Energetic particle effects on FRC stability

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Fast ion effects in prolate FRC with $S^* \gg 1$ and $E \gg 1$

Initial studies on fast ions: use beams to stabilize the $n=1$ tilt mode (internal)
- large $S^*$, MHD-like FRC (confinement)
- large $E$, for reduced $n=1$ tilt growth rate
- $V_b \gg V_A$, $n_b \ll n_e$ but $J_b \sim J_{pl}$

• Generalized energy principle [Lovelace’75, Finn and Sudan’93] calculated
  bulk plasma-beam interaction term: $I_b \sim 1/[n^2 \Omega^2 - \omega_z^2]$,
  where $\Omega$ is average toroidal rotation frequency of beam ions,
  $\omega_z$ is axial betatron frequency (assuming $\Omega \gg \omega$, rigid-tilt mode structure, and $f_b = f_b(\epsilon - \Omega \rho_\phi)$)
- The beam ion contribution is stabilizing $I_b > 0$ provided: $n \Omega > \omega_z$

• 3D simulations [D. Barnes, R. Milroy’91] found the stabilization of the tilt
  instability for prolate FRC with $n_b \sim 1$-2% and $V_0 \sim 10V_A$; destabilization of the
  $n=4$ kink mode has been found for cold beams [Nishimura`99].
Problems with beam stabilization of tilt in MHD

- Need cold super-Alfvenic beams to stabilize tilt in MHD regime when\[ \gamma_{\text{tilt}} \sim \gamma_{\text{MHD}} \sim V_A/Z_s = O(1) \]

- Cold (\( E_b \gg T_b \)) beams can de-stabilize other modes in MHD:
  - in addition to \( n=1 \) tilt, there are 3 types of global modes unstable for each \( n \),
  - if stability condition \( n \Omega - \omega > \omega_Z \) is satisfied for one mode (\( n=1 \)), other modes will be de-stabilized by the beam,
  - a more general resonant condition: \( n \Omega - \omega = l \omega_Z \) (assuming \( E \gg 1 \) and \( n \Omega \sim \omega_Z \ll \omega_R \))
  - modes with different symmetry relative to midplane (i.e. axial vs radial modes) have different resonances (\( l = \text{odd/even} \)) → different stability conditions.

- Experimental (prolate) FRCs have \( S^* \lesssim 10 \)
  - no need to worry about high-\( n \) modes (thermal ion FLR stabilization)
  - but have \( \omega_Z \ll n \Omega \sim \omega_R \), so need to account for radial beam ion motion
    \[ n \Omega - \omega = l \omega_Z + m \omega_R \]
Generalized energy principle ($S^* \gg 1$): $\omega_Z, \Omega << \omega_R$

In general, $I = \frac{1}{2} \int \delta J_b \cdot B_0 \times \xi^* \, d^3 x$ – self-ajoint provided $\omega << \Omega$ or $\text{Im}(\omega) << \text{Re}(\omega)$.

$$I = \frac{1}{2i\omega^*} \iint (v \cdot \delta E^*) \left[ - \frac{\partial f_0}{\partial \varepsilon} - \frac{n}{\omega} \frac{\partial f_0}{\partial p_\phi} \right] \text{Int}(t) \, d^3 v \, d^3 x - \frac{1}{2 |\omega|^2} \int R \left| \delta E_\phi \right|^2 \int v_\phi \frac{\partial f_0}{\partial p_\phi} \, d^3 v \, d^3 x$$

where $\text{Int}(t) = \int (v \cdot \delta E) \, dt'$. First term is responsible for resonant interactions, and the resonance condition includes secondary resonances: $n\Omega - \omega = l\omega_Z + m\omega_R$.

For $f_0 \sim \exp(-(|\varepsilon - \Omega p_\phi|)/T)$, averaging over radial motion, and assuming modes with odd/even symmetry relative to midplane, it can be shown that [Belova, IAEA 2002]:

$$I_{\text{odd}} = \frac{A_1}{\Omega_n^2 - \omega_Z^2} + \frac{A_3}{\Omega_n^2 - 9\omega_Z^2} + \ldots$$

for odd modes (antisymmetric relative to the midplane, i.e. with tilt-like polarization)

$$I_{\text{even}} = \frac{A_2}{\Omega_n^2 - 4\omega_Z^2} + \frac{A_4}{\Omega_n^2 - 16\omega_Z^2} + \ldots$$

for even modes (symmetric, with radial polarization, radial shift)

where $\Omega_n = n\Omega - \omega$
Linear simulation results: n=1 tilt mode

$F(\Omega, \omega_\beta) – equilibrium\ distribution\ function\ in\ (\Omega, \omega_\beta)\ phase-space.$

$\Omega$ is beam ion toroidal rotation frequency, $\omega_\beta$ is axial betatron (bounce) frequency, normalized by the ion cyclotron frequency.

Most beam ions satisfy condition: $\Omega > \omega_\beta$, ie beam ion effect is stabilizing. For the n=1 tilt mode, the growth rate is reduced to $\gamma = 1.27 \frac{V_a}{Z_s}$ compared to MHD growth rate $\gamma = 1.47 \frac{V_a}{Z_s}$. 
Linear simulation results: n=2 mode

Dependence of the particle weight $w=\delta f/f$ on the frequency ratio $2\Omega/\omega_\beta$ for n=2 odd mode. The resonant behavior is seen at $2\Omega/\omega_\beta=3$. The beam effect is destabilizing.

Most beam ions satisfy condition: $2\Omega<3\omega_\beta$, ie beam ion effect is destabilizing. For the n=2 odd mode, the growth rate is increased to $\gamma=2V_a/Z_s$ compared to MHD growth rate $\gamma=1.5V_a/Z_s$. 
Linear simulation results: n=3 mode

Dependence of the particle weight $w = \delta f / f$ on the frequency ratio $3\Omega / \omega_{\beta}$ for n=3 odd mode.

Most beam ions satisfy condition: $3\Omega > 3\omega_{\beta}$, but $3\Omega < 5\omega_{\beta}$.

For the n=3 odd mode, the growth rate is increased to $\gamma = 2.4V_a/Z_s$ compared to MHD growth rate $\gamma = 1.55V_a/Z_s$. 
Generalized energy principle \( (S^* \lesssim 10) \): \( \omega_Z \ll \Omega \sim \omega_R \)

Radial resonances are more important for low \( S^* \) configurations (because \( V_b \gg v_{th} \) implies \( \Omega \sim \omega_R \sim \omega_{ci} \)), and for modes with even polarization (radial modes).

\[
I = \frac{R_0^2 \Omega^2}{2 |\omega|^2 T} \int n_b \left| \frac{\Omega_n}{\Omega^2 - \omega_R^2} \right|^2 \left| \frac{\partial \partial E_\phi}{\partial R} \right|^2 \delta R^2 d^3 x \sim \frac{1}{\Omega_n^2 - \omega_R^2}
\]

where resonant condition: \( n \Omega - \omega = \omega_R \).
Numerical results using HYM code

- One-fluid MHD for thermal plasma; full-f PIC for beam ions
  - beam injection with finite temperature, $V_b > V_{th}$
  - scaling with $S^*$
  - using monoenergetic beam with $F_b = \delta(\epsilon - \epsilon_0) \delta(p_\phi - p_{\phi 0})$

- Kinetic description for both thermal and beam ions
  - slowing down beam distribution with large $T_b$

- Oblate ($E \sim 1$) FRC + beam ions
HYM – Parallel Hybrid/MHD Code

HYM used to investigate FRC formation and stability properties; beam-driven sub-cyclotron Alfvén eigenmodes in NSTX(-U) and DIII-D

- 3-D nonlinear.
- Three different physical models:
  - Resistive MHD & Hall-MHD.
  - Hybrid (fluid electrons, particle ions).
  - MHD/particle (one fluid thermal plasma, + energetic particle ions)
- Full-orbit kinetic ions.
- For particles: delta-f / full-f numerical scheme.
- Parallel (3D domain decomposition, MPI)\(^1\).
- Self-consistent equilibria, including beam ion effects.

**Grids**

- **Field grid:** arbitrary orthogonal coordinate system (xyz, cylindrical, dipole, or nonuniform).
- **Particle grid:** xyz, uniform to avoid problems at R=0. Could be made same as field grid (incomplete).

**Boundary conditions**

- **Fields:** periodic, symmetry, \(\partial I/\partial n=0\), etc. easy to change – specified by set of integer numbers.
- **Particles:** periodic, reflection, or loss.

\(^1\)Simulations are performed at NERSC.
Hybrid scheme (fluid electrons, particle ions)

Electron-fluid and fields equations

\[ \mathbf{E} = -\mathbf{V}_e \times \mathbf{B} - \nabla p_e / n + \eta \mathbf{J} \]
\[ \mathbf{V}_e = -(\mathbf{J} - \mathbf{J}_i) / e n_e \]
\[ n_e = n_i \]
\[ \mathbf{B} = \mathbf{B}_0 + \nabla \times \mathbf{A} \]
\[ \partial \mathbf{A} / \partial t = -\mathbf{E} \]
\[ \mathbf{J} = \nabla \times \mathbf{B} \]
\[ \partial p_e^{1/\gamma} / \partial t = -\nabla \cdot (\mathbf{V}_e p_e^{1/\gamma}) \]

Kinetic ions – delta-F scheme:

\[ \frac{dx}{dt} = \mathbf{v} \]
\[ \frac{d\mathbf{v}}{dt} = \mathbf{E} - \eta \mathbf{j} + \mathbf{v} \times \mathbf{B} \]

\[ w = \delta F / F \quad \text{- particle weight} \]
\[ \frac{dw}{dt} = -(1 - w) \frac{d(\ln F_0)}{dt} \]
\[ F_0 = F_0(\varepsilon, \mu, p_\phi) \]
\[ \delta n_i(x) = \sum_m w_m S(x_m - x) \]

Where \( \varepsilon = m_i v^2 / 2 + e \phi \), \( p_\phi = m_i R v_\phi \) - \( e \psi \) is the azimuthal angular momentum, \( \psi \) is the poloidal flux, and \( \phi \) is the electrostatic potential.

Equilibrium distribution function is assumed to be known, so the zero-order ion density \( n_{i0} \) and current \( J_{i0} \) are calculated analytically.
Self-consistent MHD + fast ions coupling scheme

Background plasma - fluid:

\[ \rho \frac{dV}{dt} = -\nabla p + (j - j_i) \times B - n_i (E - \eta j) \]

\[ E = -V \times B + \eta j \]

\[ B = B_0 + \nabla \times A \]

\[ \frac{\partial A}{\partial t} = -E \]

\[ j = \nabla \times B \]

\[ \frac{\partial p^{1/\gamma}}{\partial t} = -\nabla \cdot (V p^{1/\gamma}) \]

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (V \rho) \]

Fast ions – delta-F scheme:

\[ \frac{dx}{dt} = v \]

\[ \frac{dv}{dt} = E - \eta j + v \times B \]

\[ w = \delta F / F \] - particle weight

\[ \frac{dw}{dt} = -(1 - w) \frac{d(ln F_0)}{dt} \]

\[ F_0 = F_0 (\varepsilon, \mu, p_\phi) \]

\( \rho, V \) and \( p \) are bulk plasma density, velocity and pressure, \( n_i \) and \( j \) are fast ion density and current, \( n_i \ll n \) – is assumed.
**Numerical model:** one-fluid MHD for thermal plasma; full-f PIC for beam ions.

Initial FRC configuration: \( E = 3-4, \ S^* = 9.5, \ x_s = 0.7. \)

Beam parameters: \( V_0 = 3.5-5 \ V_A, \ v_{th} = 0.5-2 \ V_A, \ n_b = 1-2\% \ n_0. \)
3D stability simulations: n=3 is beam-destabilized

- Beam ions are loaded with $V_0 = 3.5v_A$ and $v_{th} = 0.5v_A$ at $t = 0 - 4t_A$.
- n=3 is the most unstable mode with $\gamma = 4 V_A/Z_s$.
- More unstable than n=3 in MHD: $\gamma_{mhd} = 1.7 V_A/Z_s$.
- n=3 is a beam-driven mode.

Ratio of radial orbit frequency $\omega_R$ to toroidal frequency $\Omega$ is $\omega_R / \Omega \sim 2.5-3$, so that resonant condition is approximately satisfied: $n\Omega \approx \omega_R$ for the n=3 mode.
3D stability simulations (larger $V_{\text{beam}}$)

Beam parameters: $V_0 = 5.6 V_A$, $v_{th} = 1.5 V_A$, $n_b = 1\% n_0$.

For these beam parameters, the $n=1$ tilt mode and $n=3$ modes are completely stabilized, but the $n=2$ rotational mode is still unstable with $\gamma = \gamma_{\text{mhd}}(n=2) - \text{beam has little effect on } n=2 \text{ mode.}$

$\omega_R / \Omega \sim 2$, so that resonant condition is satisfied: $n\Omega \approx \omega_R$ for the $n=2$ mode.
Dependence of betatron frequencies on $S^*$ and $\Omega_{\text{tor}}$

$S^*=20$, $\Omega_0=0.8$, $n_b/n_0=0.013$, $J_b/J_0=0.7$

The ratio of radial orbit frequency $\omega_R$ to toroidal frequency $\Omega$ is $\omega_R/\Omega \sim 2$, and the $n=2$ mode is the most unstable mode.
Dependence of betatron frequencies on $S^*$ and $\Omega_{\text{tor}}$

$S^*=20$, $\Omega_0=0.2$, $n_b/n_0=0.05$, $J_b/J_0=0.7$

\[ \omega_Z : \Omega : \omega_R = 1:1:4 \]

Ratio of radial orbit frequency $\omega_R$ to toroidal frequency $\Omega$ is $\omega_R / \Omega \sim 4$, so that resonant condition is satisfied: $n\Omega \approx \omega_R$ for the $n=4$ mode.
Simulations with beam $F_b=\delta(\varepsilon-\varepsilon_0)\delta(p_\phi-p_\phi0)$

FRC parameters: $S^*=9$, $E\sim3$; beam parameters: $V_0=5V_A$, $v^*=4.5\ V_A$, $n_{beam}<1\%\ n_0$.

Equilibrium is calculated and beam ions are loaded with delta-function distribution function: $F_b=\delta(\varepsilon-\varepsilon_0)\delta(p_\phi-p_\phi0)$, where $\varepsilon_0=v_0^2/2$ and $p_\phi0=R_0v^*-\psi_0$, $v^*<v_0$.

$$n_b = n_{b0}(R_0/R), \quad |\psi + p_\phi|<Rv_0,$$
$$J_b = n_{b0}(R_0/R^2)(\psi + p_\phi0),$$

Beam parameters were chosen to satisfy resonant condition: $n\Omega \approx \omega_R + \omega_z$ for $n=2$ and $n=3$ modes.

Beam ions with $F_b=\delta(\varepsilon-\varepsilon_0)\delta(p_\phi-p_\phi0)$, have very narrow distribution in orbit frequencies.

For given beam parameters:
$$<\Omega> = 0.54\ \omega_{ci}$$
$$<\omega_R> = 1.2\ \omega_{ci}$$
$$<\omega_z> = 0.25\ \omega_{ci}$$
Stabilization of MHD n=3 radial mode

Beam ions have strong stabilizing effect on n=3 radial MHD mode for very small values of \( n_{\text{beam}}/n_0 \).

Larger values of \( n_{\text{beam}}/n_0 \) result in growth of different mode with large radial and axial mode numbers.

Beam ion parameters satisfy stability condition: \( n\Omega > \omega_R + \omega_Z \) for n=3.

Growth rate of n=3 radial mode vs beam ion density.

From 3D nonlinear MHD-hybrid (full-f) simulations of FRC with E~3. Modes other than n=3 radial mode are filtered out.

Beam ion parameters:
\( V_0 = 5V_A, \nu^* = 4.5 V_A, \) 
\( n_{\text{beam}} < 1\% n_0, \ I_b/I_{\text{tot}} < 2.5\% . \)
n=3 radial MHD mode vs beam-driven mode

- MHD mode is stabilized by beam which satisfy stability condition: \( n\Omega > \omega_R + \omega_Z \).
- Beam-driven mode is excited for larger values of \( n_{\text{beam}} \) due to high-order resonances: \( n\Omega = l\omega_R + k\omega_Z \).
Beam effect on n=3 MHD mode (no symmetry filter)

- The n=3 radial mode growth rate is reduced by a factor of 3 compared to MHD.

- At later time, the n=3 axial mode grows with MHD-like growth rate.

- Stabilizing effect of the beam depends on the mode symmetry.

- Beam ion parameters: $V_0 = 5V_A$, $v^* = 4.5V_A$, $n_{beam} = 0.004\%\, n_0$, $I_b/\, I_{tot} = 1.4\%$.

Time evolution of kinetic energy of n=3 mode, initial conditions correspond to radial mode.

Mode structure (pressure perturbation) of the n=3 axial mode.
FRC stability including close-fitting conducting shell and energetic beam ion effects: I. Linear results

- Close-fitting conducting shell stabilizes all low-$n$ radially-polarized (even) modes.

- Due to localization, the ion beams are effective in stabilizing the residual low-$n$ instabilities, except for relatively cold beams which have a destabilizing effect on $n \geq 3$ modes.

- The NBI effects are stronger for lower-$n$ modes ($n=1$ and $n=2$), and smaller $V_0$.

- The $n=1$ tilt mode and the $n=2$ mode are stabilized, and the growth rate of the $n=3$ mode is reduced for $E \approx 1$, $S* \approx 18$, $n_b/n_i = 0.03$, and $V_0 = 6V_A$.

Normalized growth rates of the $n=1$-$4$ modes from 3D hybrid simulations including the effects of conducting shell and NBI stabilization.
FRC stability including close-fitting conducting shell and energetic beam ion effects: II. Nonlinear simulations

- Nonlinear 3D simulations show that the residual instabilities (n=3 mode) saturate at small amplitudes.
- FRC remains stable with respect to all MHD modes, as long as it is sustained.

Nonlinear hybrid simulations of an FRC with $E=1.1$, including the effects of the beam ions and the close-fitting conducting shell. (a) Time evolution of $n=0$-$4$ modes kinetic energy; and (b) contour plots of plasma density in the toroidal cross sections.

Simulations have been performed in support of MRX-FRC experimental proposal.
HYM simulations for C2-like equilibrium

Plasma parameters: $n_e = 2 \cdot 10^{19} \text{ m}^{-3}$, $\lambda_i = 7.4 \text{ cm}$, $R_s = 0.3 \text{ m}$, $\beta_s = 0.84$. Normalized: $S^* = 4$, $E \sim 3$, mirror ratio $\sim 3$. 
Hybrid simulations (no beam)

- The $n=1$ wobble mode is unstable (no saturation) in simulations with C2-like equilibrium with mirror ratio $\sim3$, and $S^*=4$.
- Periodic BCs are used for $E$, so no end-shorting.
- Similar simulations with mirror ratio $=1$ do not show the wobble mode, instead there is weak instability of the $n=1$ tilt mode, and $n=2$ rotational mode.

Time evolution of different Fourier harmonics from full-f hybrid simulations using HYM code for $S^*=4$. The $n=1$ wobble mode is unstable.
Injection of fast ions with slowing-down distribution

Beam ions were loaded into simulation region between $t=0$ and $t=5t_A$.

Particles were loaded with slowing-down distribution: $f_b = f(v, p_\phi)$, where

$$f(v, p_\phi) \sim \frac{1}{v^3 + v'^3} \cdot \left[ \frac{(p_\phi - p_m)/(Rv - \psi_0 - p_m)}{(Rv - \psi_0 - p_m)} \right]^4$$

$p_\phi = Rv_\phi - \psi$ is toroidal angular momentum, $p_m = -0.1\psi_0$, and $\psi$ is poloidal flux.

(a-b) Contour plots of poloidal flux and beam ion toroidal current; (c-d) radial profiles of beam ion density and current corresponding to slowing-down distribution function $f_b = f(v, p_\phi)$. 
Injection of fast ions with slowing-down distribution

C2 beam parameters: $\varepsilon \sim 10\text{keV}$, $v_0 \sim 2.2\ V_A$, $n_b/n_e \leq 0.3$. 
Simulations for C2 equilibrium including fast ions with slowing-down distribution

Polarization of unstable $n=1$ mode corresponds to radial shift mode, as can be seen from $\delta n$ and $\delta V$ plots. Perturbed axial velocity is small compared to $\delta V_\phi$ and $\delta V_R$. Beam injection does not cause beam-driven modes, and reduces the growth rate of $n=1$ wobble mode.
Conclusions

• For large beam ion toroidal rotation frequencies (small $S^*$) radial resonances are more important than axial betatron resonances.

• Simulations for the $n=2$ and $n=3$ radial modes validate stability condition

• Full-f simulations for a delta-function distribution of the beam ions and the MHD description of the thermal plasma demonstrate why cold beams ($T_b \ll E_b$) are always de-stabilizing.

• MHD growth rates are too large for beam ion stabilization for reasonable beam parameters – need kinetic description of thermal plasma.

• A stable regime can be found for oblate FRCs with a close-fitting if conducting shell and energetic beam ions are used stabilization.

• In lower $S^*$ FRCs, beams with large thermal spread could be used for stabilization. Beam injection does not cause beam-driven modes, and reduces the growth rate of $n=1$ wobble mode in C2-like equilibrium.