From Chirikov's island overlap criterion, to cantori, and ghost-surfaces.

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Abstract

A brief review of the KAM theorem, Chirikov's island overlap criterion and Greene's residue criterion will show how these widely-quoted ideas can be simply understood by considering how far a given irrational number is from nearby low-order rationals.

Flux surfaces are broken by islands and chaos, but in a very meaningful sense they do not completely disappear, at least not immediately. Graphical evidence showing the importance of cantori in restricting both fieldline transport and heat transport in partially chaotic magnetic fields will be given.

Two classes of almost-invariant surfaces, namely quadratic-flux minimizing surfaces and ghost surfaces, which serve as “replacement” flux surfaces after the destruction of invariant surfaces, can be derived quite simply from classical action principles and are shown to be equivalent.

In the beginning, there was Hamiltonian mechanics

Hamilton, Lagrange, et *al*. identified integrals of motion,
Boltzmann's postulated the ergodic hypothesis,
Poincaré described the "chaotic tangle".

Quotations from ["Regular and Chaotic Dynamics", by Lichtenberg & Lieberman]

1) "These deep contradictions between the existence of integrability and the existence of ergodicity were symptomatic of a fundamental unsolved problem of classical mechanics."

2) "Poincaré contributed to the understanding of these dilemmas by demonstrating the extremely intricate nature of the motion in the vicinity of the unstable fixed points, a first hint that regular applied forces may generate stochastic motion in nonlinear oscillator systems."

3) "Birkhoff showed that both stable and unstable fixed points must exist whenever there is a rational frequency ratio (resonance) between two degrees of freedom."

4) "..the question of the ergodic hypothesis, whether a trajectory explores the entire region of the phase space that is energetically available to it, or whether it is constrained by the existence of constants of the motion, was not definitively answered until quite recently. The KAM theorem, originally postulated by Kolomogorov (1954), and proved under different restrictions by Arnold (1963) and Moser (1962) . .

Classical Mechanics 101:

The action integral is a functional of a curve in phase space.

1. The action, S, is the line integral along an arbitrary "trial" curve $\{\mathcal{C} : q \equiv q(t)\}$, of the Lagrangian,

$$\mathcal{L} \equiv \underbrace{T(\dot{q}, q)}_{\text{kinetic}} - \underbrace{U(q, t)}_{\text{potential}}, \quad S \equiv \int_{\mathcal{C}} \mathcal{L}(q, \dot{q}, t) dt$$

2. For magnetic fields, $\mathbf{B}$, the action is the line integral, of the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$,

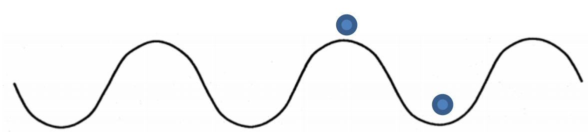
$$S \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}, \ \text{ along } \{\mathcal{C} : \theta \equiv \theta(\zeta), \rho \equiv \rho(\theta)\}.$$

3. Physical trajectories (magnetic fieldlines) extremize the action:

$$\delta S = \int_{\mathcal{C}} d\zeta \left(\delta\theta \frac{\partial S}{\partial \theta} + \delta\rho \frac{\partial S}{\partial \rho} \right), \text{ where } \boxed{\frac{\partial S}{\partial \theta} \equiv \sqrt{g} B^\rho - \dot{\rho} \sqrt{g} B^\zeta} \text{ and } \boxed{\frac{\partial S}{\partial \rho} \equiv \dot{\theta} \sqrt{g} B^\zeta - \sqrt{g} B^\theta}.$$

extremal curves satisfy $\dot{\rho} = B^\rho / B^\zeta$, and $\dot{\theta} = B^\theta / B^\zeta$.

4. Action-extremizing, periodic curves may be minimizing or minimax.



5. [Ghost surfaces are defined by an action-gradient flow between the minimax and minimizing periodic orbit.]

1954 : Kolmogorov, Dokl. Akad. Nauk SSSR 98, 469 ,1954

1963 : Arnold, Russ. Math. Surveys 18, 9,1963

1962 : Moser, Nachr. Akad. Wiss. Goett. II, Math.-Phys. Kl. 1, 1,1962

1. A dynamical system is integrable if there exists action-angle (ψ, θ) s.t. $H = H_0(\psi)$.
2. Arbitrary perturbation $H = H_0(\psi) + \sum_{m,n} H_{m,n}(\psi) \exp[i(m\theta - n\zeta)]$, where $\zeta \equiv t$ is "time".
3. Generating function to new action-action coordinates, $(\bar{\psi}, \bar{\theta})$, is

$$S(\bar{\psi}, \bar{\theta}) = \bar{\psi} \cdot \bar{\theta} + i \sum \frac{H_{m,n}}{(m\,\dot{\theta} - n)} \exp[i(m\theta - n\zeta)]. \tag{1}$$

   i. small denominators: rationals are dense; $\exists (m,n)$ s.t. $m\,\dot{\theta} - n$ is arbitrarily small.

4. KAM: adjust ψ, iteratively, to ensure that $\iota \equiv \dot{\theta}$ is sufficiently irrational,

Diophantine condition $\boxed{\left| \iota - \frac{n}{m} \right| > \frac{r}{m^k}}$, for all n & m, where $r \geq 0$ and $k > 1$.

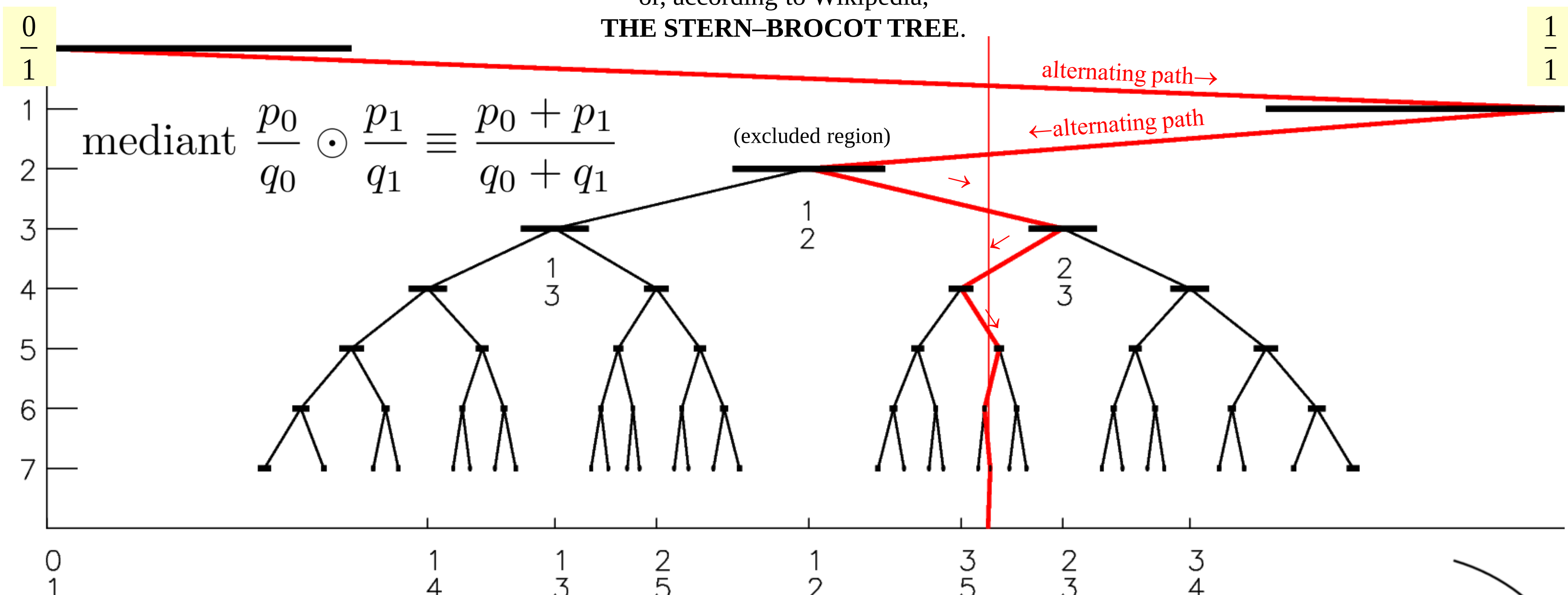
5. If ι is sufficiently irrational then for sufficiently small $H_{m,n}(\psi)$, Eqn(1) converges.

   i. action-angle coordinates can be constructed locally if $\iota \equiv \dot{\theta}$ is irrational.

"one of the most important concepts is labelling orbits by their frequency" [J. D. Meiss, Reviews of Modern Physics, 64(3):795 (1992)]

The structure of phase space is related to the structure of rationals and irrationals.

THE FAREY TREE;
or, according to Wikipedia,
THE STERN–BROCOT TREE.



1. Islands, and chaos, emerge at every rational:

   about each rational, n/m, introduce "excluded region" with width r/m^k; if excluded regions don't overlap, then

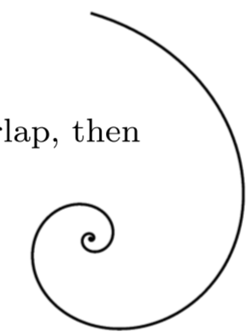
2. KAM theorem: irrational flux surface can survive if $\underbrace{|\iota - n/m| > r/m^k \text{ for all } n, m.}_{\text{Diophantine condition}}$

   Call ι *strongly irrational.*

3. Greene's residue criterion: the most robust flux surfaces have "noble" transform:

   noble irrationals $\equiv$ limit of ultimately alternating paths $\equiv$ limit of Fibonacci ratios;

   e.g. $\frac{0}{1}, \frac{1}{0}, \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \cdots \to \gamma \equiv$ golden mean $\equiv \frac{(1+\sqrt{5})}{2}$; e.g. $\frac{1}{0}, \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \cdots \to \gamma^{-1}$.



The standard map is a simple and widely-used model of chaotic dynamics . . .

1. Standard map: (sometimes also called Chirikov-Taylor map);

$$\begin{aligned} r_{n+1} &= r_n + k \sin\theta_n, \quad \text{where } k \text{ is perturbation} \\ \theta_{n+1} &= \theta_n + r_{n+1} \end{aligned}$$

   i. For $k = 0$, motion is integrable: $r = const.$, $\theta = \theta_0 + n\, r$;

   ii. For $k \neq 0$, islands and "chaotic" $\equiv$ irregular trajectories emerge;

   iii. For $k > k_c$, no invariant surfaces (except inside islands);

   iv. Definition : (m, n)-periodic orbit: $r_m = r_0 \quad \theta_m = \theta_0 + n$.

2. Linearized motion: $$\delta z_{n+1} \equiv \begin{pmatrix} \delta r \\ \delta\theta \end{pmatrix}_{n+1} = \underbrace{\begin{pmatrix} 1, & k\sin\theta_n \\ 1, & 1 + k\sin\theta_n \end{pmatrix}}_{\text{tangent map, } M_n} \begin{pmatrix} \delta r \\ \delta\theta \end{pmatrix}_n$$

   i. mapping is area preserving, therefore $det|M_n| = 1$.

3. Linearized motion of (m, n)-periodic orbits, $\delta z_m = M_m \ldots M_1 M_0, \delta z_0$

   i. "stability" of periodic orbits determined by eigenvalues, λ_i, of $M^m \equiv M_m \ldots M_1 M_0$;

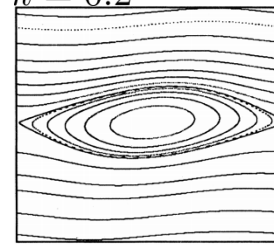
      X: hyperbolic: $\lambda_2 = 1/\lambda_1$; λ_i real; $|\lambda_1| > 1$; unstable;

      O: elliptic: λ_i complex congugates; $|\lambda_i| = 1$; stable;

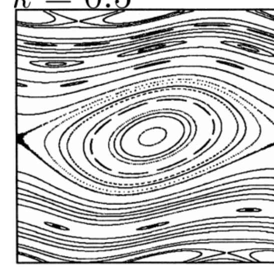
   ii. As k increases, eventually all periodic orbits become unstable.

4. For given k, which flux surfaces exist? What is k_c?

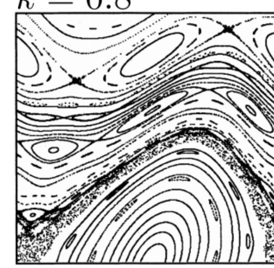
$k = 0.2$



$k = 0.5$



$k = 0.8$

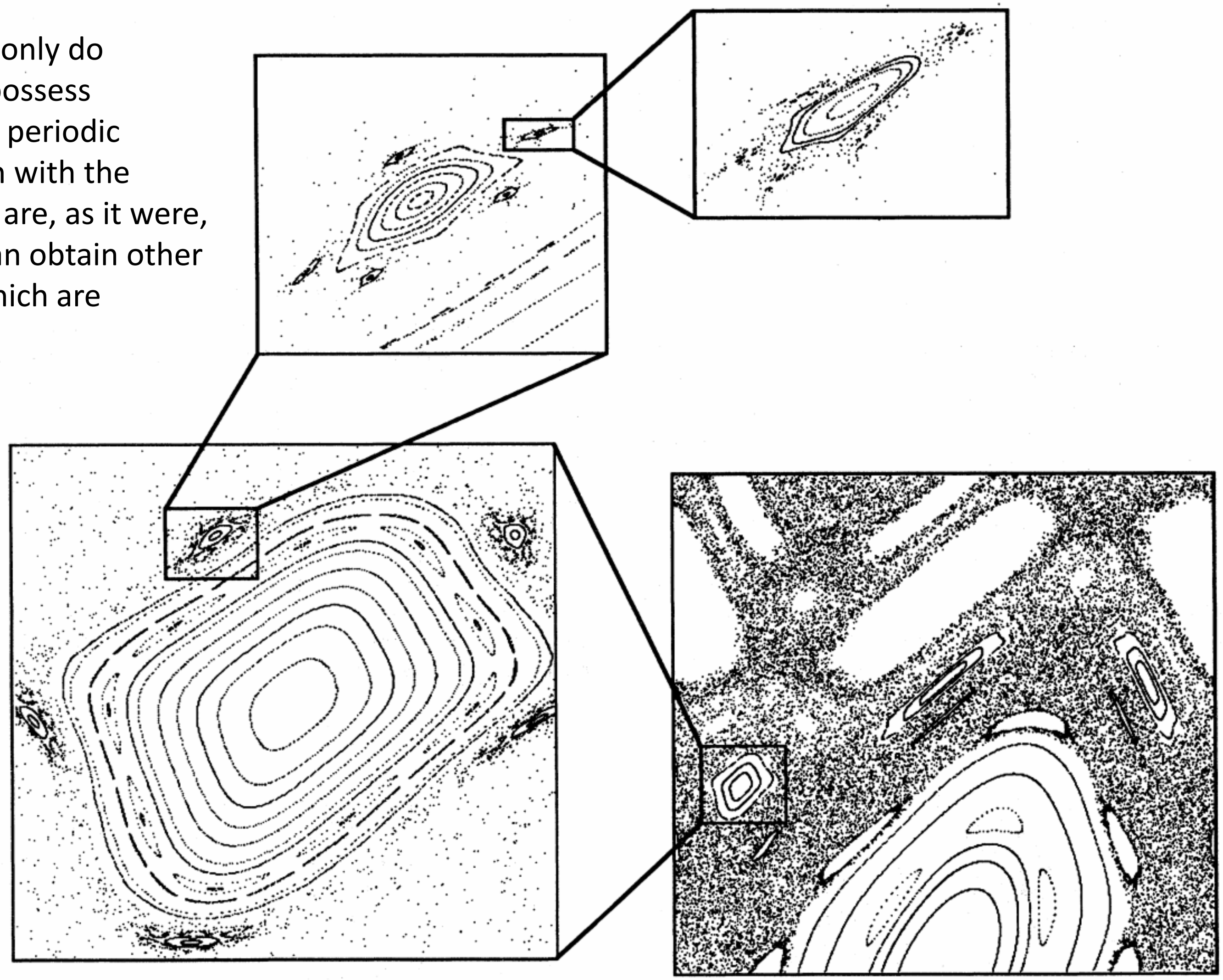


$k_c = 0.9716354$

The standard map is very simple, but the trajectories are very complicated.

There are islands around islands around islands . . .

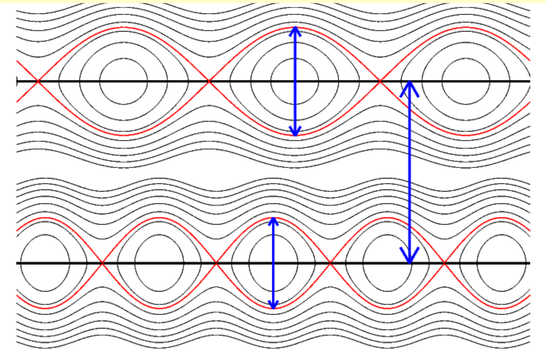
Birkhoff (1935) "It is clear that not only do general elliptic periodic solutions possess neighboring elliptic and hyperbolic periodic solutions, but also, beginning again with the neighboring elliptic solutions, who are, as it were, satellites of these solutions, one can obtain other elliptic and hyperbolic solutions which are secondary satellites."



["Symplectic maps, variational principles, and transport", J. D. Meiss, Reviews of Modern Physics, 64(3):795 (1992)]

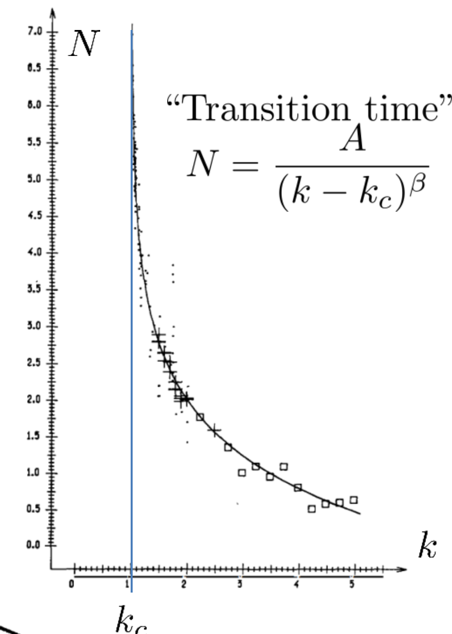
1979 : Chirikov’s Island-Overlap Criterion

PHYSICS REPORTS 52, 5(1979)263-37



1. Can estimate resonance $\equiv$ “island” width
   - i. single resonance Hamiltonian $H = \frac{1}{2}\psi^2 + \epsilon\cos(m\theta) = E = const.$
   - ii. $\psi = \pm\sqrt{2[E - \epsilon\cos(m\theta)]}$
   - iii. for separatrix, choose $E = \epsilon$; island width $w = 4\sqrt{\epsilon}$

2. Introduce island overlap criterion: if $\boxed{\frac{w_1}{2} + \frac{w_2}{2} > \Delta\psi}$, then chaos.
   - i. for standard map, predicted $k_c \approx 2.5$.



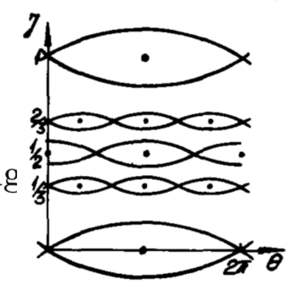
“Transition time”

$$N = \frac{A}{(k - k_c)^\beta}$$

3. However, numerical experiments show $k_c \approx 0.989$
   - i. plot $N \equiv$ # iterations to leave given domain against k;
   - ii. two free parameters, k_c and β;

4. “The overlap is not only provided by integer resonances, but also by higher harmonic resonances.”
   - i. If additional resonances are considered, the predicted k_c chang

     e.g. $k_{1,1} = 2.5$, $k_{1,2} = 1.46$, $k_{1,3} = 0.90$, $k_{2,3} = 1.35$, . .



1979 : Greene's Residue Criterion

J. Math. Phys. 20(6), 1979, 1183 (1979)

1. The existence of an irrational flux surface is related to the stability of nearby periodic orbits!

   i. periodic orbits are convenient because they have finite length

   and ii. because they are guaranteed to exist [Poincaré-Birkhoff theorem]

2. Construct a sequence of rationals that converge to the irrational, $\lim\limits_{i\to\infty} \dfrac{n_i}{m_i} = \iota\!\!\!-$

   $$\text{e.g. } \frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34}, \frac{89}{55}, \frac{144}{89}, \frac{233}{144}, \frac{377}{233} \cdots \to \frac{1+\sqrt{5}}{2} = \text{golden mean}$$

3. Introduce the residue, $R_{n/m}$, defined on the periodic orbits, which measures stability:

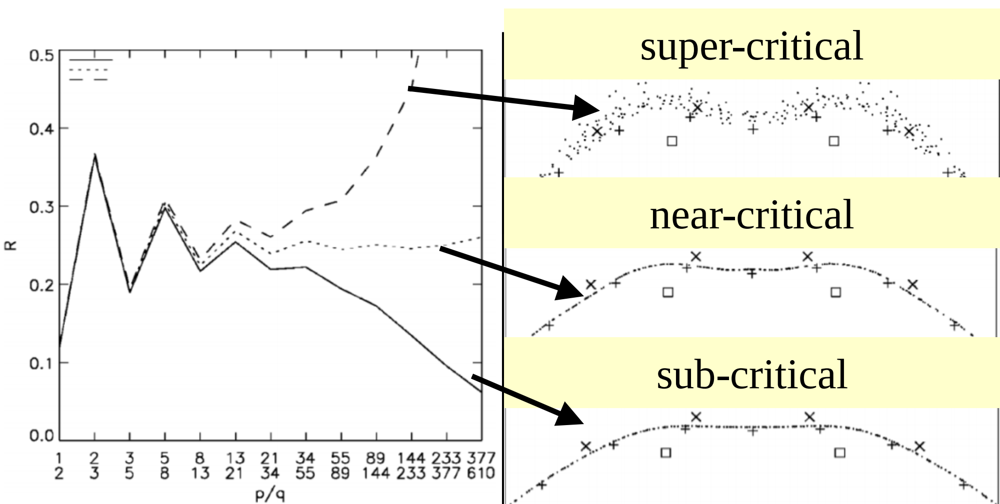
   $$\boxed{R_{n/m} \equiv \frac{1}{4}\left[2 - \lambda_1 - \lambda_2\right]}\text{, where } \lambda_i \text{ are eigenvalues of tangent map, } M^m.$$

   i. if $R_{n_i/m_i} \to 0$, surface exists

   ii. if $R_{n_i/m_i} \to \frac{1}{4}$, surface is critical

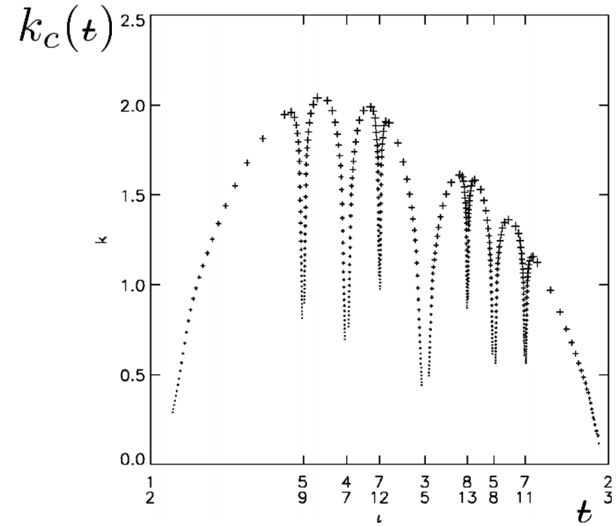
   iii. if $R_{n_i/m_i} \to \infty$, surface destroyed

4. $k_c = 0.971635406...$



Standard Map critical function is similar to the Bruno function

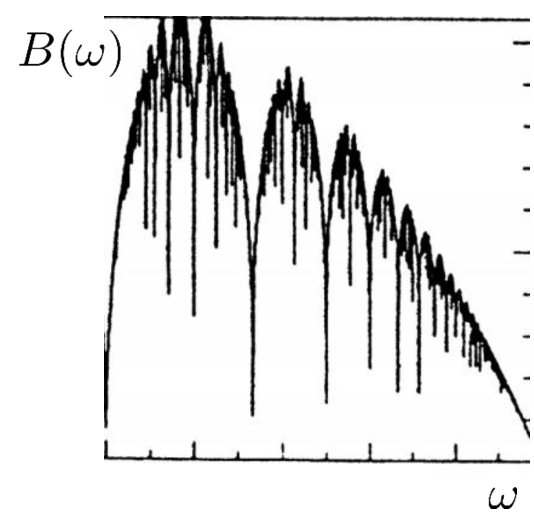
1. Define $k_c(\iota)$ as the largest value of k for which an ι invariant curve exists;
   - i. the critical function peaks on strongly irrationals:
     $k_c(\iota) > 0$ if ι is irrational;
   - ii. rational surfaces do not exist:
     $k_c\,(\iota) = 0$ if $\iota = n/m$.
   - iii. k_c is everywhere discontinuous.



2. The critical function has a form very similar to the Bruno function,

$$B(\omega) = -\log\omega + \omega B(\omega^{-1}), \quad B(\omega) = B(\omega+1) = B(-\omega).$$

   - i. $B(\omega)$ is more simply calculated as a function of the continued fraction representation.

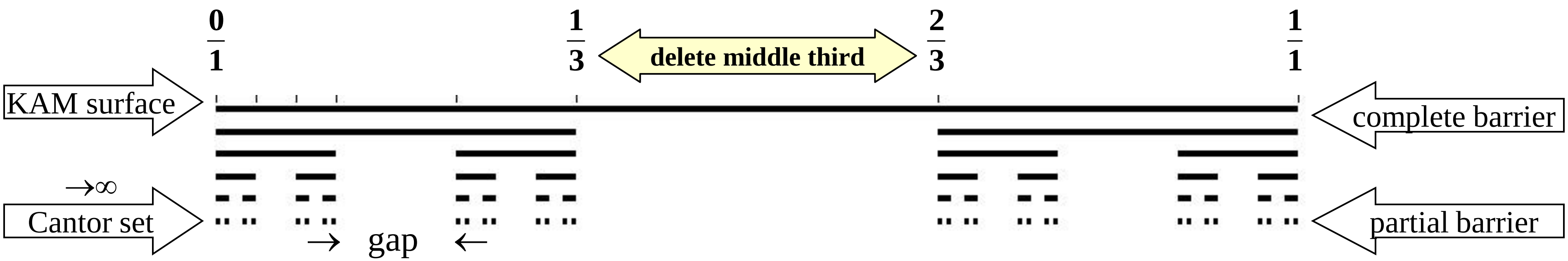


3. Marmi & Stark [Nonlinearity, 1992] gave evidence that

$$C_\beta \equiv \log(\,k_c(\omega)\,) + \beta B(\omega)$$

   is continuous.
   - i. i.e., that the critical function and the Bruno function have the same fractal structure.

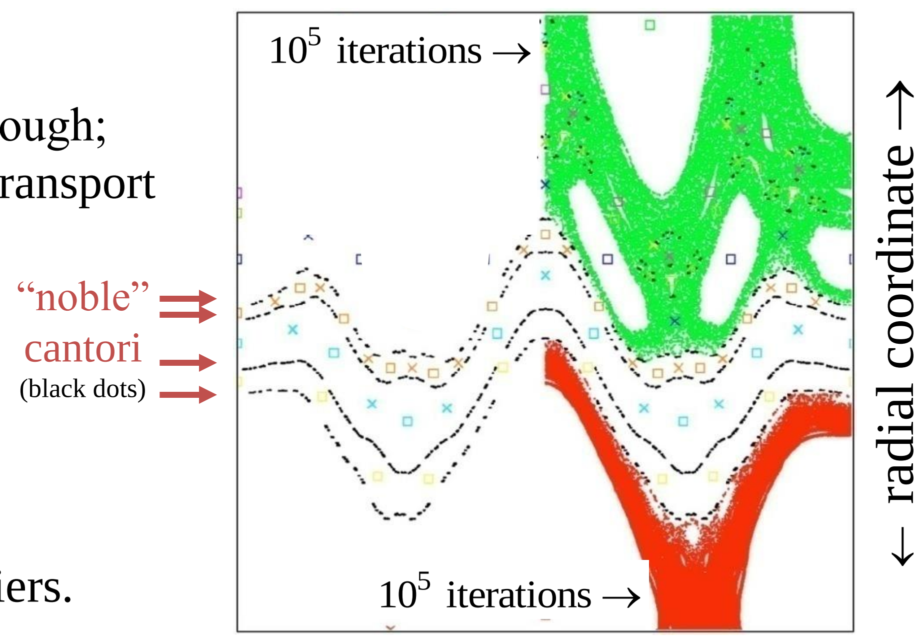
Irrational KAM surfaces break into cantori when perturbation exceeds critical value.
Both KAM surfaces and cantori restrict transport.



→ KAM surfaces are closed, toroidal surfaces that **stop** radial field line transport

→ Cantori have "gaps" that fieldlines can pass through; however, **cantori can severely restrict** radial transport

→ Example: all flux surfaces destroyed by chaos, but even after **100 000 transits** around torus the fieldlines **don't get past cantori !**

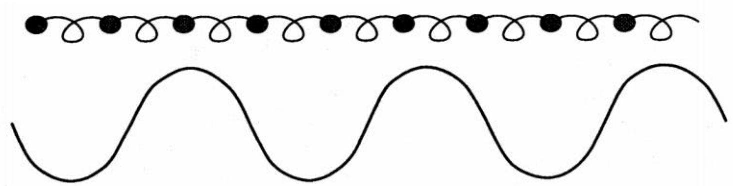


→ Regions of chaotic fields can provide some confinement because of the cantori partial barriers.

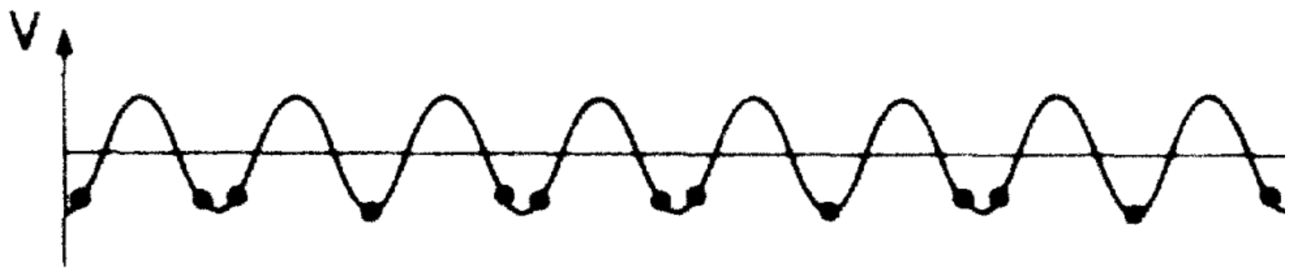
Simple physical picture of cantori

[Percival, 1979]

1. Consider masses, m, linked by springs in a periodic potential.
2. For $m = 0$, potential is irrelevant: minimum energy state has masses equally spaced.



3. For large m, springs are irrelvant: all the masses lie at the potential minimum, and there are “gaps”.



[Schellnhuber, Urbschat & Block, Physical Review A, 33(4):2856 (1986)]

The construction of extremizing <u>**curves**</u> of the <u>**action**</u> generalized extremizing <u>**surfaces**</u> of the <u>**quadratic-flux**</u>

1. $\delta S = \int_{\mathcal{C}} d\zeta \left(\delta\theta \frac{\partial S}{\partial \theta} + \delta\rho \frac{\partial S}{\partial \rho} \right)$, where $\boxed{\frac{\partial S}{\partial \theta} \equiv \sqrt{g} B^\rho - \dot{\rho} \sqrt{g} B^\zeta}$ and $\boxed{\frac{\partial S}{\partial \rho} \equiv \dot{\theta} \sqrt{g} B^\zeta - \sqrt{g} B^\theta}$.
2. Extremal curves satisfy $\frac{\partial S}{\partial \theta} = 0$, i.e. $\dot{\rho} = B^\rho / B^\zeta$, and $\frac{\partial S}{\partial \rho} = 0$, i.e. $\dot{\theta} = B^\theta / B^\zeta$.
3. Introduce toroidal surface, $\rho \equiv P(\theta, \zeta)$, and *family* of angle curves, $\theta_\alpha(\zeta) \equiv \alpha + p\,\zeta/q + \tilde{\theta}(\zeta)$, where α is a fieldline label; p and q are integers that determine periodicity; and $\tilde{\theta}(0) = \tilde{\theta}(2\pi q) = 0$.
4. On *each* curve, $\rho_\alpha(\zeta) = P(\theta_\alpha(\zeta), \zeta)$ and $\theta_\alpha(\zeta)$, can <u>enforce</u> $\frac{\partial S}{\partial \rho} = 0$; generally $\nu \equiv \frac{\partial S}{\partial \theta} \neq 0$.
5. The *pseudo* surface dynamics is defined by $\dot{\theta} \equiv B^\theta / B^\zeta$ and $\dot{\rho} \equiv \partial_\theta P \, \dot{\theta} + \partial_\zeta P$.
6. Corresponding *pseudo* field $\mathbf{B}_\nu \equiv \dot{\rho} \, B^\zeta \, \mathbf{e}_\rho + \dot{\theta} \, B^\zeta \, \mathbf{e}_\theta + B^\zeta \mathbf{e}_\zeta$; simplifies to $\mathbf{B}_\nu = \mathbf{B} - \frac{\nu}{\sqrt{g}} \mathbf{e}_\rho$.
7. Introduce the <u>**quadratic-flux functional**</u>: $\boxed{\varphi_2 \equiv \frac{1}{2} \iint d\theta d\zeta \left(\frac{\partial S}{\partial \theta} \right)^2}$
8. Allowing for δP, the first variation is $\delta \varphi_2 = \iint d\theta d\zeta \; \delta P \, \sqrt{g} \, \underbrace{\left(B^\theta \partial_\theta + B^\zeta \partial_\zeta \right) \nu}_{\text{Euler-Lagrange for QFMs}}$.

The action gradient, ν, is constant along the pseudo fieldlines; construct Quadratic Flux Minimzing (QFM) surfaces by pseudo fieldline (local) integration.

1. The *true* fieldline flow along $\mathbf{B}$ around q toroidal periods from (θ_0, ρ_0) produces a mapping, $\begin{pmatrix} \theta_q \\ \rho_q \end{pmatrix} = M^q \begin{pmatrix} \theta_0 \\ \rho_0 \end{pmatrix}$.

2. Periodic fieldlines are fixed points of M^q, i.e. $\theta_q = \theta_0 + 2\pi p,\ \rho_q = \rho_0$.

3. In integrable case: given θ_0, a one-dimensional search in ρ is required to find the *true* periodic fieldline.

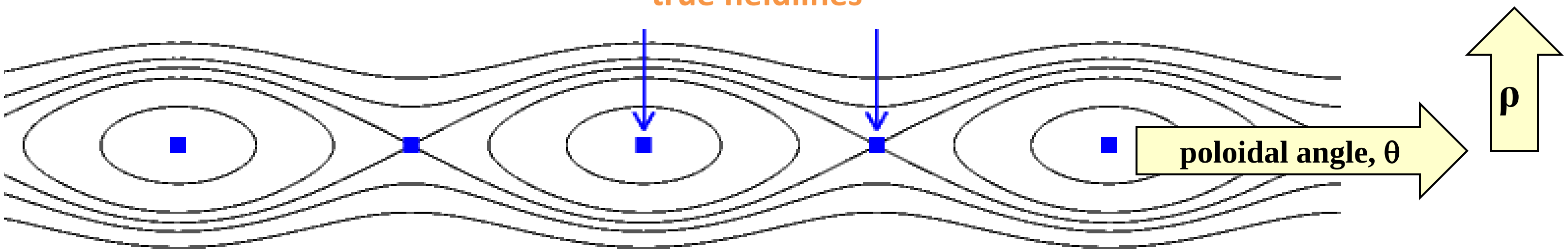
4. In non-integrable case, only the
   (i) "stable" (action-minimax), O, (which is not always stable), and the
   (ii) unstable (action minimizing), X, periodic fieldlines are guaranteed to survive.

5. The *pseudo* fieldline flow along $\mathbf{B}_\nu = \mathbf{B} - \dfrac{\nu}{\sqrt{g}}\mathbf{e}_\rho$ around q periods from (θ_0, ρ_0) produces a mapping, $\begin{pmatrix} \theta_q \\ \rho_q \end{pmatrix} = P^q \begin{pmatrix} \nu \\ \rho_0 \end{pmatrix}$, but ν is not yet known.

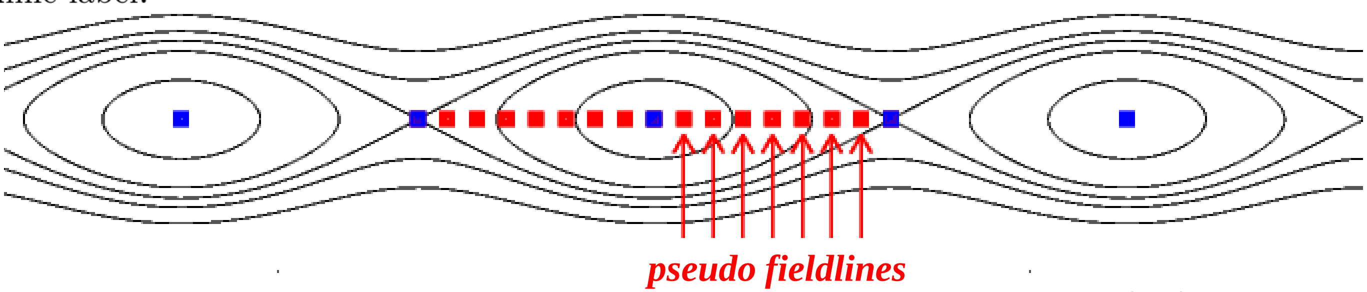
6. In general case: given θ_0, a two-dimensional search in (ν, ρ) is required to find the periodic *pseudo* fieldline.

At each poloidal angle, compute radial "error" field that must be subtracted from **B** to create a periodic curve, and so create a rational, pseudo flux surface.

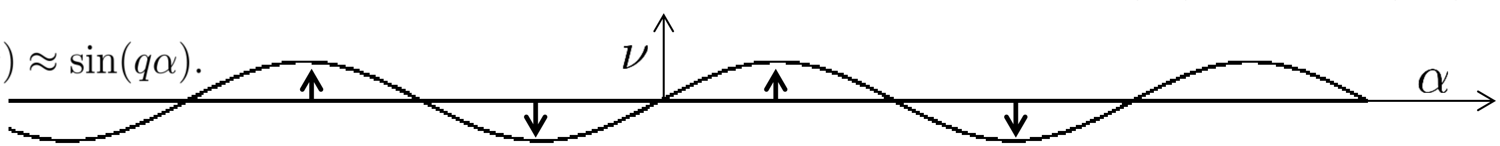
0. Usually, there are only the "stable" periodic fieldline and the unstable periodic fieldline,



1. At every $\theta = \alpha$, determine $\nu(\alpha)$ via numerical search so that $\mathbf{B} - \nu\, \mathbf{e}_\rho/\sqrt{g}$ yields a periodic integral curve; where α is a fieldline label.



2. At the true periodic fieldlines, the required additional radial field is zero: i.e. $\nu(\alpha_0) = 0$ and $\nu(\alpha_X) = 0$.
3. Typically, $\nu(\alpha) \approx \sin(q\alpha)$.



4. The pseudo fieldlines "capture" the true fieldlines; QFM surfaces pass through the islands.

Alternative Lagrangian integration construction:

QFM surfaces are families of extremal curves of the constrained-area action integral.

1. Introduce $F(\boldsymbol{\rho}, \boldsymbol{\theta}) \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l} - \nu \left(\int_{\mathcal{C}} \theta \nabla \zeta \cdot d\mathbf{l} - a \right)$, where $\boldsymbol{\rho} \equiv \{\rho_i\}$, $\boldsymbol{\theta} \equiv \{\theta_i\}$;
   where ν is a Lagrange multiplier, and a is the required "area", $\int_0^{2\pi q} \theta(\zeta)\, d\zeta$.

2. An identity of vector calculus gives $\delta F = \int_{\mathcal{C}} d\mathbf{l} \times (\nabla \times \mathbf{A} - \nu \nabla \theta \times \nabla \zeta) \cdot \delta \mathbf{l}$,
   extremizing curves are tangential to $\mathbf{B} - \nu \nabla \theta \times \nabla \zeta = \mathbf{B} - \dfrac{\nu}{\sqrt{g}} \mathbf{e}_\rho = \mathbf{B}_\nu$.

3. Constrained-area action-extremizing curves satisfy $\dfrac{\partial F}{\partial \rho_i} = 0$ and $\dfrac{\partial F}{\partial \theta_i} = 0$.

4. The piecewise-constant representation for $\rho(\zeta)$ and $\partial_{\rho_i} F = 0$ yields $\rho_i = \rho_i(\theta_{i-1}, \theta_i)$, so the trial curve is completely described by θ_i, i.e. $F \equiv F(\boldsymbol{\theta})$.

5. The piecewise-linear representation for $\theta(\zeta)$ gives $\dfrac{\partial F}{\partial \theta_i} = \partial_2 F_i(\theta_{i-1}, \theta_i) + \partial_1 F_{i+1}(\theta_i, \theta_{i+1})$,
   so the Hessian, $\nabla^2 F(\boldsymbol{\theta})$, is tridiagonal (assuming ν is given) and is easily inverted.

6. Multi-dimensional Newton method: $\delta \boldsymbol{\theta} = -\left(\nabla^2 F\right)^{-1} \cdot \nabla F(\boldsymbol{\theta})$;
   global integration, <u>much</u> less sensitive to "Lyapunov" integration errors.

Ghost surfaces, another class of almost-invariant surface, are defined by an action-gradient flow between the action minimax and minimizing fieldline.

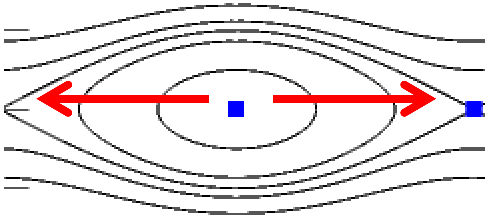
1. Action, $S[\mathcal{C}] \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$, and action gradient, $\dfrac{\partial S}{\partial \theta} \equiv \sqrt{g} B^\rho - \dot{\rho} B^\zeta$.

2. Enforce $\dfrac{\partial S}{\partial \rho} \equiv \dot{\theta} B^\zeta - \sqrt{g} B^\theta = 0$, i.e. invert $\dot{\theta} \equiv B^\theta / B^\zeta$ to obtain $\rho = \rho(\dot{\theta}, \theta, \zeta)$; so that trial curve is completely described by $\theta(\zeta)$, and the action reduces from $S \equiv S[\rho(\zeta), \theta(\zeta)]$ to $S \equiv S[\theta(\zeta)]$

3. Define action-gradient flow: $\boxed{\dfrac{\partial \theta(\zeta;\tau)}{\partial \tau} \equiv -\dfrac{\partial S[\theta]}{\partial \theta}}$, where τ is an arbitrary integration parameter.

4. Ghost-surfaces are constructed as follows:

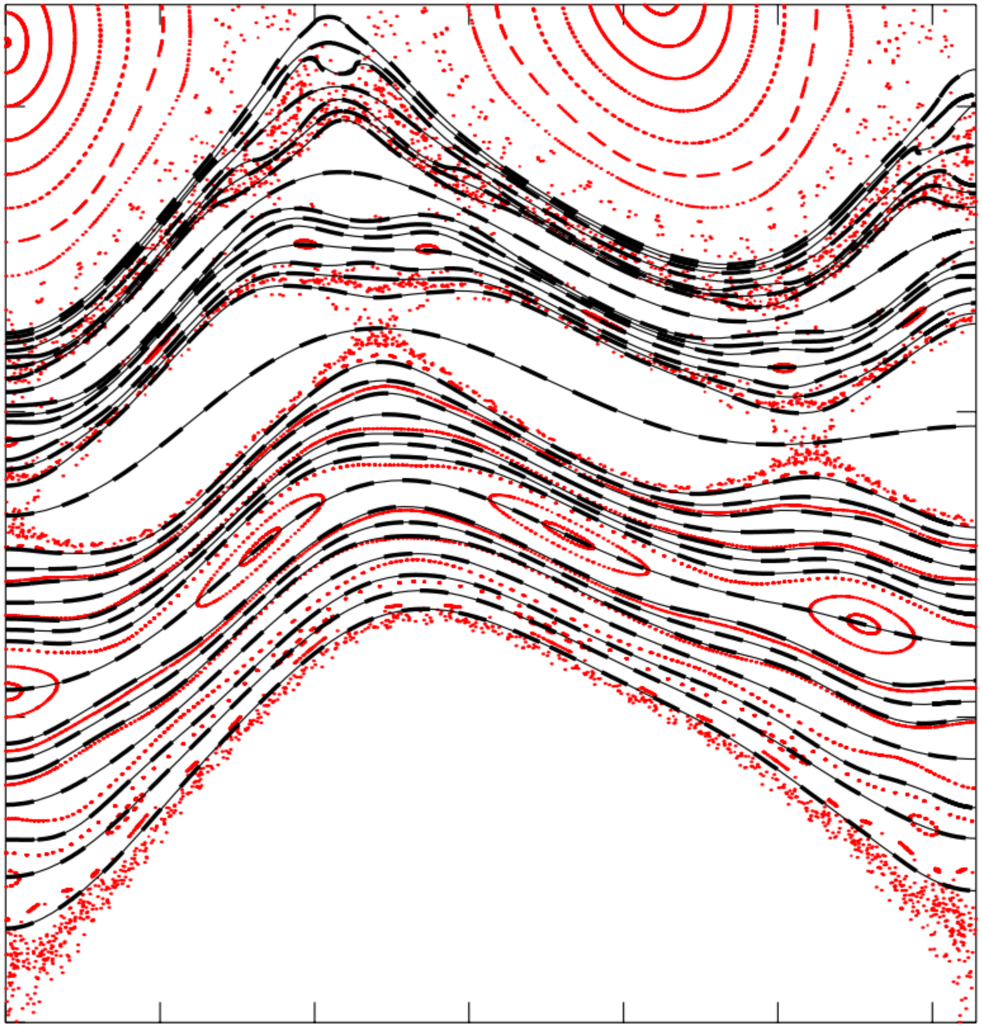
    - Begin at action-minimax ("O", "not-always-stable") periodic fieldline, which is a saddle;
    - initialize integration in decreasing direction (given by negative eigenvalue/vector of Hessian);
    - the entire curve "flows" down the action gradient, $\partial_\tau \theta = -\partial_\theta S$;
    - action is decreasing, $\partial_\tau S < 0$;
    - finish at action-minimizing ("X", unstable) periodic fieldline.
    - ghost surface described by $\mathbf{x}(\zeta, \tau)$, where τ is a fieldline label.



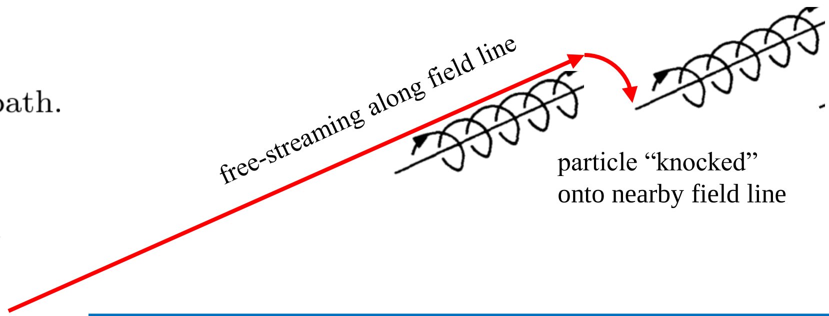
Ghost surfaces are (almost) indistinguishable from QFM surfaces

can redefine poloidal angle to unify ghost surfaces with QFMs.

1. Ghost-surfaces are defined by an (action gradient) flow.
2. QFM surfaces are defined by minimizing $\int$(action gradient)$^2 d\mathbf{s}$.
3. Not obvious if the different definitions give the same surfaces.
4. For model chaotic field:
   (a) ghosts = thin solid lines;
   (b) QFMs = thick dashed lines;
   (c) agreement is excellent;
   (d) difference = $\mathcal{O}(\epsilon^2)$, where ϵ is perturbation.
5. Can redefine θ to obtain unified theory of ghosts & QFMs; straight *pseudo* fieldline angle.



Isotherms of the steady state solution to the anisotropic diffusion coincide with ghost surfaces; analytic, 1-D solution is possible.



1. Transport along the magnetic field is unrestricted:
   e.g. parallel random walk with long steps ≈ collisional mean free path.

2. Transport across the magnetic field is very small:
   e.g. perpendicular random walk with short steps ≈ Larmor radius.

3. Simple transport model: anisotropic diffusion,
   $\boxed{\kappa_\parallel \nabla_\parallel^2 T + \kappa_\perp \nabla_\perp^2 T = 0}$, $\quad \kappa_\perp / \kappa_\parallel \sim 10^{-10}$, grid $= 2^{12} \times 2^{12}$.
   steady state, no source, inhomogeneous boundary conditions.

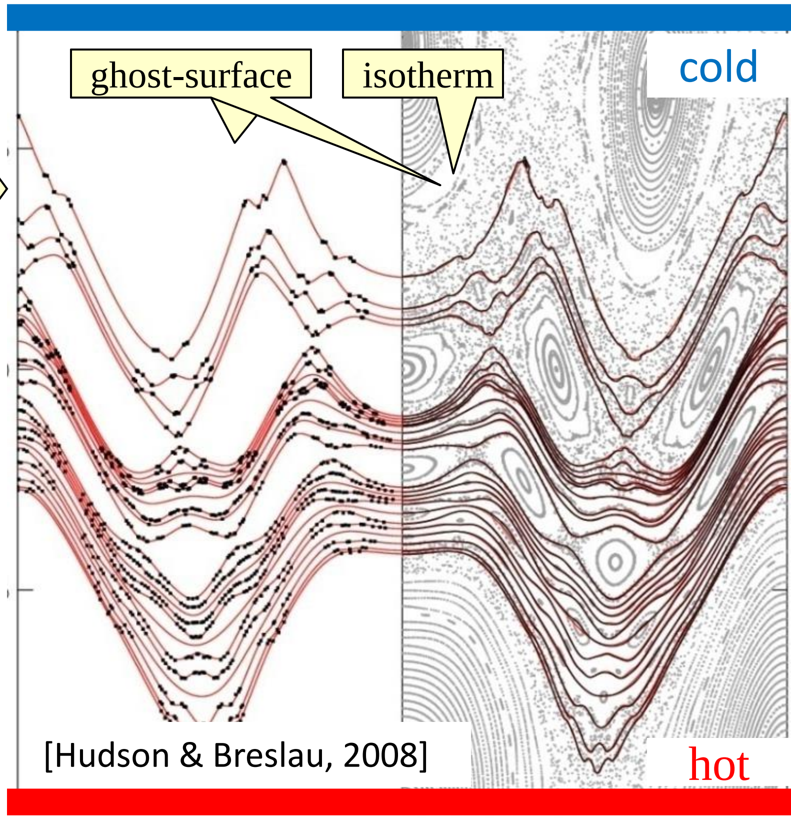
4. Compare numerical solution to "irrational" ghost-surfaces

5. The temperature adapts to KAM surfaces, cantori, **and ghost-surfaces!**, i.e. $T = T(\rho)$.

6. From $T = T(\rho, \theta, \zeta)$ to $\boxed{T = T(\rho)}$ allows an expression for the temperature gradient in chaotic fields:

$$\boxed{\frac{dT}{d\rho} \propto \frac{1}{\kappa_\parallel \varphi_2 + \kappa_\perp G}},$$

where $\varphi_2 \equiv \underbrace{\int B_n^2 \, d\mathbf{s}}_{\text{quadratic flux}}$, and $G \equiv \underbrace{\int \nabla\rho \cdot \nabla\rho \, d\mathbf{s}}_{\text{metric}}$.



Summary: Timeline of topics addressed in talk

(*not* a comprehensive history of Hamiltonian chaos!)

- Poincaré — unstable manifold (i.e. chaos)

- 1954 Kolmogorov — KAM theorem
- 1962 Moser
- 1963 Arnold

- 1979 Chirikov — island overlap criterion
- 1979 Greene — residue criterion [see also 1991 MacKay]

- 1979 Percival — can(tor + tor)us = cantorus
- 1982 Mather — Aubry-Mather theorem (showing existence of cantori)
- 1983 Aubry

- 1991 Angenent & Golé — ghost-circles
- 1991 Meiss & Dewar — quadratic-flux minimizing curves
- 2008 Hudson & Breslau — isotherms = ghost-surfaces
- 2009 Hudson & Dewar — ghost-surfaces = quadratic-flux minimizing surfaces

and texts:

- 1983 Lichtenberg & Lieberman — [Regular and Stochastic Motion]
- 1992 Meiss — [Reviews of Modern Physics]