Stellarator Coil Design: Past, Present and Future.

S. R. Hudson + thanks to many.

Research & Review Seminar, PPPL Theory Department, 6/2/2020

1. The magnetic field required to confine a plasma must be provided externally, provided (historically) by a set of external current-carrying “coils”.

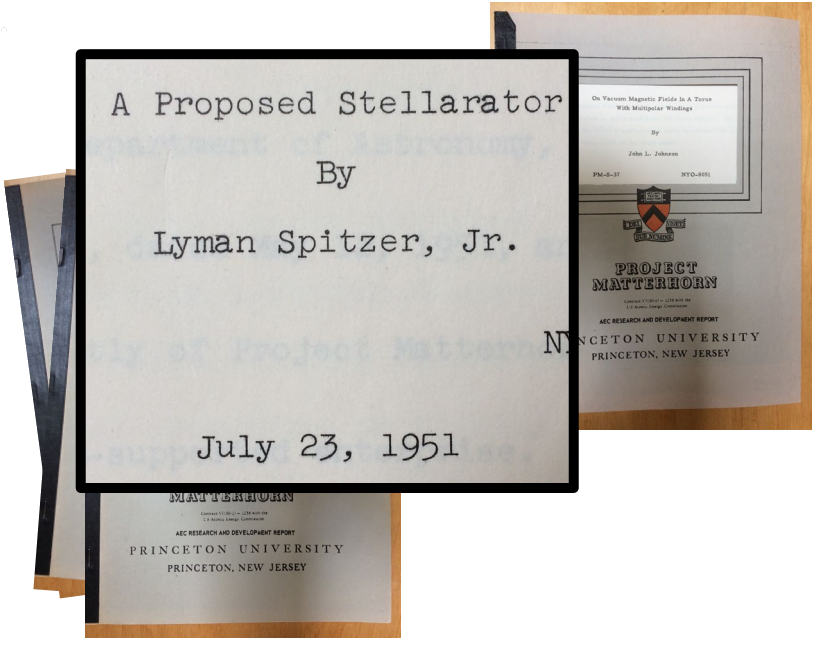
2. Very early coil designs were simplistic; but, advanced stellarators were constructed, e.g. LHD.

3. The idea of “optimized stellarators” was conceptual breakthrough, but leads to complex coils.

4. In this talk, the mathematics of modern coil-design algorithms & trends shall be described.

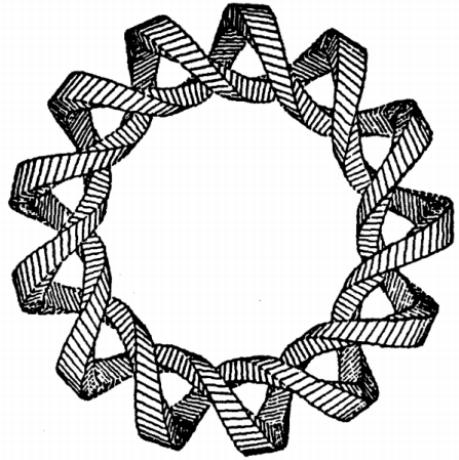
Lyman Spitzer's invented the stellarator in 1951. Model A began operation sixty-six years ago in 1953.

1. Original documents available in PPPL Furth Library (and they are very worth reading).
2. Model A, PPPL, began operation in 1953.

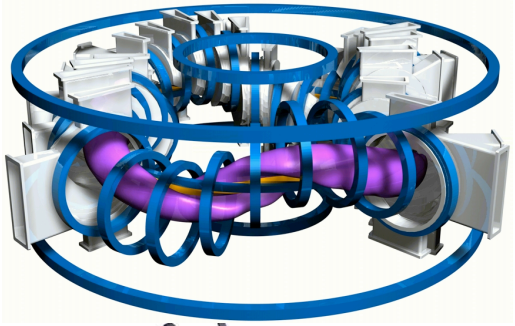


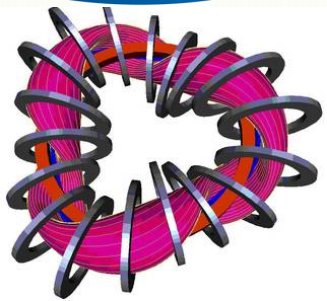
Many coil arrangements were investigated, and the stellarator programme made great progress.

torsatrons: e.g. ATF

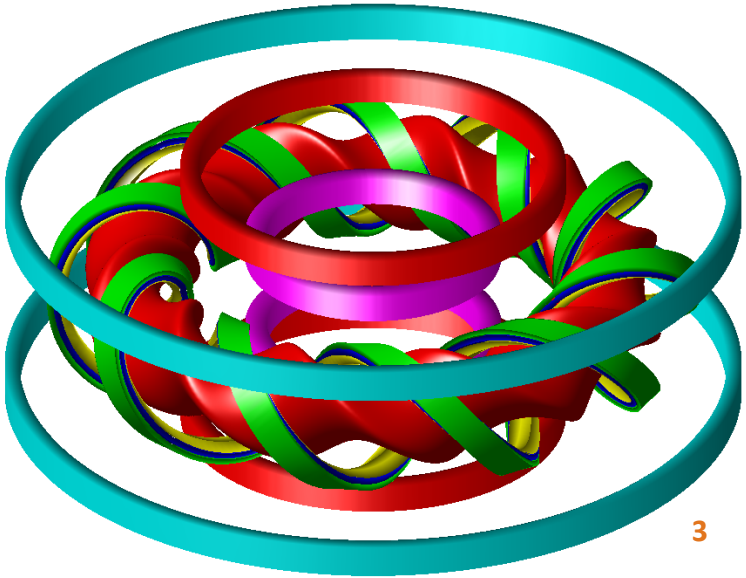


heliacs: H-1, TJ-II





heliotrons: e.g. LHD



Q. Why do we need external current-carrying coils?
A. Plasmas cannot create their own confinement.

1. Stress tensor: $\mathbf{T} \equiv p_\perp (\mathbf{I} - \mathbf{bb}) + p_\parallel \mathbf{bb}$, where $p_\perp = p + B^2/2$ and $p_\parallel = p - B^2/2$.

2. Equilibrium: $\nabla \cdot \mathbf{T} = 0$, reduces to $\nabla p = \mathbf{j} \times \mathbf{B}$.

3. Identity: $\nabla \cdot (\mathbf{r} \cdot \mathbf{T}) = \mathbf{r} \cdot (\nabla \cdot \mathbf{T}) + Tr(\mathbf{T})$.

4. Integrate over volume: $\int_{\mathcal{V}} \left(3p + \frac{1}{2}B^2\right) dv = \oint_{\mathcal{S}} \left[\left(p + \frac{1}{2}B^2\right)(\mathbf{n} \cdot \mathbf{r}) - \frac{1}{2}B^2(\mathbf{r} \cdot \mathbf{b})(\mathbf{n} \cdot \mathbf{b})\right] ds.$

5. Take $\mathcal{S} \equiv \partial\mathcal{V} \to \infty$, assume no external currents:

   l.h.s. : l.h.s. $\to$ positive,

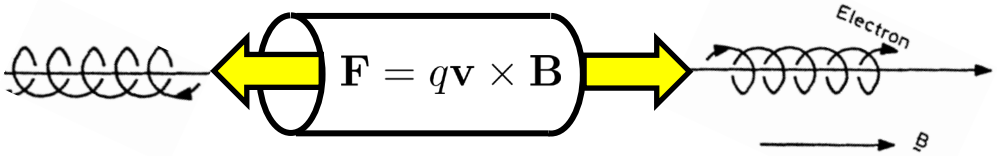
   r.h.s. : $p = 0$, $B \sim 1/r^3$, so r.h.s $\sim 1/r^3 \to 0$.

6. Virial Theorem: an MHD equilibrium cannot confine itself with $\mathbf{B}$ generated internally.

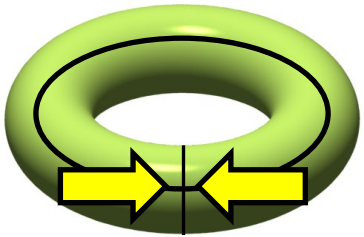
Shafranov (1966). For more detailed derivation of MHD virial theorem, see J. Freidberg, *Ideal MHD*.

To prevent end losses, magnetic field must be closed. Hairy Ball theorem: closed magnetic fields are toroidal.

1. An open-ended cylinder has good perpendicular confinement of charged particles, but particles are lost through the ends.



2. For $\mathbf{B}$, where $\mathbf{B} \neq 0$, to be everywhere tangential to a closed surface, the surface must be a torus.

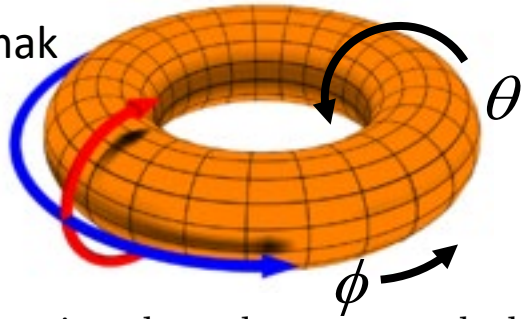


3. Hairy Ball Theorem: closed magnetic fields are toroidal. (Generally, a torus can be knotted.)

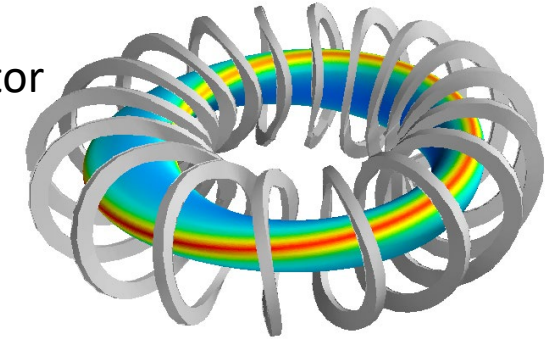
Why do stellarators look so complicated? Confinement requires plasma current or 3D shaping.

1. Rotational-transform, $\iota \equiv \lim_{\Delta\zeta\to\infty} \frac{\Delta\theta}{\Delta\phi}$, is essential for confinement.

tokamak



stellarator



2. In axisymmetric tokamaks, ι can only be generated by internal plasma currents; but, self-organized plasma currents are not easy to control (disruptions).

3. Not relying on internally self-organized plasma currents is a great idea (stellarators).

4. Rotational transform, ι, can be generated by (i) shaping the magnetic axis (torsion), or by (ii) shaping the boundary, i.e. rotating ellipse. Leads to geometrical complexity.

C. Mercier, Nucl. Fusion **4**, 213 (1964)

Magnetic field is given by Biot-Savart law. One might think that coil design might be quite simple.

1. The only physics equation that we care about for coil design is the Biot-Savart law:

   $\mathbf{B}(\bar{\mathbf{x}}) = \int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{x}) \times \mathbf{r}}{r^3} dv$, where $\mathbf{j}(\mathbf{x})$ is current density, and $\mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}$.

   i. If the current density is a sheet current on a surface, the integral becomes a surface integral.

   ii. If the current density is filament lying on closed curve, the integral becomes a line integral.

2. If magnetization, $\mathbf{M}$, is included, $\mathbf{B}_m(\bar{\mathbf{x}}) = \int_{\mathcal{V}} \frac{(\nabla \times \mathbf{M}) \times \mathbf{r}}{r^3} dv + \int_{\partial\mathcal{V}} \frac{(\mathbf{M} \times \mathbf{n}) \times \mathbf{r}}{r^3} ds.$

3. It becomes challenging because we have an inverse problem:

   i. The physics optimization tells us what $\mathbf{B}$ is good for confinement,

   ii The coil design must invert the Biot-Savart law to determine $\mathbf{j}$, and $\mathbf{M}$.

Optimized stellarators: first design *B*, then design coils. But how is the magnetic field to be prescribed?

Q. How is the desired magnetic field to be defined? Must we give $\mathbf{B}(\mathbf{x})$, $\forall \mathbf{x} \in \mathcal{V}$?

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Q. How is the desired magnetic field to be defined? Must we give $\mathbf{B}(\mathbf{x})$, $\forall \mathbf{x} \in \mathcal{V}$?

1. Given volume $\mathcal{V}$, with closed boundary, $\mathcal{S} \equiv \partial\mathcal{V}$.
2. Vacuum fields satisfy $\nabla \times \mathbf{B} = 0$, which implies $\mathbf{B} = \nabla\Phi$.
3. Given boundary condition, e.g. $\mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$. $\nabla \cdot \mathbf{B} = 0$ implies $\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$.
4. Toroidal flux $\Psi \equiv \oint_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{l}$, (require one loop integral per "hole").
5. In $\mathcal{V}$, solution to $\nabla \cdot \nabla\Phi = 0$ is unique.
6. Task is to design coils that provide required $B_n \equiv \mathbf{B} \cdot \mathbf{n}$ on given closed surface, $\mathcal{S}$.

Consider an external "winding surface" that surrounds the plasma, on which a sheet current is allowed.

1. Consider a "winding surface", $\mathcal{W}$, with prescribed geometry that surrounds the plasma, that carries a sheet current.

2. Current-density $\mathbf{K} = \mathbf{n} \times \nabla\Phi$, where $\Phi = \tilde{\Phi}(\theta, \zeta) + \frac{I}{2\pi}\theta + \frac{G}{2\pi}\zeta$ is the current potential.

   i. $\tilde{\Phi}$ is single valued, I and G are enclosed currents.

3. Current density, $\mathbf{K}$, is parallel to contours of current potential, Φ.

   i. The current potential $\equiv$ infinitely many current carrying filaments parallel to $\mathbf{K}$.

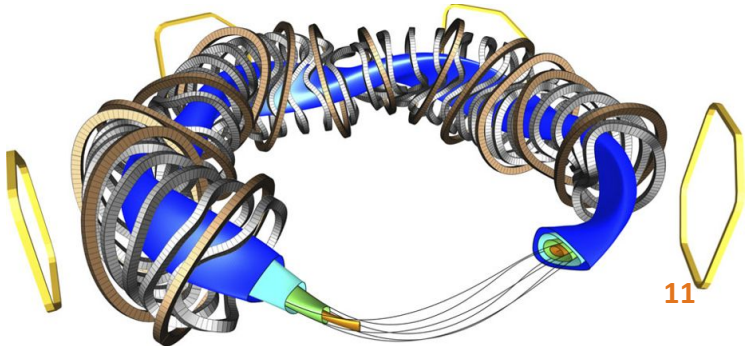
   ii. A discrete set of contours $\equiv$ a discrete set of coils.

5. Magnetic field $\mathbf{B} = \int_{\mathcal{W}} \frac{\mathbf{K} \times \mathbf{r}}{r^3} ds = \int_{\mathcal{W}} \frac{(\mathbf{n} \times \nabla\Phi) \times \mathbf{r}}{r^3} ds.$

For a given winding surface, calculate current potential that minimizes "error" functional.

1. Let $\mathcal{W}$ = winding surface, Φ = current potential on $\mathcal{W}$, and $\mathcal{S}$ = plasma boundary.
2. For arbitrary $\mathcal{S}$, $\mathcal{W}$, cannot generally produce exactly the required field.
2. Introduce "error functional" $\mathcal{F}_0[\mathcal{W}, \Phi, \mathcal{S}] \equiv \oint_{\mathcal{S}} \frac{1}{2} (\mathbf{B}_T \cdot \mathbf{n})^2 \, ds$,

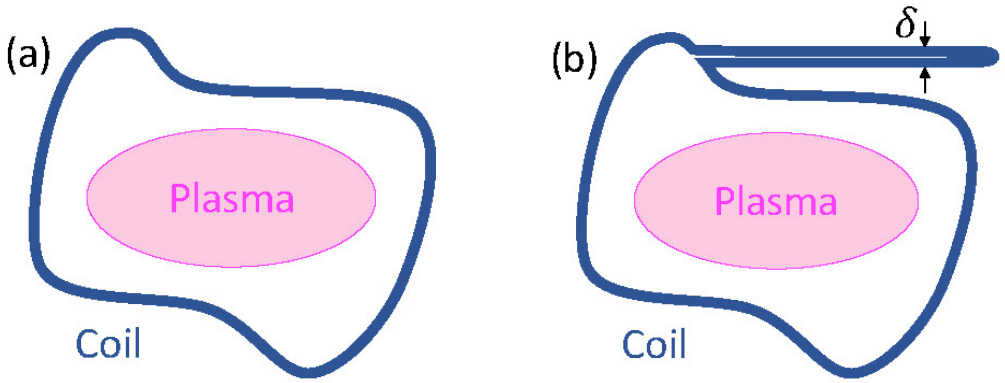
   where $\mathbf{B}_T = \mathbf{B}_P + \mathbf{B}_C$ (plasma and coils), and $\mathbf{B}_C = \int_{\mathcal{W}} \frac{(\mathbf{n} \times \nabla\Phi) \times \mathbf{r}}{r^3}$.
3. $\mathcal{F}_0$ is quadratic in Φ. Setting $\left.\frac{\partial \mathcal{F}_0}{\partial \Phi}\right|_{\mathcal{W},\mathcal{S}} = 0$ leads to $\mathcal{A} \cdot \mathbf{x} = \mathbf{d}$, where $\mathbf{x} \equiv \{\Phi_{m,n}\}$.
4. *NESCOIL* used to design HSX, W7-X, . . .



P. Merkel, Nucl. Fusion **27**, 867 (1987)

Coil design is inverse problem; maybe not well-defined. Large change in geometry gives small change in field.

1. Large changes in coil geometry can create almost identical magnetic fields.



2. EM101: high-order harmonics of vacuum fields decay rapidly.

3. Small changes in high (m, n) harmonics of B_n on $\mathcal{S}$ leads to large changes in $\Phi_{m,n}$ on $\mathcal{W}$.

4. This ill-posed-ness was often expediantly "removed" by Fourier truncation.

Can regularize the minimization problem by introducing a penalty on the current density.

1. $\mathcal{F}_0 \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 \, ds + \lambda \oint_{\mathcal{S}} \frac{1}{2} K^2 \, ds,$

   where $K = |\mathbf{K}| = |\mathbf{n} \times \nabla \Phi|$,
   (K= inverse distance between contours),
   and λ is a user-supplied "weight".

2. $\mathcal{F}_0$ is still quadratic in Φ.

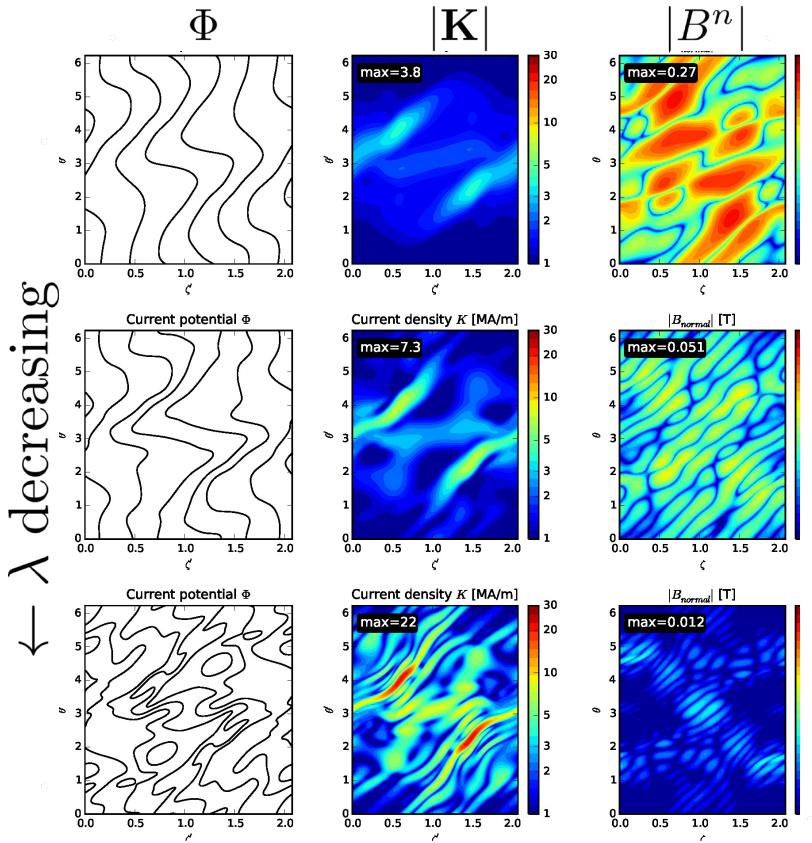
3. Implemented in *REGCOIL*

   i. increasing $\lambda \rightarrow$ smoother contours,

   ii. decreasing $\lambda \rightarrow$ smaller B_n, but fractal contours.

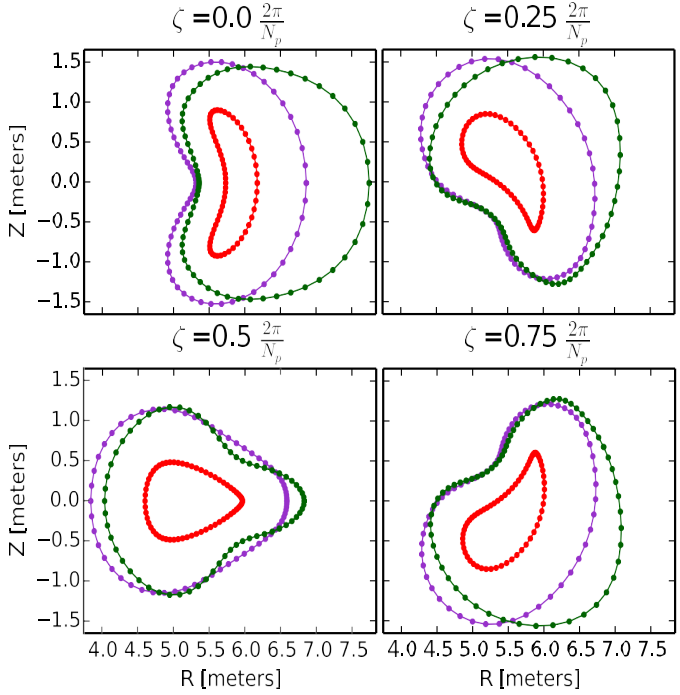
4. Other regularization terms are possible, and can penalize gradients in certain areas (i.e., allow ports).

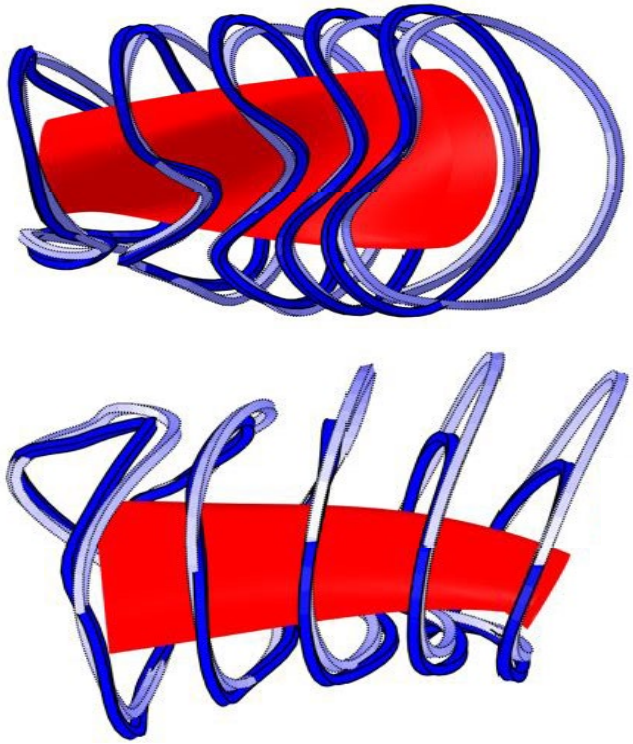
M. Landreman, Nucl. Fusion **57**, 046003 (2017)



The sheet current / coil contours lie on winding surface. What is the optimal geometry of the winding surface?

1. Adjoint methods can be used to optimize the shape of the winding surface to obtain simpler coils.



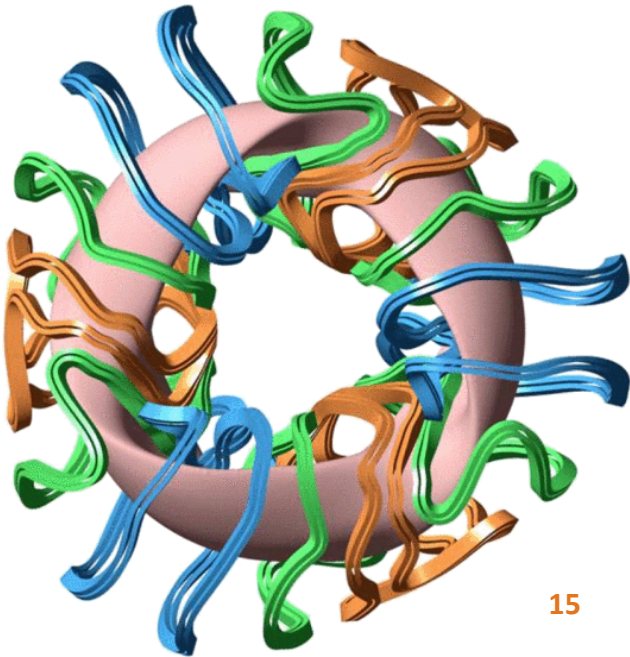


E. J. Paul, M. Landreman *et al.*, Nucl. Fusion **58**, 076015 (2018)

The continuous current potential on is useful approximation, but we need discrete coils.

1. As number of coils decreases, continuous approximation deterioates. Leads to "coil ripple".
2. Consider coils as zero-width filaments that lie on a prescribed winding surface, $\mathcal{W}$.
3. $\mathcal{F}[\mathcal{W}, \boldsymbol{\Theta}, \mathcal{S}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \sum_i \lambda_i \{\text{engineering constraints}\}_i$,

   where $\boldsymbol{\Theta} \equiv$ angular variation of current filaments,
   λ_i are weights,
   engineering constraints include coil curvature, coil-coil distance,
4. Dependence on coil geometry is nonlinear.
5. Nonlinear optimization of geometry implemented in *COILOPT*, *NESCOIL* + *COILOPT* used to design NCSX.



COILOPT: D. J. Strickler *et al.,* Fusion Sci. Technol. 41, 107 (2002);
ONSET: M. Drevlak, Fusion Technol. 33, 106 (1998).

Some normal fields, namely resonant fields, are more important than others.

1. Fields that "resonate" with closed fieldlines create islands, overlapping islands create chaos; degrades confinement.

2. $\mathcal{F} \equiv \mathcal{F}_0 + \oint_{\mathcal{S}_{m,n}} \frac{1}{2}(B^n_{m,n})^2 ds,$

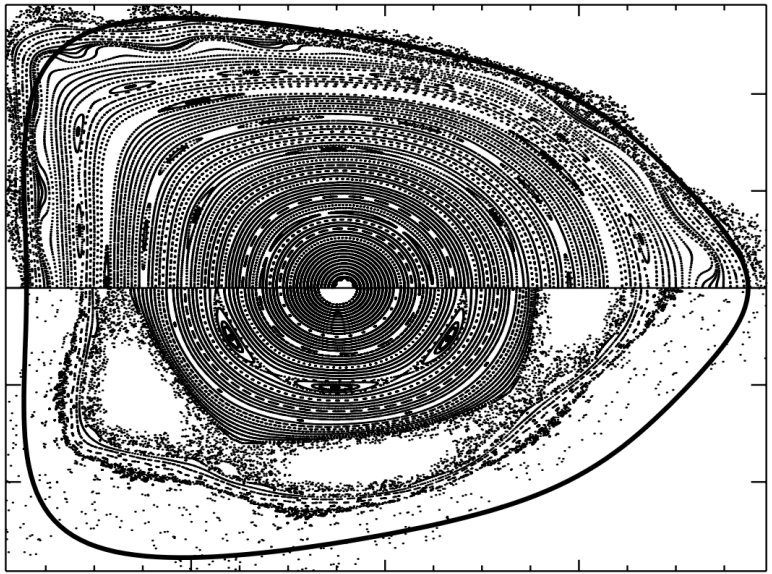
   where $B^n_{m,n}$ is resonant field at rational surface, $\mathcal{S}_{m,n}$.

3. Fast methods for computing $B^n_{m,n}$ have been introduced.

4. Island-healing techniques were applied to NCSX.

5. Non-resonant fields are potentially less important,

   i. need equilibrium & transport calculations to determine most important error fields.



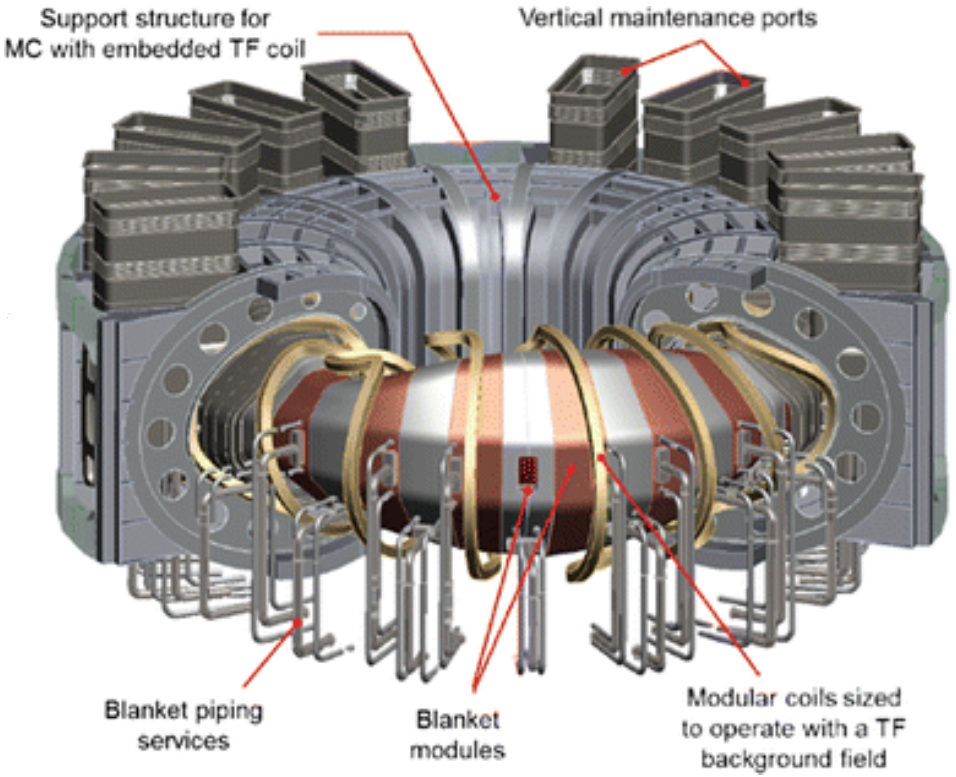
"healing" of $\iota = 3/5$ island in NCSX

J. R. Cary, Phys. Rev. Lett. **49**(4), 276 (1982), J. R. Cary & J. D. Hanson, Phys. Fluids **29**, 2464 (1986)
S. R. Hudson, D. A. Monticello *et al.*, Phys. Rev. Lett, **89**, 275003 (2002)

Can construct stellarators with planar sections on outboard side, which provides easy access.

1. Complicated coil shapes can restrict access, for maintenance etc.

2. COILOPT modifed to enforce planar outer sections.



D. A. Gates *et al.*, Nucl. Fusion 57, 126064 (2017).

Beyond the winding surface: the coil geometry is completely free.

1. Introduce $\mathbf{x}_i(l),\ i = 1, \ldots, N_C,$ to represent arbitrary (closed) current-carrying curves.

2. Include penalty on length, $\mathcal{F}[\mathbf{x}_i, S] = \int_S \frac{1}{2} B_n^2 ds + \lambda \sum_i L_i$.

3. $L = \oint_i dl$, $\delta L_i = \oint_i \frac{\delta L}{\delta \mathbf{x}} \cdot \delta \mathbf{x}_i \, dl$, where $\frac{\delta L}{\delta \mathbf{x}} = \kappa \times \mathbf{x}'$ and κ is curvature.

4. $\mathbf{B}_i = I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3} dl$, $\delta \mathbf{B}_i = I_i \oint_i \frac{\delta \mathbf{B}}{\delta \mathbf{x}} \cdot \delta \mathbf{x}_i \, dl$, where $\frac{\delta \mathbf{B}}{\delta \mathbf{x}} = \mathbf{R} \times \mathbf{x}_i'$ and $\mathbf{R} = \frac{3\,\mathbf{r}\,\mathbf{r}}{r^5} - \frac{(\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k})}{r^3}$.

5. $\delta \mathcal{F} = \sum_i I_i \oint_i \delta \mathbf{x}_i \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} dl$, where $\frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} = \mathbf{x}_i' \times \left(\int_S \mathbf{R}_{i,n} B_n \, ds + \lambda \kappa_i \right)$. Implemented in *FOCUS*.

6. Steepest descent, $\frac{\partial \mathbf{x}_i}{\partial \tau} = -\left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_S$, preserves linking integral, $\frac{1}{4\pi} \oint_i \oint_a \frac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a$.

C. Zhu, S. R. Hudson *et al.*, Nucl. Fusion **58**, 016008 (2018)

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5. $\delta \mathcal{F} = \sum_i I_i \oint_i \delta \mathbf{x}_i \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} dl,$ where $\frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} = \mathbf{x}_i' \times \left(\oint_S \mathrm{R}_{i,n} B_n \, ds + \lambda \kappa_i \right)$. Implemented in *FOCUS*.

6. Steepest descent, $\frac{\partial \mathbf{x}_i}{\partial \tau} = -\left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_S$, preserves linking integral, $\frac{1}{4\pi} \oint_i \oint_a \frac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a.$

C. Zhu, S. R. Hudson *et al.*, Nucl. Fusion **58**, 016008 (2018)

Beyond the winding surface: the coil geometry is completely free.

1. Introduce $\mathbf{x}_i(l),\ i = 1, \ldots, N_C,$ to represent arbitrary (closed) current-carrying curves.

2. Include penalty on length, $\mathcal{F}[\mathbf{x}_i, \mathcal{S}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds + \lambda \sum_i L_i.$

3. $L_i \equiv \oint_i dl,\ \delta L_i = \oint_i \frac{\delta L}{\delta \mathbf{x}} \cdot \delta \mathbf{x}_i \, dl,$ where $\frac{\delta L}{\delta \mathbf{x}} = \boldsymbol{\kappa} \times \mathbf{x}'$ and $\boldsymbol{\kappa}$ is curvature.

4. $\mathbf{B}_i = I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3} dl,\ \delta \mathbf{B}_i = I_i \oint_i \frac{\delta \mathbf{B}}{\delta \mathbf{x}} \cdot \delta \mathbf{x}_i \, dl,$ where $\frac{\delta \mathbf{B}}{\delta \mathbf{x}} = \mathbf{R} \times \mathbf{x}_i'$ and $\mathbf{R} = \frac{3\,\mathbf{r}\,\mathbf{r}}{r^5} - \frac{(\mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k})}{r^3}.$

5. $\delta \mathcal{F} = \sum_i I_i \oint_i \delta \mathbf{x}_i \cdot \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} dl,$ where $\frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} = \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbf{R}_{i,n} \, B_n \, ds + \lambda \, \boldsymbol{\kappa}_i \right).$ Implemented in *FOCUS*.

6. Steepest descent, $\frac{\partial \mathbf{x}_i}{\partial \tau} = - \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\mathcal{S}},$ coils cannot pass through boundary.

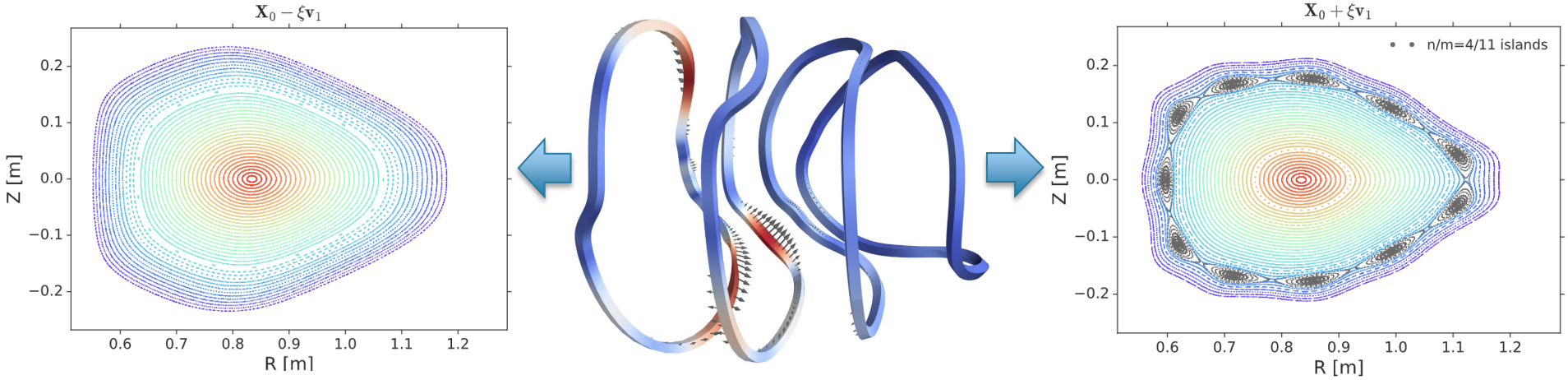
C. Zhu, S. R. Hudson *et al.*, Nucl. Fusion **58**, 016008 (2018)

Can calculate the second variation w.r.t. coil variations. This determines sensitivity to coil variations.

1. An eigenvalue analysis of the Hessian, $\mathcal{H}$, at the minimum determines the important directions,

$$\mathcal{F}(\mathbf{x}+\delta\mathbf{x}) \approx \mathcal{F}(\mathbf{x}) + \nabla\mathcal{F}\cdot\delta\mathbf{x} + \tfrac{1}{2}\,\mathbf{x}^T\cdot\mathcal{H}\cdot\mathbf{x}, \quad \mathcal{H}\cdot\mathbf{v}_i = \lambda_i\mathbf{v}_i, \quad \mathbf{x} = \textstyle\sum_i \alpha_i\mathbf{v}_i, \quad \delta\mathcal{F} \approx \textstyle\sum_i \alpha_i^2\lambda_i.$$

2. Figure shows principal deformation for 4/11 islands in CFQS.
   Island width sensitive to this perturbation. This determines construction tolerances.

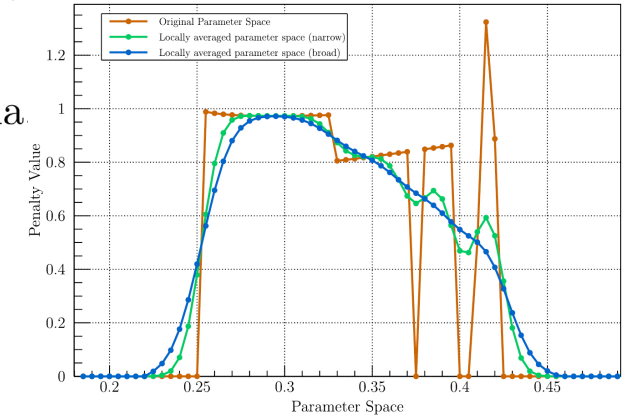


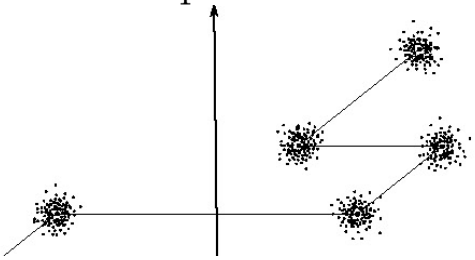
C. Zhu *et al.*, Plasma Phys. Control. Fusion **60** 054016 (2018), C. Zhu *et al.*, Nucl. Fusion **59**, 126007 (2019).

Stochastic Optimization: optimize geometry of coils so that small errors do not degrade performance

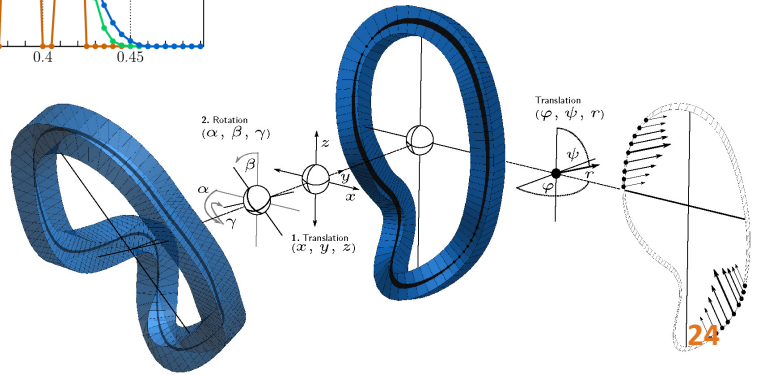
1. Stochastic optimization: $\mathcal{F}_N \equiv \frac{1}{N}\sum_i^N \mathcal{F}(\mathbf{x}+\delta\mathbf{x}_i)$, where $\{\delta\mathbf{x}_i\}$ is a cloud of displacements.

2. Smoothes non-smooth functionals.
   Avoids getting stuck in local minima.





2. Can explicity consider most important/likely errors in construction process;
   for example, rigid rotations and shifts.



J-F. Lobsien *et al.*, Nucl. Fusion **57**, 102020 (2018)

Variational calculus of surface integrals: The target functional is analytic.

1. Surface area $\mathcal{A} \equiv \oint_{\mathcal{S}} ds$. $\delta\mathcal{A} = \oint_{\mathcal{S}} \frac{\delta\mathcal{A}}{\delta\mathcal{S}} \cdot \delta\mathcal{S}\, ds$, where $\frac{\delta\mathcal{A}}{\delta\mathcal{S}} = H\,\mathbf{n}$ and $H = \nabla \cdot \mathbf{n}$ is mean curvature.

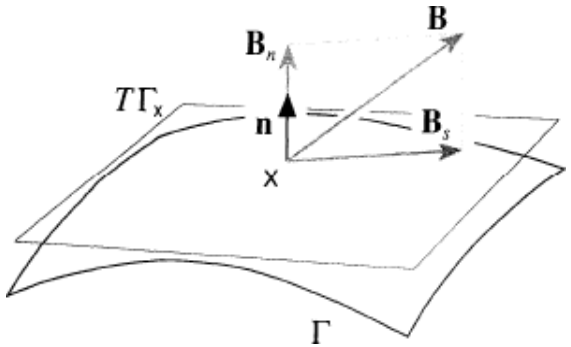
2. Penalized quadratic flux, $\mathcal{F}[\mathbf{x}_i, \mathcal{S}] = \oint_{\mathcal{S}} \frac{1}{2} B_n^2\, ds + \lambda \sum_i L_i$, depends on coil and surface geometry.

i. $\delta\mathcal{F} = \oint \delta\mathbf{x} \cdot \frac{\delta F}{\delta \mathbf{x}}\, dl$, where $\frac{\delta F}{\delta \mathbf{x}_i} = \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbb{R}_{i,n}\, B_n\, ds + \lambda\, \kappa_i \right)$

ii. $\delta\mathcal{F} = \oint_{\mathcal{S}} \frac{\delta F}{\delta \mathcal{S}} \cdot \delta\mathcal{S}\, ds$, where $\frac{\delta F}{\delta \mathcal{S}} = \left(\frac{1}{2} \mathbb{H} \cdot \mathrm{n}\, B_n^2 + \mathbb{B}_S \cdot \nabla B_n \right) \mathrm{n}$.

iii. $\delta^2\mathcal{F} = \oint_i \delta\mathbf{x}_i \cdot \oint_{\mathcal{S}} \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \mathcal{S}} \cdot \delta\mathcal{S}\; ds\, dl$, where

$\frac{\delta^2 \mathcal{F}}{\delta \mathbf{x}_i \delta \mathcal{S}} = \mathbf{x}_i' \times (\mathbb{R}_S \cdot \nabla B_n - \mathbb{B}_S \cdot \nabla \mathbb{R}_n + B_n\, \mathbb{R} \cdot \mathbb{H})\, \mathrm{n}$, and $\mathbb{B}_S \equiv \mathbb{B} - B_n \mathrm{n}$, and $\mathbb{R}_S \equiv \mathbb{R} - \mathbb{R}_n \mathrm{n}$.



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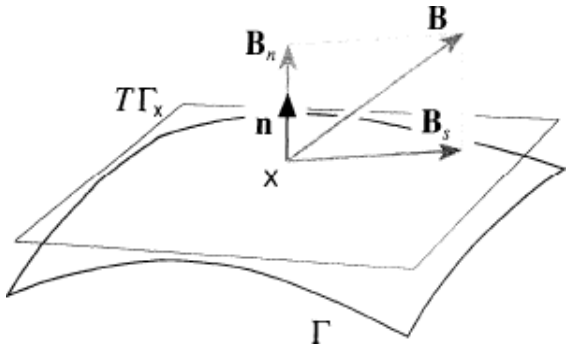

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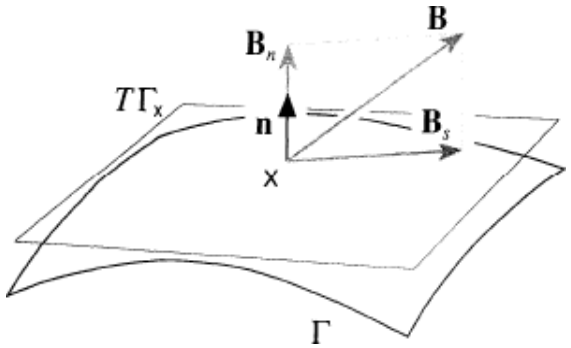

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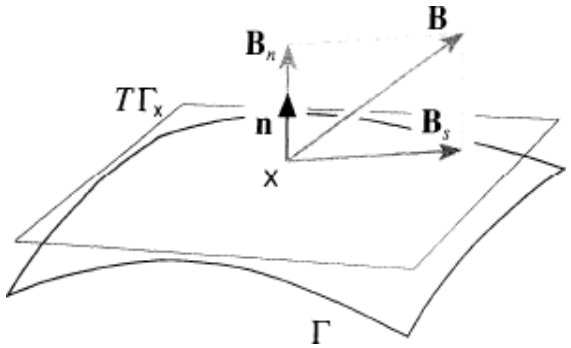

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R.L. Dewar *et al.*, Phys. Lett. A **194**, 49 (1994)

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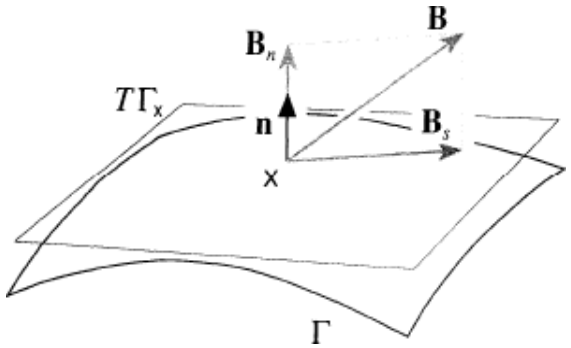


R.L. Dewar *et al.*, Phys. Lett. A **194**, 49 (1994), S. R. Hudson *et al.*, Phys. Lett A **382**, 2732 (2018).

Can differentiate the geometry of the optimal coils w.r.t. the shape of the plasma boundary.

1. Write $\mathcal{F} = \mathcal{F}[\mathbf{c}, \mathcal{S}]$, where $\mathbf{c}$ = coil geometry and $\mathcal{S}$ = surface geometry.

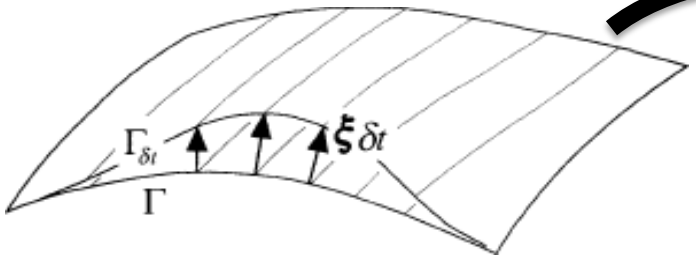
2. For given $\mathcal{S}$, optimal coil geometry satisfies $\nabla_{\mathbf{c}}\mathcal{F} = \left.\dfrac{\partial \mathcal{F}}{\partial \mathbf{c}}\right|_{\mathcal{S}} = 0.$

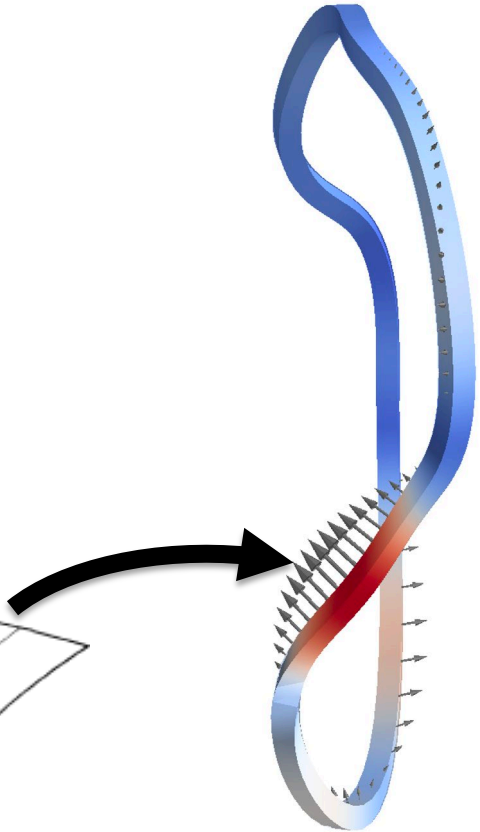
3. If $\mathcal{S}$ changes, $\mathbf{c}$ must change to preserve $\nabla_{\mathbf{c}}\mathcal{F} = 0$,

   $$\nabla_{\mathbf{c}}\mathcal{F}(\mathbf{c} + \delta\mathbf{c}, \mathcal{S} + \delta\mathcal{S}) \approx \nabla^2_{\mathbf{cc}}\mathcal{F} \cdot \delta\mathbf{c} + \nabla^2_{\mathbf{c}\mathcal{S}}\mathcal{F} \cdot \delta\mathcal{S} = 0.$$

4. Derivative of coil geometry w.r.t. surface geometry:

   $$\frac{\partial \mathbf{c}}{\partial \mathcal{S}} = -\left(\nabla^2_{\mathbf{cc}}\mathcal{F}\right)^{-1} \cdot \nabla^2_{\mathbf{c}\mathcal{S}}\mathcal{F}.$$





Changing the shape of the plasma can simplify the coil complexity.

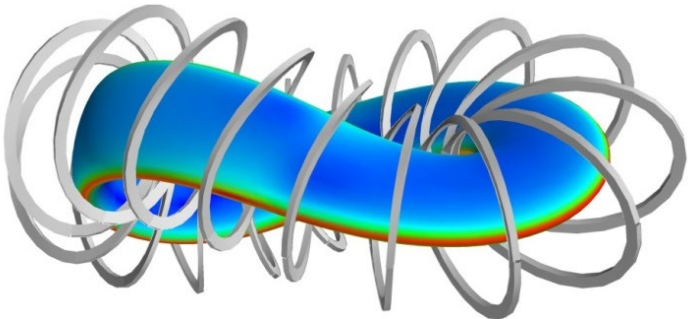
1. Rotational-transform on axis, ι_a, can be produced in vacuum by [Mercier (1964)]
   i. shaping the boundary, e.g. rotating ellipse, ii. torsion of the magnetic axis, or iii. both.

2. There is freedom to change the boundary at $\iota_a = const.$. This can be used to simplify the coils.

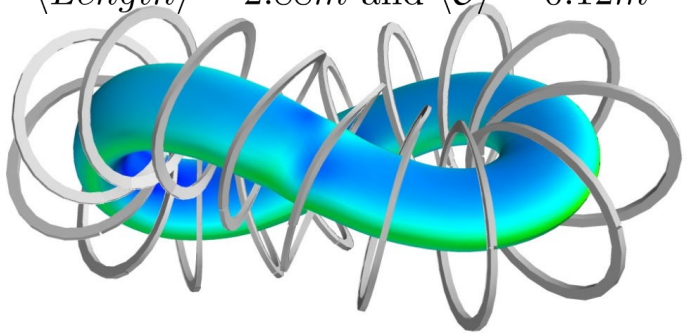
3. Introduce some measure of "complexity", e.g. $\mathcal{C}_i \equiv \oint_i \frac{1}{2}\tau^2 dl$, where $\tau = \dfrac{\mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}'''}{|\mathbf{x}' \times \mathbf{x}''|^2}$.

3. Both coil sets shown below produce $\iota_a \approx 0.276$, with plasma volume $0.8m^3$, using 18 coils.

higher ellipticity, lower torsion, higher coil complexity
$\langle Length \rangle = 3.07m$ and $\langle \mathcal{C} \rangle = 0.66m^{-1}$

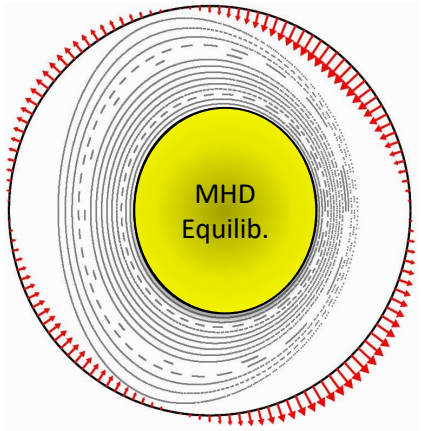


lower ellipticity, higher torsion, lower coil complexity
$\langle Length \rangle = 2.88m$ and $\langle \mathcal{C} \rangle = 0.12m^{-1}$



Beyond optimized stellarators: integrated plasma + coil optimization.

1. Optimizing the plasma properties, without regard to coil complexity, can lead to complex coils.
   Need to consider both plasma properties and coil complexity.



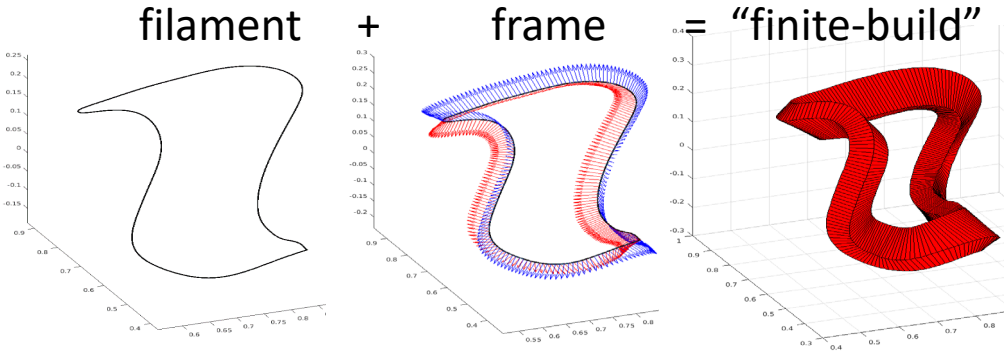
2. Introduce "matching surface", $\mathcal{S}$, that lies outside the plasma, inside coils.
   i. The free-boundary plasma equilibrium depends on B^n on $\mathcal{S}$.
   ii. The coil geometry depends on B^n on $\mathcal{S}$.
3. Integrated plasma coil optimization: $\mathcal{F} = \lambda_P \, \mathcal{P}[\,\mathbf{B}[B^n]\,] + \lambda_C \, \mathcal{C}[\,\mathbf{x}_i[B^n]\,]$,
   i. $\mathcal{P}$ = plasma property, depends on equilibrium $\mathbf{B}$, depends on B^n,
   ii. $\mathcal{C}$ = coil complexity, depends on coil geometry $\mathbf{x}_i$, depends on B^n on $\mathcal{S}$, and
   iii. λ_P and λ_C are weights (and as λ_P increases, coil complexity & cost increases ?).
4. Can simultaneously optimize plasma and coil properties.

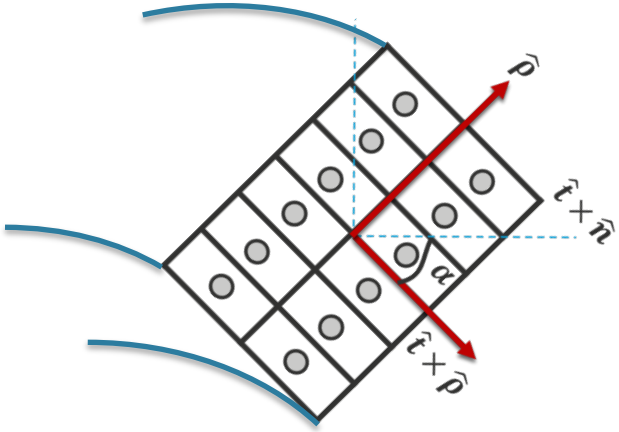
Beyond the single-filament approximation: optimization of non-zero cross-section coils.

1. Rectangle rotating by some angle $\alpha(l)$ w.r.t. e.g. Frenet frame, $(\mathbf{t}, \mathbf{n}, \mathbf{b})$, of a given curve, $\mathbf{x}(l)$.

2. Inside the rectangle, place $N_X \times N_Y$ filaments.

   i. Obtain multi-filament approx. to non-zero volume coils.





3. Can optimize over geometry of central filament, $\mathbf{x}_i(l)$, and rotation, $\alpha(l)$, to minimize $\mathcal{F}$,
   L. Singh, T. Kruger *et al. OMIC*

4. The multi-filaments complicates the analytic derivatives; use automatic differentiation,
   N. McGreivy *et al.*, $FOCUS_{ADD}$

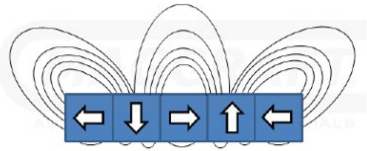
Magnetic field can be provided by permanent magnets. Can optimize shape of coils + magnets.

1. $\mathbf{B}_m(\bar{\mathbf{x}}) = \int_{\mathcal{V}} \frac{(\nabla \times \mathbf{M}) \times \mathbf{r}}{r^3} dv + \int_{\partial\mathcal{V}} \frac{(\mathbf{M} \times \mathbf{n}) \times \mathbf{r}}{r^3} ds.$

2. No unique solution: $\mathbf{M} \to \mathbf{M} + \nabla\chi$ with $\chi = 0$ on $\partial\mathcal{V}$ leaves $\mathbf{B}$ unchanged.

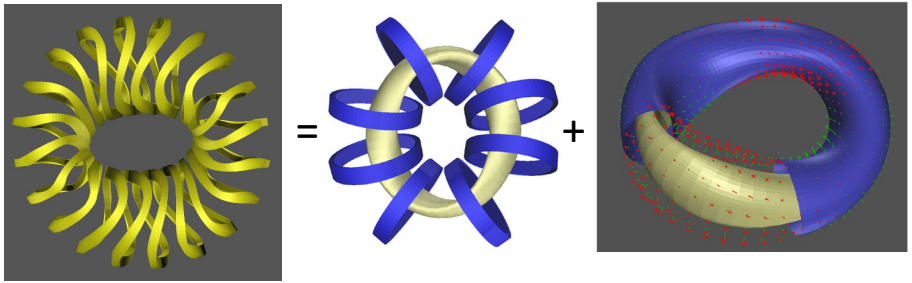
   i. Choose $\nabla \times \mathbf{M} = 0$, magnetization skin current, $\mathbf{M} = -\nabla \left[f(r,u,v)\tilde{\Phi}(u,v) \right]$, related to current potential used in NESCOIL, REGCOIL,

   ii. Choose $\mathbf{M}(\mathbf{r}) = M(\mathbf{r})\mathbf{n}$, iii. some combination, e.g. "Halbach" array, to minimize $\int_{\mathcal{V}} M^2 ddv$

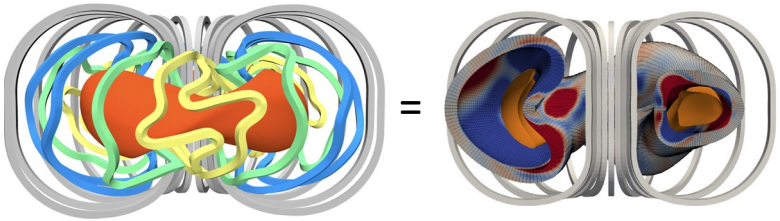


Halbach

ESTELL



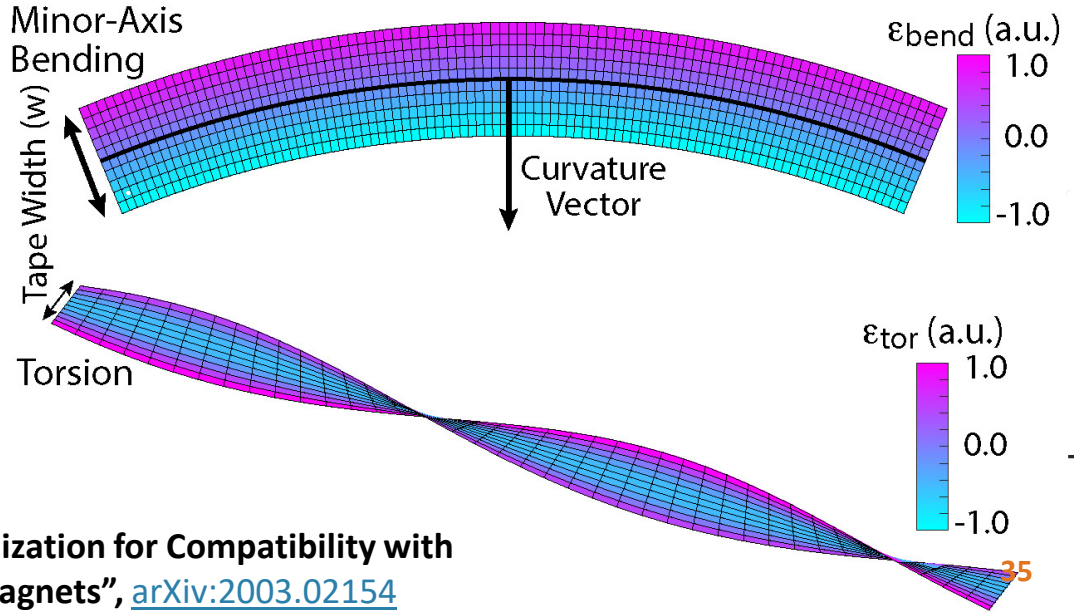
NCSX



P. Helander, M. Drevlak, M. Zarnstorff, S. C. Cowley, Phys. Rev. Lett. **124**, 095001 (2020),
C. Zhu et al., Nucl. Fusion (2020); FAMUS: C. Zhu et al., arXiv: 2005.05504; REGCOIL_PM: Landreman,

Flexible high-temperature superconductoring tape can provide stronger magnetic fields.

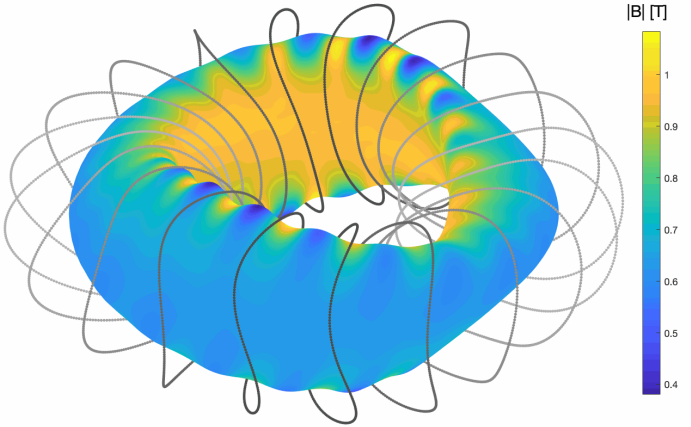
1. Technological advances: e.g. high-temperature superconducting tape.
   i. tight curvature should be OK, but curvature + torsion problematic,
   ii. need appropriate penalties in *FOCUS*, *REGCOIL* etc. to find optimal geometry,
2. Possibility of "de-mountable" coils, i.e. coils that can be taken apart.



C. Paz-Soldan, "**Non-Planar Coil Winding Angle Optimization for Compatibility with Non-Insulated High-Temperature Superconducting Magnets**", arXiv:2003.02154

Beyond the target surface: optimizing *B* directly from the coil geometry.

1. Let the geometry of N_C coils (closed curves) be given by $\mathbf{x}_i$.
2. Forget target surface. Define relevant "properties", $\mathcal{P}_i = \mathcal{P}_i[\mathbf{B}]$, of the magnetic field.
3. Seek minima of, for example, $\mathcal{F} = \sum_i \lambda_i \frac{1}{2} \left(\mathcal{P}_i - \mathcal{P}_{i,0}\right)^2$, where λ_i are weights and $\mathcal{P}_{i,0}$ are targets.
4. Example relevant properties include
   i. position, length of magnetic axis,
   ii. rotational-transform, ι, on axis,
   iii. "ripple" $= |\mathbf{B}| - \langle |\mathbf{B}| \rangle$ on axis,
5. Optimization problem: min $\mathcal{F}[\mathbf{x}_i]$ subject to $\mathbf{g}[\mathbf{x}_i] = \mathbf{0}$.



A. Guiliani, G. Stadler, M. Landreman *et al.*,

Future work: Optimization of “island divertor”. Outside the “last closed flux surface” is a chaotic mix.

1. The structure of the magnetic field outside the plasma region determines how escaping particles hit the inner wall. Heat and particle fluxes can cause damage.

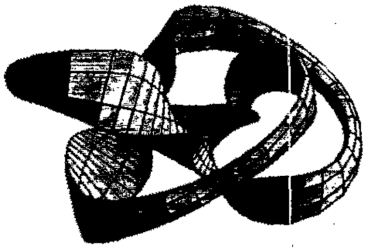
2. The magnetic field in the edge is typically chaotic, i.e. magnetic islands, ergodic fieldlines, cantori,

Closed magnetic fields are toroidal.
Most general torus is knotted. Quasi-helical knots exist.

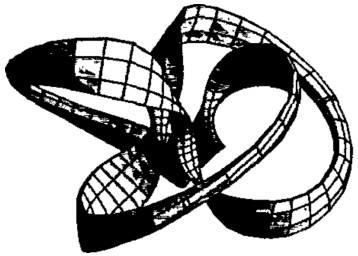
1. Hairy ball theorem: $\mathbf{B}$ must be tangential to a torus. The most general torus is a knot.

2. Quasi-helical knots were imagined in 1991. Coils for "knotatron" imagined in 2014.

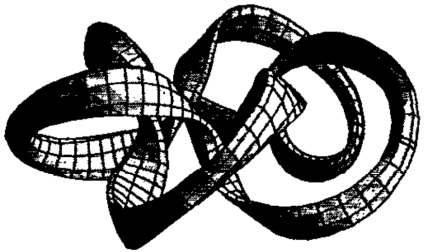
Garren & Boozer Fig.9



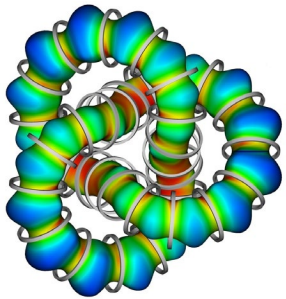
Garren & Boozer Fig.11



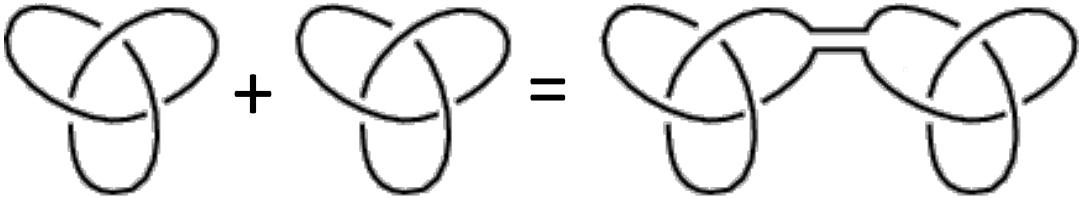
Garren & Boozer Fig.13



"knotatrons"



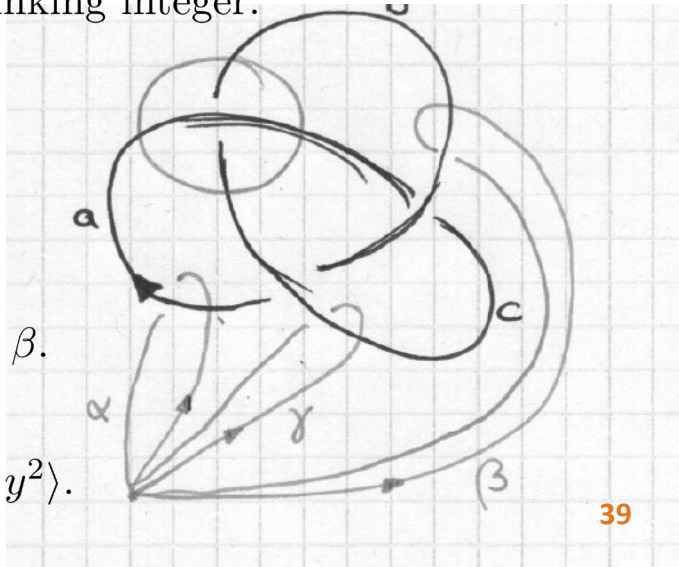
3. There is an infinite family of knots. Is there a knot that is optimal for simple coils?



D. A. Garren & A. H. Boozer, Phys. Fluids B **3**, 2822 (1991) , S.R. Hudson *et al.*, Phys. Plasmas **21**, 010705 (2014)

Coils for knotatrons: How can a given knot be linked?

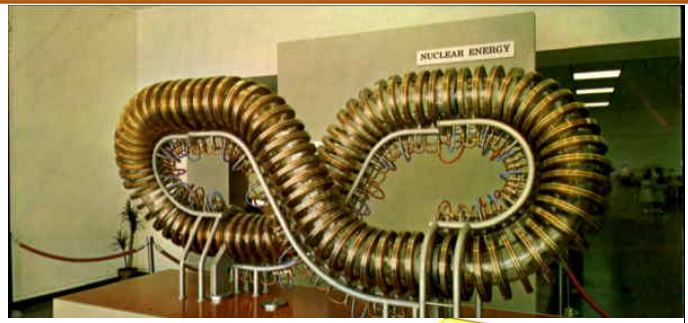
1. A knot, $\mathcal{K}$, is an embedding of a circle in $\mathbb{R}^3$. The trivial embedding, S^1, is called the unknot.
2. A closed plasma tube is a "fattened" knot. An external coil is a curve in $\mathbb{R}^3 - \mathcal{K}$ that links $\mathcal{K}$.
3. The fundamental knot group, $\pi_1(\mathbb{R}^3 - \mathcal{K})$, classifies the unique closed links.
   i. For S^1, $\pi_1(\mathbb{R}^3 - S^1) = \mathbb{Z}$, i.e. each coil is classified by a linking integer.
4. The knot group of complicated knots found by analysing relations between different linking paths.
   For example: the trefoil, $\mathcal{T}$,
   i. Distinct paths α, β and γ, shown in figure.
   ii. Looping γ then α gives the same curve as looping α then β.
   iii. Identify distinct links: $\pi_1(\mathbb{R} - \mathcal{T}) = \langle \alpha, \beta | \alpha\beta\alpha = \beta\alpha\beta \rangle$.
   iv. Simplification: $x = \alpha\beta$, $y = \alpha\beta\gamma$, $\pi_1(\mathbb{R} - \mathcal{T}) = \langle x, y | x^3 = y^2 \rangle$.

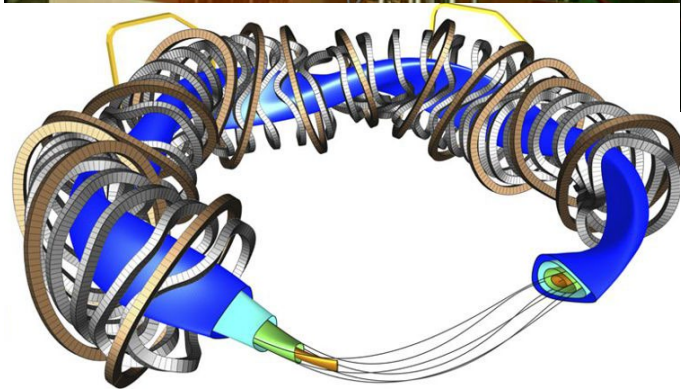


L. Gunderson, C. Smiet *et al.*,

So, where are we now?
This is the 3rd era of stellarator design.

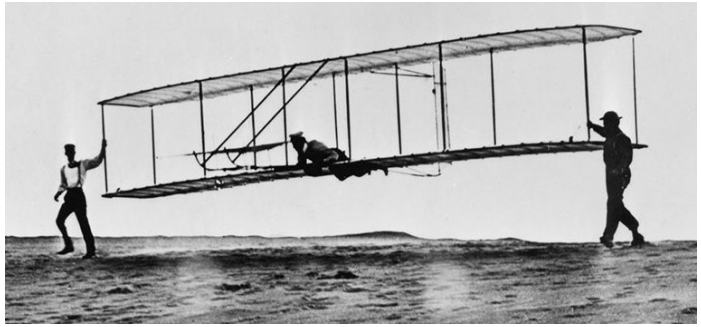
1. The early days:
   i. combination of divine insight, and
   ii. wrap currents around anything that doesn't move.
2. Optimized stellarators:
   i. first, optimize plasma properties;
   ii. second, determine coil geometry.
3. Integrated design:
   i. integrated plasma + coil optimization,
   ii. recent theoretical/engineering developments,
   iii. massive computational resources now available.
4. The next stellarator experiment will demonstrate the viability of fusion energy.



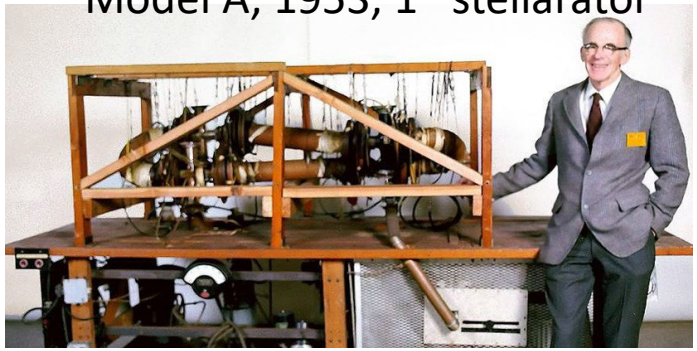


No commercial flights to the moon; no fusion reactor. But there has been impressive progress.

Kittyhawk, 1903, 1st flight

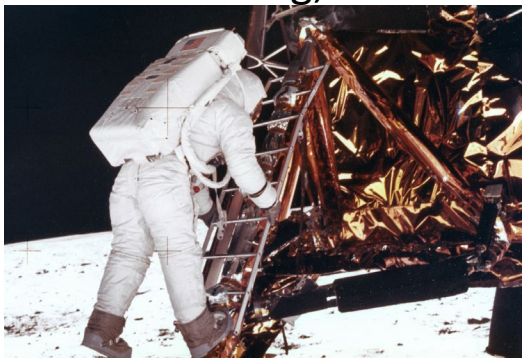


Model A, 1953, 1st stellarator

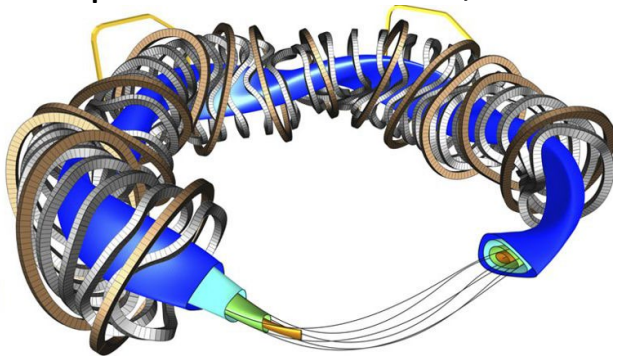


≈ 66 years

Moon landing, 1969



Optimized stellarator, W7-X



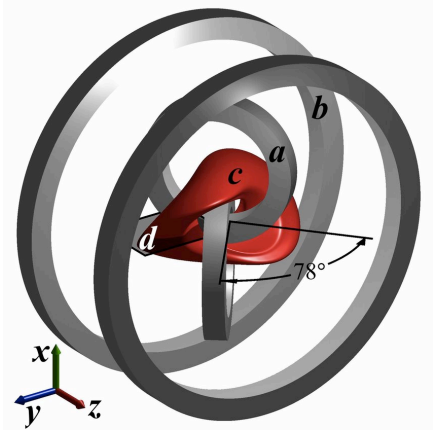
Thank You.

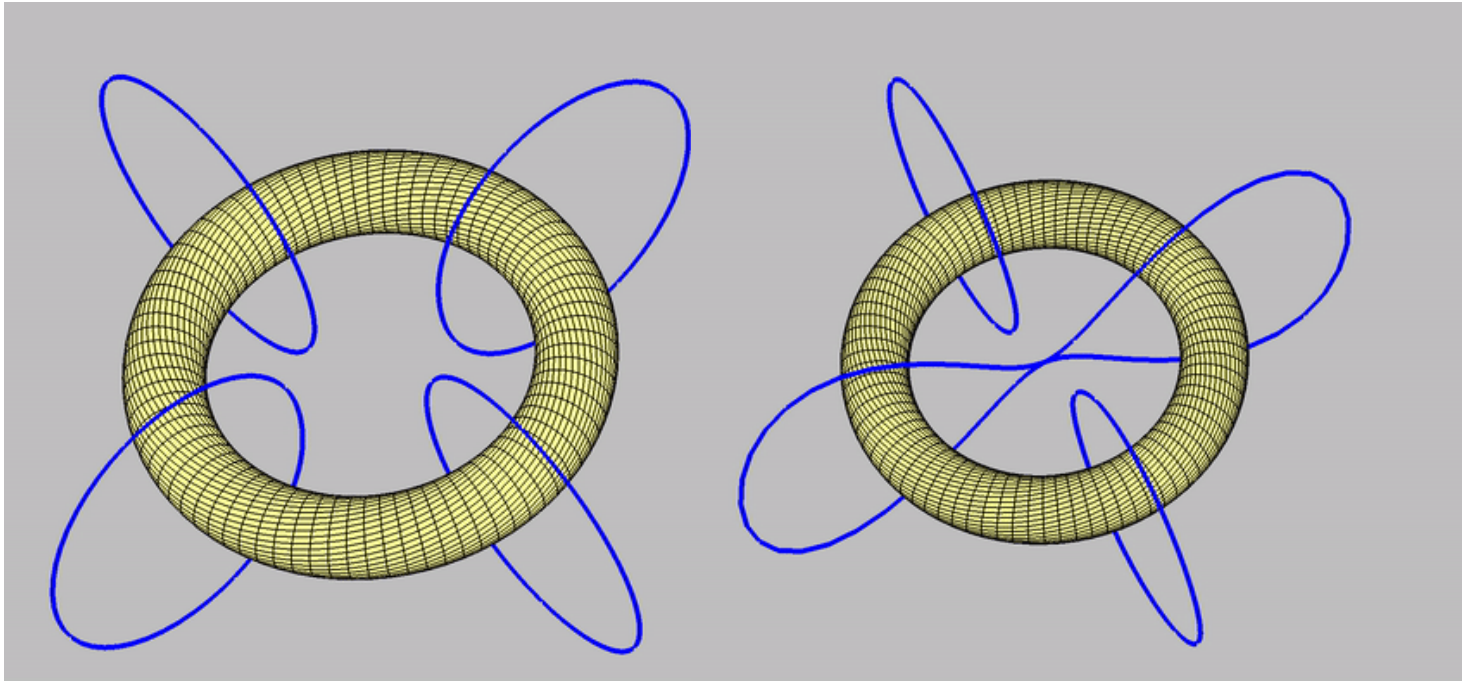
Fewer coils = less $$$ + more space.
What is the simplest possible stellarator?

1. Toroidal vacuum fields can be created with very few coils.

2. Increasing torsion of axis provides rotational transform.

Columbia
Non-neutral Torus
(CNT)





Variation in length of curve depends only on normal variations and curvature.

$$L \equiv \oint (\mathbf{x}' \cdot \mathbf{x}')^{1/2}\, dl \tag{1}$$

$$\delta L = \oint (\mathbf{x}' \cdot \mathbf{x}')^{-1/2} (\mathbf{x}' \cdot \delta\mathbf{x}')\, dl \tag{2}$$

$$= \oint \delta\mathbf{x} \cdot \mathbf{x}' (\mathbf{x}' \cdot \mathbf{x}')^{-3/2} \mathbf{x}' \cdot \mathbf{x}''\, dl - \oint \delta\mathbf{x} \cdot \mathbf{x}'' (\mathbf{x}' \cdot \mathbf{x}')^{-1/2}\, dl \tag{3}$$

Correct, but not transparent. Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change length?

Use $(\delta\mathbf{x} \times \mathbf{x}') \cdot (\mathbf{x}' \times \mathbf{x}'') = (\delta\mathbf{x} \cdot \mathbf{x}') \cdot (\mathbf{x}' \cdot \mathbf{x}'') - (\delta\mathbf{x} \cdot \mathbf{x}'') \cdot (\mathbf{x}' \cdot \mathbf{x}')$.

$$\delta L = -\oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \boldsymbol{\kappa}, \quad \text{where} \quad \underbrace{\boldsymbol{\kappa} \equiv \frac{\mathbf{x}' \times \mathbf{x}''}{(\mathbf{x}' \cdot \mathbf{x}')^{3/2}}}_{\text{curvature}} \tag{4}$$

Variation in magnetic field w.r.t. variations in the coil geometry (for filamentary currents).

$$\mathbf{B} = \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\, dl, \quad \text{where} \quad \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \quad r \equiv \sqrt{\mathbf{r}\cdot\mathbf{r}}, \quad \mathbf{x}' \equiv \partial_l \mathbf{x} \tag{1}$$

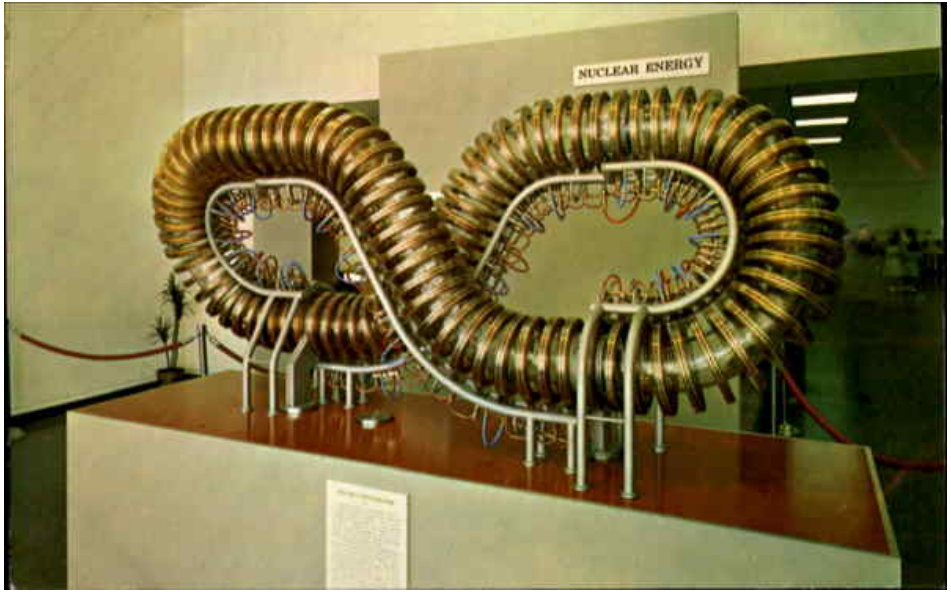
$$\begin{aligned}
\delta\mathbf{B} &= \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl \\
&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl \\
&= 2\oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl
\end{aligned} \tag{2}$$

Use $\mathbf{r} \times [\mathbf{r} \times (\delta\mathbf{x} \times \mathbf{x}')] = \mathbf{r} \times [\delta\mathbf{x}\,(\mathbf{r}\cdot\mathbf{x}') - \mathbf{x}'\,(\mathbf{r}\cdot\delta\mathbf{x})] = (\mathbf{r}\times\delta\mathbf{x})\,(\mathbf{r}\cdot\mathbf{x}') - (\mathbf{r}\times\mathbf{x}')\,(\mathbf{r}\cdot\delta\mathbf{x})$

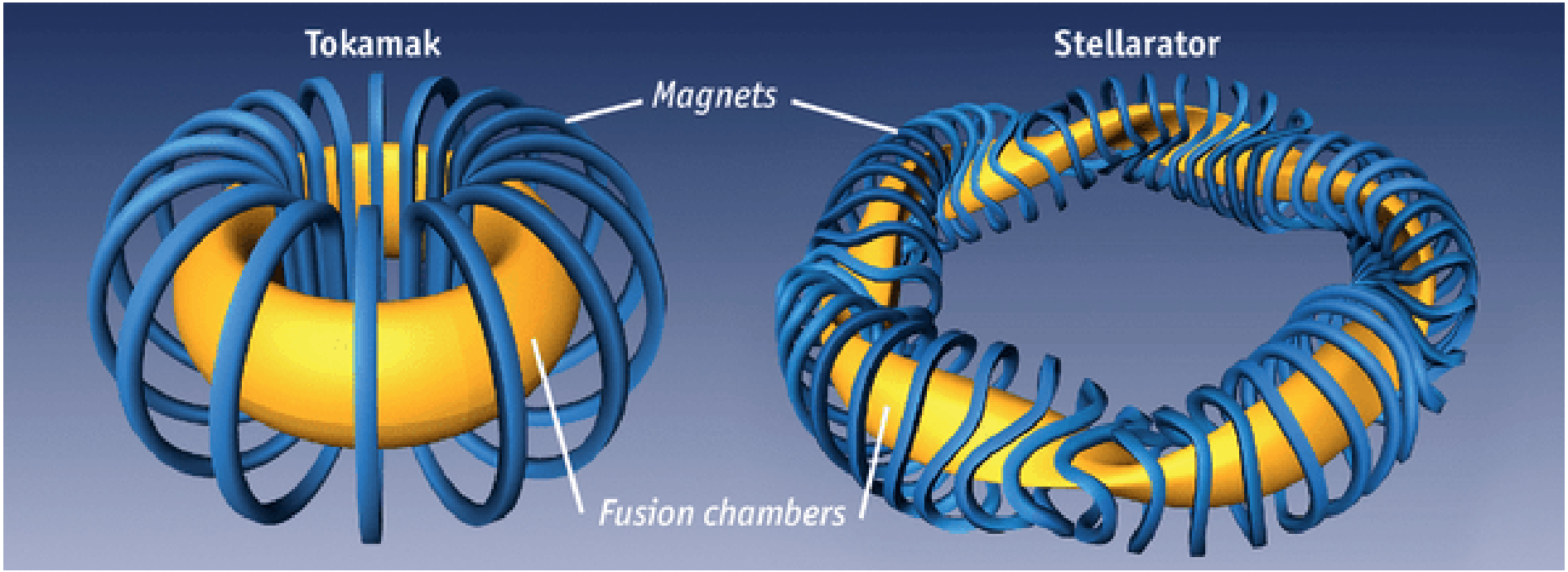
$$\delta\mathbf{B} = \oint \left[\frac{(\delta\mathbf{x} \times \mathbf{x}' \cdot \mathbf{r})\, 3\,\mathbf{r}}{r^5} - \frac{\delta\mathbf{x} \times \mathbf{x}'}{r^3}\right] dl \tag{3}$$

$$= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\,\mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3}\right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, \tag{4}$$

Early Stellarators.



In stellarators, the coils provide the poloidal field.

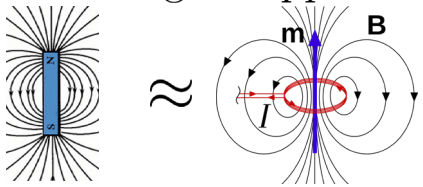


Magnetic field can be provided by permanent magnets. Can optimize shape of coils and positions of dipoles.

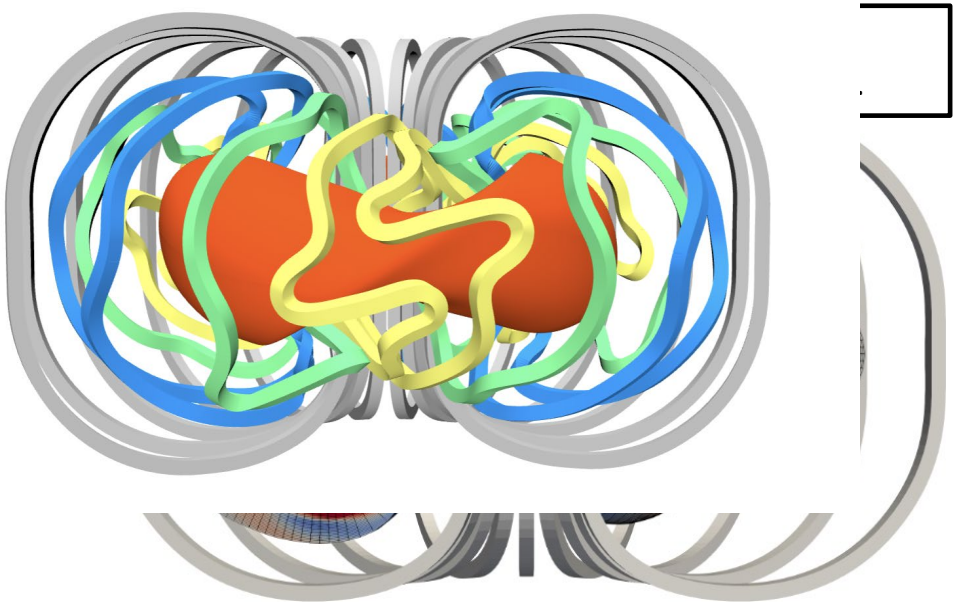
1. Vector potential from sheet current on $\mathcal{W}$ equivalent to continuous normal magnetization,

$$\mathbf{A} = \int_{\mathcal{W}} \frac{\mathbf{K}}{r} ds = \int_{\mathcal{W}} \frac{\mathbf{n} \times \nabla \Phi}{r} ds = \int_{\mathcal{W}} \frac{\Phi\, \mathbf{n} \times}{r^3}$$

2. Permanent magnet approximated by dipole (



4. $$\mathbf{B} = \underbrace{\sum_i^{N_C} I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3} dl}_{coils\ (curves)} + \nabla \times \underbrace{\sum_i^{N_D} \frac{\mathbf{m}_i \times \mathbf{r}}{r^3}}_{dipoles\ (arrows)}$$



5. Can optimize (i) the coil geometry, and (ii) the position, orientation and magnitude of dipoles.

M. Zarnstorff, C. Zhu, P. Helander, M. Drevlak, M. Landreman *et al.*,