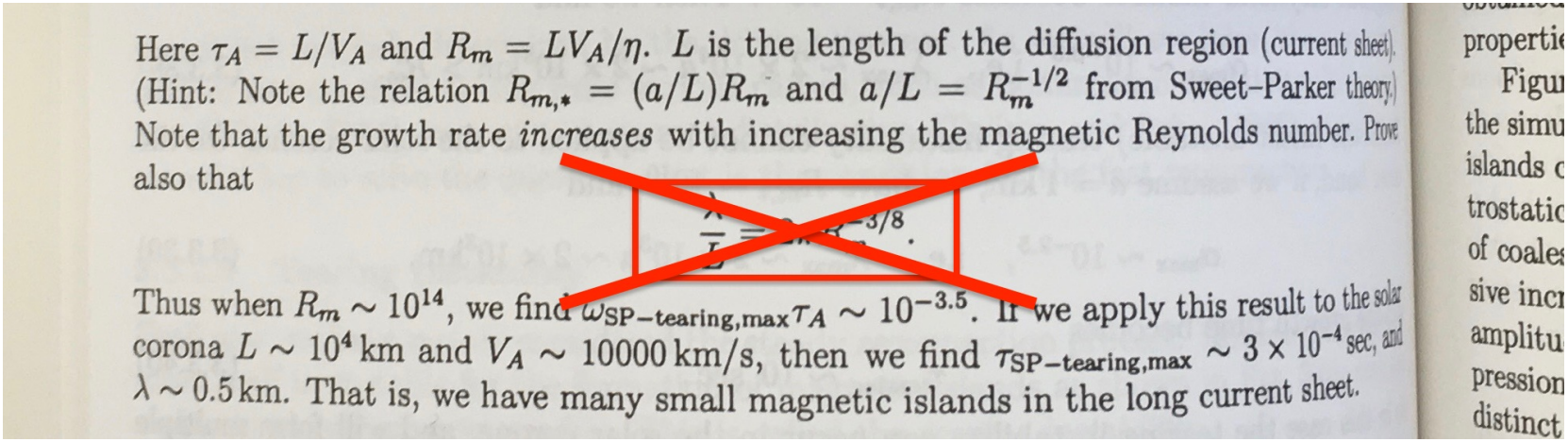

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Assuming a Sweet-Parker aspect ratio with viscosity...

- Plasma viscosity modifies the previous scalings as (Comisso & Grasso, PoP 2016)

$$\gamma\tau_A \sim S^{1/4}(1+P_m)^{-5/8}\,, \tag{1}$$

$$kL \sim S^{3/8}(1+P_m)^{-3/16}\,. \tag{2}$$

- The growth rate γ decreases for increasing $P_m = \nu/\eta$, but... for realistic values of perpendicular P_m $(10^{-4} \div 10^4)$ and $S >>>> 1$, the linear growth of the instability is still extremely fast (super-Alfvénic).

- The chosen equilibrium current sheet (Sweet-Parker sheet) can not be attained as current layers are unstable and disrupt before the equilibrium is achieved.

Need for a New Theory... [Comisso *et al.* (2016)]

- In a general (time-evolving) current sheet, tearing modes become unstable at different times and exhibit different instantaneous growth rates $\gamma(k,t)$.
- Their amplitude changes in time according to

$$\psi(k,t) = \psi_0 \exp\left(\int_{t_0}^{t} \gamma(t')dt'\right). \tag{3}$$

- Their linear evolution ends when the plasmoid half-width

$$w(k,t) = 2(\psi a/B)^{1/2} \tag{4}$$

grows to the same order of the inner resistive layer width

$$\delta_{\rm in}(k,t) = \left[\eta\gamma a^2/(kv_A)^2\right]^{1/4}. \tag{5}$$

[1] Other recent theories: Pucci, Velli (2014); Uzdensky, Loureiro (2016)

Need for a New Theory... [Comisso *et al.* (2016)]

- In such a complex scenario, we need to determine:
  - the mode (k_*) that emerges from the linear phase and transitions into the nonlinear phase
  - the time (t_*) at which this transition occurs

- Principle of Least Time for the Plasmoid Instability, i.e., *the mode of the plasmoid instability that emerges from the linear phase is the one that traverses it in the least time.*

- Mathematically, to implement our formulation, we introduce the function

$$F(k,t) := \delta_{\rm in}(k,t) - w(k,t)\,. \tag{6}$$

- Then, the least time principle yields the relations

$$F(k,t)|_{k_*,t_*} = 0\,, \qquad dt_*/dk_* = 0\,. \tag{7}$$

[2] A phenomenological model that takes into account the contributions of different modes will be presented by Yi-Min Huang this afternoon.

Need for a New Theory... [Comisso *et al.* (2016)]

- From these relations we can obtain the *least time plasmoid equation*

$$\left\{\left(\gamma\bar{t}-\frac{1}{2}\right)\frac{\partial\gamma}{\partial k}+\frac{\gamma}{k}+2\frac{\gamma}{w_0}\frac{\partial w_0}{\partial k}\right\}\Bigg|_{k_*,t_*}=0 \qquad (8)$$

and

$$\left\{\ln\left(\frac{\delta_{\rm in}}{w_0}\frac{g^{1/2}}{f^{1/2}}\right)-\frac{1}{2}\int_{t_0}^{t}\gamma(t')dt'\right\}\Bigg|_{k_*,t_*}=0 \qquad (9)$$

where $f=a(t)/a_0$, $g=B(t)/B_0$, $\bar{t}=\partial/\partial\gamma\int_{t_0}^{t}\gamma(t')dt'$, and $w_0=2\sqrt{\psi_0 a_0/B_0}$.

- From these equations it is possible to arrive at

γ_*	(final growth rate)	L_*/a_*	(final aspect ratio)
k_*	(final wavenumber)	t_*	(elapsed time from t_0)
$\delta_{\rm in*}$	(final inner layer)	τ_p	(time from $\hat{\gamma}(\hat{k}_*,\hat{t}_{\rm on})>1/\tau$)

Need for a New Theory... [Comisso *et al.* (2016)]

- For convenience we now use dimensionless quantities:
  - lengths normalized by the current sheet half-length L
  - magnetic field normalized by the upstream field B_0
  - time normalized by the Alfvén time $\tau_A = L/v_A$
- Although our framework is altogether general, we are interested in the case where
  - L and B_0 remain approximately constant
  - The current sheet width decreases in time via $\hat{a}(\hat{t}) = \hat{a}_0 f(\hat{t})$
- For the moment we consider an exponentially thinning current sheet (this will be generalized later) of the form

$$\hat{a}(\hat{t})^2 = (\hat{a}_0^2 - \hat{a}_\infty^2) e^{-2\hat{t}/\tau} + \hat{a}_\infty^2 \,, \qquad \text{with } \hat{a}_\infty = S^{-1/2}$$

- Note that the plasmoid width $w(k,t) = 2\sqrt{\psi a/B_0}$ starts to grow only when $\hat{\gamma} > 1/\tau$.

Need for a New Theory... [Comisso *et al.* (2016)]

- For $S < S_T$: where $S_T = \frac{1}{c_a^6}\left[\frac{4}{3\tau}\, W\left(\frac{3}{4}\frac{c_a^{3/2}\hat{a}_0^{3/8}\tau^{9/8}}{\hat{w}_0^{3/4}}\right)\right]^4$

$$\hat{\gamma}_* \simeq c_\gamma S^{1/4}\,, \quad \hat{k}_* \simeq c_k S^{3/8}\,, \quad \hat{\delta}_{\mathrm{in}*} \simeq c_\delta S^{-5/8}\,. \tag{10}$$

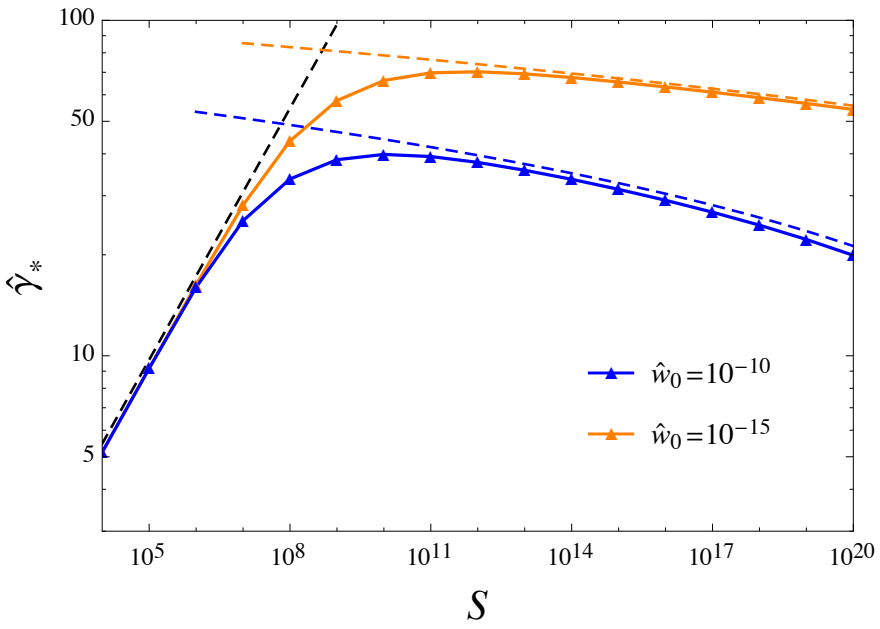
- For $S > S_T$:

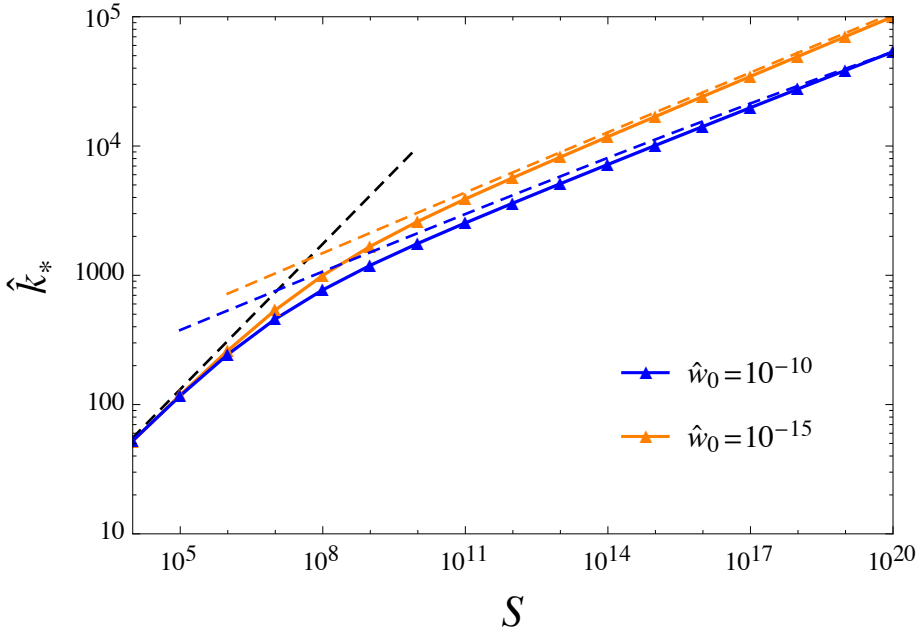
$$\hat{k}_* \simeq c_k c_a^{-5/4}\frac{S^{1/6}}{\tau^{5/6}}\left[\ln\left(\frac{\tau^{1/6}}{S^{1/3}}\frac{\hat{a}_0^{1/2}}{\hat{w}_0}\right)\right]^{5/6}, \tag{11}$$

$$\hat{\gamma}_* \simeq c_\gamma c_a^{-3/2}\frac{1}{\tau}\ln\left(\frac{\tau^{1/6}}{S^{1/3}}\frac{\hat{a}_0^{1/2}}{\hat{w}_0}\right), \tag{12}$$

$$\hat{\delta}_{\mathrm{in}*} \simeq c_\delta c_a^{3/4}\left(\frac{\tau}{S}\right)^{1/2}\left[\ln\left(\frac{\tau^{1/6}}{S^{1/3}}\frac{\hat{a}_0^{1/2}}{\hat{w}_0}\right)\right]^{-1/2}. \tag{13}$$

Need for a New Theory... [Comisso *et al.* (2016)]





$\hat{\gamma}_*$ and $\hat{k}_*$ vs. S for $\hat{w}_0 = 10^{-10}$ and $\hat{w}_0 = 10^{-15}$ $[\tau = 1,\ \hat{a}_0 = \frac{1}{\pi}]$.

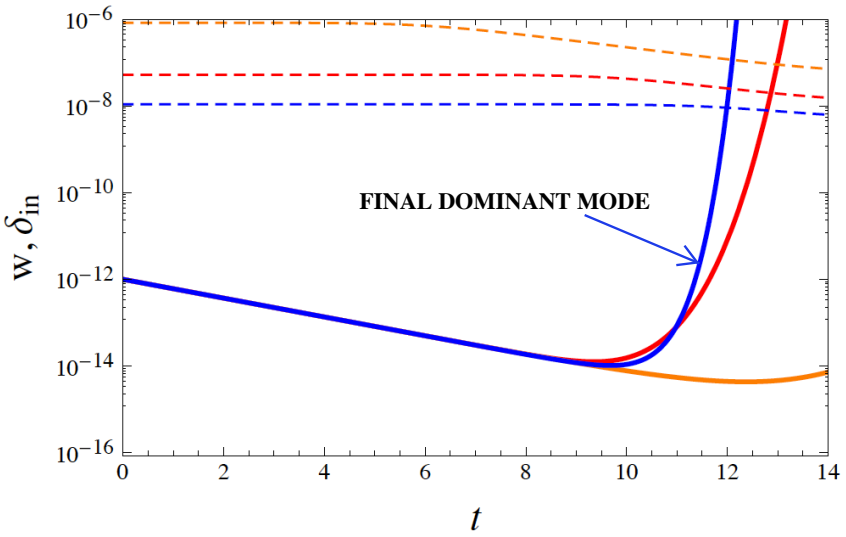
- The black dashed lines represent the earlier scalings, which are not applicable for large-S plasmas.
- The solid curves represent the results obtained with the principle of least time for the plasmoids.

(Stay tuned for more details in a forthcoming Comisso *et al.* (2017))

Need for a New Theory... [Comisso *et al.* (2016)]

- The total time scale of the plasmoid instability is

$$\hat{t}_* \simeq \tau \ln \left\{ \frac{\hat{a}_0}{c_a} \frac{S^{1/3}}{\tau^{2/3}} \left[\ln \left(\frac{\tau^{1/6}}{S^{1/3}} \frac{\hat{a}_0^{1/2}}{\hat{w}_0} \right) \right]^{2/3} \right\} . \qquad (14)$$



The emergent mode effectively starts growing when

$$\hat{\gamma}(\hat{k}_*, \hat{t}_{\rm on}) > 1/\tau$$

(Occurs when $\hat{a} < \hat{k}_* S^{-1/2} \tau^{3/2}$)

- This implies that the *intrinsic* time scale of the instability is

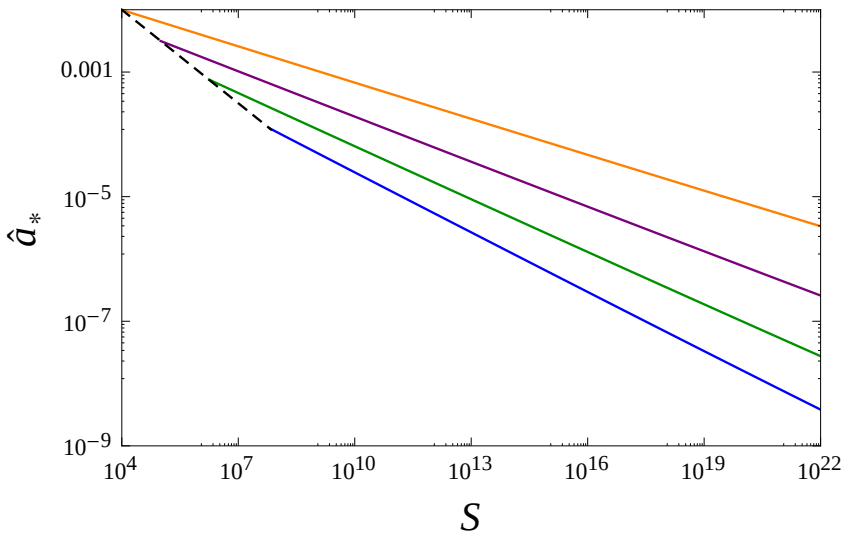
$$\tau_p \simeq \tau \ln \left\{ \frac{c_k}{c_a^{9/4}} \left[\ln \left(\frac{\tau^{1/6}}{S^{1/3}} \frac{\hat{a}_0^{1/2}}{\hat{w}_0} \right) \right]^{3/2} \right\} . \qquad (15)$$

Need for a New Theory... [Comisso *et al.* (2016)]

- To generalize the previous scaling laws, we consider the generalized current thinning function

$$\hat{a}(\hat{t})^2 = (\hat{a}_0^2 - \hat{a}_\infty^2)\left(\frac{\tau}{\tau + 2\hat{t}/\chi}\right)^\chi + \hat{a}_\infty^2 \,, \qquad \text{with } \hat{a}_\infty = S^{-1/2}$$

- The final inverse-aspect-ratio depends on thinning process:



$$\hat{a}_* \sim \hat{a}_0^{\frac{2\zeta}{\chi}} \frac{\tau^\zeta}{S^{\frac{\zeta}{2}}} \left[\ln\left(\frac{\tau^{\frac{\zeta}{4}}}{S^{\frac{2+\zeta}{8}}} \frac{\hat{a}_0^{\frac{\zeta}{2\chi}}}{2\hat{\psi}_0^{\frac{1}{2}}}\right)\right]^{-\zeta}$$

where $\zeta := \frac{2\chi}{4+3\chi}$.

- Different colors refer to $\chi = 2$, $\chi = 4$, $\chi = 10$, $\chi \to \infty$ $[\tau = 1,\ \hat{w}_0 = 10^{-12},\ \hat{a}_0 = 1/2\pi]$.

Need for a New Theory... [Comisso *et al.* (2016)]

- The scaling laws of the plasmoid instability at large S depend on the thinning process:

$$\hat{\gamma}_* \sim S^{(3\zeta-2)/4} \left(\frac{\ln(\theta_R)}{\hat{a}_0^{2/\chi_\tau}} \right)^{3\zeta/2} , \tag{16}$$

$$\hat{k}_* \sim S^{(5\zeta-2)/8} \left(\frac{\ln(\theta_R)}{\hat{a}_0^{2/\chi_\tau}} \right)^{5\zeta/4} , \tag{17}$$

$$\hat{\delta}_{\mathrm{in}*} \sim S^{-(3\zeta+2)/8} \left(\frac{\hat{a}_0^{2/\chi_\tau}}{\ln(\theta_R)} \right)^{3\zeta/4} , \tag{18}$$

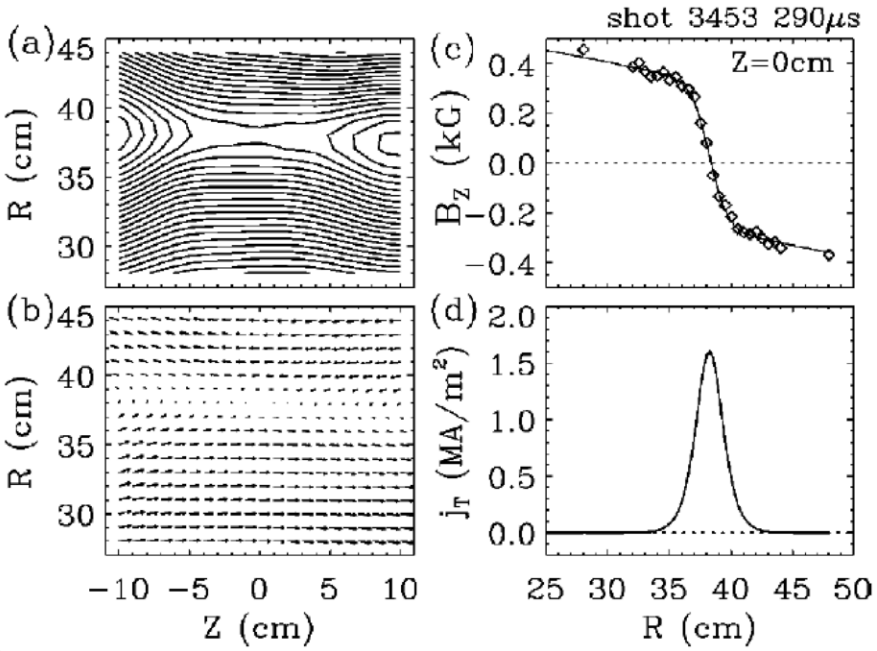
where

$$\theta_R := \frac{\tau^{\zeta/4}}{S^{(2+\zeta)/8}} \frac{\hat{a}_0^{\zeta/2\chi}}{2\hat{\psi}_0^{1/2}} . \tag{19}$$

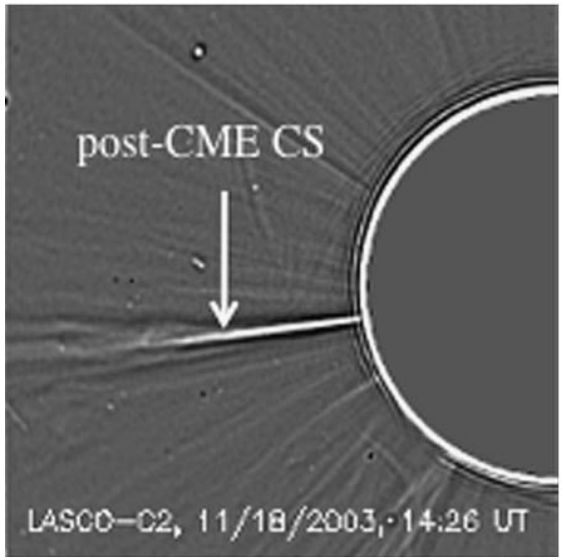
Connections with Experiments and Observations

The plasmoid instability occurs in myriad of plasma systems. Plenty of opportunities to check the theory!

- The "easiest" quantities to check should be $\hat{a}_*$ and $\hat{t}_*$.
- Next, depending on the diagnostics, it could be possible to check the number of plasmoids ($\sim \hat{k}_*$) during the very early nonlinear phase.



Ji *et al.*, PoP (1999)



Lin *et al.*, SSRv (2015)