

Resonant Pressure Driven Currents In and Near Magnetic Islands

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- Parallel transport large relative to perpendicular \rightarrow pressure flattened within flux surfaces, except in regions near
 - small magnetic islands,
 - X-lines of larger islands.
- Variation of pressure within flux surface has strong effect on pressure-driven current:
 - Can get singularity at X-line;
 - Singularity absent in MHD equilibria that are symmetric with respect to combined reflection in poloidal and toroidal angles. (“stellarator symmetry”.)
- Compare with:
 - In MHD equilibria with $\mathbf{B} \cdot \nabla p = 0$, singular component of current vanishes regardless of symmetry.
 - Simply nested flux surfaces have $1/x$ singularity, where x is distance from rational surface. (Quasi-symmetric surfaces are an exception.)

Of course, the current density is not truly singular.

Additional physics comes into play close to the rational surface, or to the X-line.

Some effects that can play this role:

- enhanced Pfirsch-Schlüter transport near the rational surface[Boozer];
 - field line stochasticity[Reiman; Krommes and Reiman];
 - FLR effects[Hazeltine and Catto].
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- Logarithmic singularity at X-line is integrable
 - physics insensitive to details of cutoff.
 - Breakup of surfaces to form islands ameliorates, but does not eliminate singularity seen in MHD equilibria with simply nested flux surfaces.

Why This May Be of Interest

- The major international stellarators are all stellarator symmetric.
 - Can we improve them further if we allow the breaking of stellarator symmetry?
 - Could potentially provide an additional degree of freedom. (Zarnstorff)
 - Effect on intrinsic equilibrium islands.
 - What are the consequences of breaking of stellarator symmetry by field errors?
- Glasser-Greene-Johnson effect on magnetic island stability, and associated term in modified Rutherford equation, mediated by the Pfirsch-Schlüter (PS) current in the neighborhood of the magnetic island. Important to get this right.
 - The widely used GGJ expressions were derived before it became widely known that $\mathbf{B} \cdot \nabla p \neq 0$.
 - Neoclassical tearing mode (NTM) threshold.
- Coupling between NTMs.
 - Single NTM can preserve symmetry, and second NTM breaks it.

Why This May Be of Interest (continued)

- DIII-D experiments see difference in response to resonant magnetic perturbations (RMPs) between stellarator-symmetric and nonsymmetric configurations.
 - Axisymmetric plasmas with balanced double-null divertors are stellarator-symmetric.
 - Axisymmetric plasmas with single-null divertors are not stellarator-symmetric.
 - Suppression in stellarator symmetric configuration has not been achieved, despite repeated efforts.
 - Was achieved in nonsymmetric configuration a decade ago.

Outline of Talk

- Why $\mathbf{B} \cdot \nabla p \neq 0$ near magnetic islands.
- What equations to use?
(Conventional MHD force balance equation implies)
- Solution of equations along field lines for j_{\parallel} via magnetic coordinates.
- Solution for magnetic coordinates and pressure driven current in two ways:
 - For analytically tractable magnetic field with an island.
 - More general solution by expansion around the X-line.
- Implications of stellarator symmetry.
- Physical Interpretation.

$\mathbf{B} \cdot \nabla p = 0$ violated near magnetic islands.

- In steady state: $\kappa_{\parallel} \nabla_{\parallel}^2 T + \kappa_{\perp} \nabla_{\perp}^2 T = 0$.
- $\kappa_{\parallel} \gg \kappa_{\perp} \Rightarrow$ pressure constant on flux surfaces through much of the plasma.
- Fitzpatrick 1995 paper: $\mathbf{B} \cdot \nabla p = 0$ not correct in neighborhood of island because long connection length reduces parallel transport.
 - Also Gorelenkov *et al* (1996), Gianakon, Hegna, J. D. Callen (1996), Hazeltine, Helander and Catto (1997), Hölzl, Günter, Yu, Lackner (2007).
- Near magnetic island separatrix:
 - Connection length $\sim 1 / (mq - n) \approx 1 / (mq' x)$ where x is distance from rational surface $\Rightarrow \nabla_{\parallel}^2 T \sim (mq' x)^2$.
 - $\delta T \sim 1 / x \Rightarrow \nabla_{\perp}^2 T \sim 1 / x^2$.
 - $\kappa_{\perp} \nabla_{\perp}^2 T$ term comparable to $\kappa_{\parallel} \nabla_{\parallel}^2 T$ when $x \approx (\kappa_{\perp} / \kappa_{\parallel})^{1/4} (mq')^{1/2} \equiv x_c$, and T then has strong variation within flux surface.

$\mathbf{B} \cdot \nabla p = 0$ violated near magnetic islands (continued)

- Variation of pressure within flux surfaces determined by width of island, w , relative to critical length scale, x_c .
 - For $w \ll x_c$, temperature profile unaffected by presence of island.
 - For islands with $w \gg x_c$, pressure flattened in island interior, but there remains region near X-line, extending radial distance of order x_c , where $\mathbf{B} \cdot \nabla p \neq 0$ significant.
- For DIII-D shot 115467, which achieved ELM suppression, and for which the relevant data has been published, $x_c \approx 1$ cm. (approximation from idealized analytical calculation). ion sound radius ≈ 0.3 cm.

Closed subset of MHD equilibrium equations imposes perpendicular force balance without constraint on $\mathbf{B} \cdot \nabla p$.

- Conventional form of MHD force balance equation, $\mathbf{j} \times \mathbf{B} = \nabla p$, implies $\mathbf{B} \cdot \nabla p = 0$. Is not valid near magnetic islands.
- Motivation: Keeping weak anisotropic terms in pressure tensor ($\mathbf{P} = p\mathbf{I} + \boldsymbol{\pi}$, with $|\nabla \cdot \boldsymbol{\pi}| \ll |\nabla p|$) and weak flow ($|\rho_m \mathbf{v} \cdot \nabla \mathbf{v}| \ll |\nabla p|$, where ρ_m is mass density):

$$\mathbf{j} \times \mathbf{B} - \nabla p = \rho_m \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \cdot \boldsymbol{\pi}.$$

- Parallel component of force balance equation:

$$\mathbf{B} \cdot \nabla p = -\mathbf{B} \cdot (\rho_m \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \cdot \boldsymbol{\pi}). \quad (1)$$

- Cannot neglect small terms on right hand side. They balance small but significant pressure gradient along the field lines.

- Writing $\mathbf{j} = j_{\parallel} \mathbf{B} / B + \mathbf{j}_{\perp}$, cross product of \mathbf{B} with Eq. (1) gives

$$\mathbf{j}_{\perp} = \mathbf{B} \times \nabla p / B^2 + \mathbf{B} \times (\rho_m \mathbf{v} \cdot \nabla \mathbf{v} + \nabla \cdot \boldsymbol{\pi}) / B^2.$$

- The small terms make a small contribution to \mathbf{j}_{\perp} .

$$\mathbf{j}_{\perp} = \mathbf{B} \times \nabla p / B^2. \quad (2)$$

MHD Equilibrium Equations (continued)

$$\mathbf{j}_{\perp} = \mathbf{B} \times \nabla p / B^2 \quad (2)$$

- Parallel component of current determined by $\nabla \cdot \mathbf{j} = 0$:

$$\mathbf{B} \cdot \nabla (j_{\parallel} / B) = -\nabla \cdot \mathbf{j}_{\perp}. \quad (3)$$

The “Pfirsch-Schlüter current”. Enforces quasi-neutrality

- The equations are closed by Ampere’s Law:

$$\nabla \times \mathbf{B} = \mathbf{j}(\mathbf{B}). \quad (4)$$

- Eqs. (2) - (4) are a closed subset of the MHD equilibrium equations.
- $\mathbf{B} \cdot \nabla p \neq 0$ drives flow. Must be included in transport equations, not equilibrium equations.
- Pressure gradient determined by solving transport equations, as before, but cannot use flux surface averaged transport equations near magnetic islands.
- Calculation of p near magnetic islands has been addressed by Günter and coworkers.
- Eq. (3) is an equation along field lines, a “magnetic differential equation” (Newcomb).
- As in any equilibrium calculation, there is a profile that must be specified that determines the constant of integration for Eq. (3).

We note that the PIES equilibrium code, which calculates 3D equilibria with magnetic islands, solves equations (2) - (4).

Use magnetic coordinates to solve equations along magnetic field lines.

- In magnetic coordinates (ψ_t, θ, ϕ) ,

$$\mathbf{B} = \nabla \psi_t \times \nabla \theta + \nabla \phi \times \nabla \psi_p(\psi_t) = \nabla \psi_t \times \nabla \theta + \iota \nabla \phi \times \nabla \psi_t,$$

$$\nabla \psi_p(\psi_t) = \left(d\psi_p / d\psi_t \right) \nabla \psi_t \equiv \iota \nabla \psi_t \equiv (1/q) \nabla \psi_t.$$

- Also called “field line following coordinates”, or “flux coordinates with straight field lines”.

$$\mathbf{B} \cdot \nabla \theta / \mathbf{B} \cdot \nabla \phi = \iota(\psi_t) = 1/q.$$

- The equation for j_{\parallel} becomes

$$\mathbf{B} \cdot \nabla (j_{\parallel} / B) = -\nabla \cdot \mathbf{j}_{\perp}.$$

$$\left(\mathbf{B} \cdot \nabla \phi \frac{\partial}{\partial \phi} + \mathbf{B} \cdot \nabla \theta \frac{\partial}{\partial \theta} \right) (j_{\parallel} / B) = -\nabla \cdot \mathbf{j}_{\perp},$$

$$\left(\frac{\partial}{\partial \phi} + \iota \frac{\partial}{\partial \theta} \right) (j_{\parallel} / B) = -\nabla \cdot \mathbf{j}_{\perp} / \mathbf{B} \cdot \nabla \phi.$$

Solve for magnetic coordinates and pressure driven current in two ways:

1. For analytically tractable magnetic field with an island.
2. More general solution by expansion around the X-line.

Analytically tractable magnetic field with an island

- Adopt the usual assumptions of analytical calculations for magnetic islands:
 - zero'th order equilibrium field with simply nested flux surfaces;
 - perturbation consisting of single resonant Fourier harmonic;
- \mathbf{B}_0 can be written in terms of magnetic coordinates as

$$\mathbf{B}_0 = \nabla \psi_{t0} \times \nabla \theta_0 + \nabla \phi \times \nabla \psi_{p0}(\psi_{t0}). \quad (5)$$

- Add resonant perturbation such that $\mathbf{B}_1 \cdot \nabla \psi_{t0} \neq 0$,

$$\mathbf{B}_1 = \varepsilon \sin(M\theta_0 - N\phi) \nabla \phi \times \nabla(M\theta_0), \quad (6)$$

- Transform to “helical magnetic coordinate system”:
 - helical angle $\theta_{h0} \equiv M\theta_0 - N\phi$;
 - helical flux function $\psi_{h0} \equiv \psi_{p0} - (N/M)\psi_{t0} + c$, $\psi_{h0} = 0$ at rational surface;
 - normalized toroidal flux: $\bar{\psi}_{t0} \equiv (\psi_{t0} - \psi_{t0,r}) / M$.

$$\begin{aligned} \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1 &= \nabla \bar{\psi}_{t0} \times \nabla \theta_{h0} + \nabla \phi \times \nabla \psi_h(\bar{\psi}_{t0}, \theta_{h0}), \\ \psi_h(\bar{\psi}_{t0}, \theta_{h0}) &= \psi_{h0}(\bar{\psi}_{t0}) - \varepsilon \cos(\theta_{h0}). \end{aligned} \quad (7)$$

- $\mathbf{B} \cdot \nabla \psi_h = 0$, so surfaces of constant ψ_h correspond to flux surfaces.

Taylor expand $\psi_h(\bar{\psi}_{t0}, \theta_{h0}) \approx \iota'_{h0} \bar{\psi}_{t0}^2 / 2 - \varepsilon \cos(\theta_{h0})$.

- Surfaces of constant ψ_h define a magnetic island.

$$\mathbf{B} = \nabla \bar{\psi}_{t0} \times \nabla \theta_{h0} + \nabla \phi \times \nabla \psi_h(\bar{\psi}_{t0}, \theta_{h0}),$$

$$\psi_h(\bar{\psi}_{t0}, \theta_{h0}) = \psi_{h0}(\bar{\psi}_{t0}) - \varepsilon \cos(\theta_{h0}) \approx \iota'_{h0} \bar{\psi}_{t0}^2 / 2 - \varepsilon \cos(\theta_{h0}).$$

- $(\psi_h, \theta_{h0}, \phi)$ is a flux coordinate system, but not a magnetic coordinate system:

$$\mathbf{B} \cdot \nabla \theta_{h0} / \mathbf{B} \cdot \nabla \phi = \partial \psi_h / \partial \bar{\psi}_{t0} = \iota'_{h0} \bar{\psi}_{t0}.$$

To find magnetic coordinates, use correspondence with action-angle variables of Hamiltonian for field line trajectories.

- Equations for field line trajectories for \mathbf{B} are Hamilton's equations,

$$d\bar{\psi}_{t0} / d\phi = \mathbf{B} \cdot \nabla \bar{\psi}_{t0} / \mathbf{B} \cdot \nabla \phi = -\partial \psi_h / \partial \theta_{h0},$$

$$d\theta_{h0} / d\phi = \mathbf{B} \cdot \nabla \theta_{h0} / \mathbf{B} \cdot \nabla \phi = \partial \psi_h / \partial \bar{\psi}_{t0}.$$

- ψ_h is Hamiltonian, $\bar{\psi}_{t0}$ momentum, θ_{h0} conjugate coordinate, ϕ is time.
- Magnetic coordinates correspond to action-angle variables.

$$\mathbf{B}_0 = \nabla \psi_{t0} \times \nabla \theta_0 + \nabla \phi \times \nabla \psi_{p0}(\psi_{t0}).$$

$$d\psi_{t0} / d\phi = \mathbf{B}_0 \cdot \nabla \psi_{t0} / \mathbf{B}_0 \cdot \nabla \phi = -\partial \psi_{p0} / \partial \theta_0 = 0,$$

$$d\theta_0 / d\phi = \mathbf{B}_0 \cdot \nabla \theta_0 / \mathbf{B}_0 \cdot \nabla \phi = \partial \psi_{p0} / \partial \psi_{t0} = \iota_0(\psi_{t0}).$$

- $\psi_h = \iota'_{h0} \bar{\psi}_{t0}^2 / 2 - \varepsilon \cos(\theta_{h0})$ corresponds to Hamiltonian for pendulum.

Known action-angle variables for pendulum provide magnetic coordinates.

For calculating \mathbf{j} , need $\iota_h = 1/q_h = 1/(d\bar{\psi}_t/d\psi_h)$ and canonical angular coordinate θ_h .

(Focus on region outside the island, $\bar{\psi}_{t0} \geq 0$, here.)

$$\theta_h = \pi F(\theta_{h0}/2, 1/\rho) / K(1/\rho)$$

$$\iota_h = \text{sgn}(\bar{\psi}_{t0}) \pi \sqrt{\varepsilon \iota'_{h0}} \rho / K(1/\rho),$$

Where

$$F(\chi, k) \equiv \int_0^\chi (1 - k^2 \sin^2 \chi')^{-1/2} d\chi'.$$

is incomplete elliptic integral of first kind,

$$K(k) \equiv F(\pi/2, k).$$

is complete elliptic integral of first kind, and

$$\rho \equiv \sqrt{(\psi_h + \varepsilon) / 2\varepsilon}.$$

Solution inside island is similar.

Having magnetic coordinates, can use them to solve for j_{\parallel} / B .

Equation for resonant component of current:

$$l_h \frac{\partial}{\partial \theta_h} (j_{\parallel} / B) = -(\nabla \cdot \mathbf{j}_{\perp} / \mathbf{B} \cdot \nabla \phi)_{n_h=0}.$$

Define $h \equiv \nabla \cdot \mathbf{j}_{\perp} / \mathbf{B} \cdot \nabla \phi$.

Solution outside island:

$$\begin{aligned} & (j_{\parallel} / B)(\psi_h, \theta_{h0}) \\ &= \frac{q_h}{4K(1/\rho)} \int_0^{\theta_{h0}} \left\{ \int_0^{2\pi} h(\psi_{t0}(\psi_h, \theta'_{h0}), \theta_0(\theta'_{h0}, \phi), \phi) d\phi \right\} \left[1 - \frac{1}{\rho^2} \sin^2 \left(\frac{\theta'_{h0}}{2} \right) \right]^{-1/2} d\theta'_{h0} \\ & \quad + (j_{\parallel} / B)_{n_h=0}^{(0)}(\psi_h). \end{aligned} \tag{8}$$

where $(j_{\parallel} / B)_{n_h=0}^{(0)}$ is constant of integration, determined through the specification of corresponding profile.

Use limiting values of elliptic integrals for determination of current density near separatrix.

Use analytical expressions for limiting value of elliptic integrals of first kind near separatrix.

$$l_h \approx \text{sgn}(\bar{\psi}_{t_0}) \pi \sqrt{\varepsilon l'_{h0}} / \ln\left(4 / \sqrt{\rho^2 - 1}\right). \quad (9)$$

In region near separatrix, but not near X-line, which we call “Region 1”,

$$\theta_h \approx \pi \ln[\sec(\theta_{h0} / 2) + \tan(\theta_{h0} / 2)] / \ln\left(4 / \sqrt{\rho^2 - 1}\right). \quad (10)$$

In region near X-line, which we call “Region 2”,

$$\theta_h / \pi \approx 1 - \ln\left(\frac{1 + \sqrt{1 + (\rho^2 - 1) \tan^2(\theta_{h0} / 2)}}{\sqrt{\rho^2 - 1} \tan(\theta_{h0} / 2)}\right) / \ln\left(\frac{4}{\sqrt{\rho^2 - 1}}\right). \quad (11)$$

Pfirsch-Schlüter current well behaved in Region 1. Does not blow up as we approach separatrix, despite fact that $l_h \rightarrow 0$ as we approach the separatrix.

Obtain upper bound near separatrix, away from X-line (Region 1):

$$\left| \frac{1}{l_h} \int_0^{\theta_{h0}^{(i)}(\psi_h)} \left[\int_0^{2\pi} h(\psi_{t0}(\psi_h, \theta'_{h0}), \theta_0(\theta'_{h0}, \phi), \phi) d\phi \right] (\partial\theta_h / \partial\theta'_{h0}) d\theta'_{h0} \right|$$

$$\leq \ln \left[\left(1 + \sin(\theta_{h0}^{(i)}(\psi_h) / 2) \right) / \cos(\theta_{h0}^{(i)}(\psi_h) / 2) \right] \max_{0 \leq \theta'_{h0} \leq \theta_{h0}^{(i)}(\psi_h)} \left| \int_0^{2\pi} h(\psi_{t0}(\psi_h, \theta'_{h0}), \theta_0(\theta'_{h0}, \phi), \phi) d\phi \right| / \sqrt{\epsilon l'_{h0}}.$$

where

$$\theta_{h0}^{(i)} \equiv \min(\pi - \delta, \theta_{h0}).$$

The current density is singular at the X-line in MHD equilibria that are not stellarator-symmetric, but nonsingular in equilibria that are stellarator-symmetric about a point on the X-line.

$$j_{\parallel} / B \approx \text{sgn}(\bar{\psi}_{t_0})(\varepsilon t'_{h_0})^{-1/2} \pi \ln \left(\frac{4 \tan(\theta_{h_0} / 2)}{1 + \sqrt{1 + (\rho^2 - 1) \tan^2(\theta_{h_0} / 2)}} \right) \left[\int_0^{2\pi} h(\psi_{t_0}(\psi_h, \pi), \theta_0(\pi, \phi), \phi) d\phi \right] + (j_{\parallel} / B)^{(0)} \quad (13)$$

where $h \equiv \nabla \cdot \mathbf{j}_{\perp} / \mathbf{B} \cdot \nabla \phi$.

- Integral nonzero, in general, for non-stellarator-symmetric 3D configurations, and PS current has a logarithmic singularity, in general, as X-line approached, $\theta_{h_0} \rightarrow \pi$ and $\rho \rightarrow 1$.
- In stellarator symmetric MHD equilibria, $\nabla \cdot \mathbf{j}_{\perp} / \mathbf{B} \cdot \nabla \phi$ antisymmetric, so $\int_0^{2\pi} h(\psi_{t_0}(\psi_h, \pi), \theta_0(\pi, \phi), \phi) d\phi$ vanishes, and singular component of Pfirsch-Schlüter current vanishes.
- If $\mathbf{B} \cdot \nabla p = 0$, $\nabla \cdot \mathbf{j}_{\perp}$ vanishes at X-line and

$$\int_0^{2\pi} h(\psi_{t_0}(\psi_h, \pi), \theta_0(\pi, \phi), \phi) d\phi = 0.$$

Can get more general solution for limiting behavior of ι_h near the separatrix, θ_h near the X-line, and j_{\parallel} / B near the X-line, by expanding about the X-line.

- Assumptions:
 - Given magnetic field \mathbf{B} near X-line, and shape of flux surfaces near X-line.
- For comparison with narrow island calculation, denote coordinates by $\tilde{\psi}_{t0}$ (radial), $\tilde{\theta}_{h0}$ (poloidal) and $\tilde{\phi}$, where $d\tilde{\phi} / d\phi = 1$ along the X-line.
- Want to find magnetic coordinates $\tilde{\psi}_t$, $\tilde{\theta}_h$, $\tilde{\psi}_h$, such that $\mathbf{B} = \nabla \tilde{\psi}_t \times \nabla \tilde{\theta}_h + \nabla \phi \times \nabla \tilde{\psi}_h$, with $\tilde{\psi}_t = \tilde{\psi}_t(\tilde{\psi}_h)$, and $d\tilde{\psi}_t / \tilde{\psi}_h = q_h = 1 / \iota_h$.
- Taylor expand $\tilde{\psi}_h$ about X-line: $\tilde{\psi}_h \approx \tilde{\psi}_{t0}^2 / a^2 - \tilde{\theta}_{h0}^2 / c^2$.
 - First derivatives vanish at X-line.
 - Cross term eliminated by coordinate rotation.
 - c / a determined by shape of flux surfaces.
 - Magnitude determined, for given B, by flux through helical ribbon about the X-line bounded by corresponding flux surface.

Have $\tilde{\psi}_h \approx \tilde{\psi}_{t0}^2 / a^2 - \tilde{\theta}_{h0}^2 / c^2$. Want $\tilde{\theta}_h$ such that $\mathbf{B} = \nabla \tilde{\psi}_t \times \nabla \tilde{\theta}_h + \nabla \phi \times \nabla \tilde{\psi}_h$.

- Transform to hyperbolic coordinate system,

$$\tilde{\psi}_{t0} = a\tilde{\rho} \cosh(\alpha), \quad \tilde{\theta}_{h0} = c\tilde{\rho} \sinh(\alpha). \quad (15)$$

Gives $\tilde{\psi}_h = \tilde{\rho}^2$.

- Expression for \mathbf{B} , and $d\tilde{\psi}_t / d\tilde{\psi}_h = q_h$, then gives

$$\mathbf{B} \cdot \nabla \phi = 2\tilde{\rho} q_h \mathfrak{T}_\rho^{-1} \partial \tilde{\theta}_h / \partial \alpha,$$

where $\mathfrak{T}_\rho = (\nabla \tilde{\rho} \times \nabla \alpha \cdot \nabla \phi)^{-1}$ is Jacobian for $(\tilde{\rho}, \alpha, \tilde{\phi})$ coordinate system.

- Eq. (15) gives $\mathfrak{T}_\rho = ac\tilde{\rho}\mathfrak{T}_0$, where $\mathfrak{T}_0 = (\nabla \tilde{\psi}_{t0} \times \nabla \tilde{\theta}_{h0} \cdot \nabla \phi)^{-1}$.
- Get

$$\tilde{\theta}_h \approx (1/2)ac\mathfrak{T}_0 B^\phi \iota_h \alpha.$$

- Expressing α in terms of $\tilde{\theta}_{h0}$ and $\tilde{\psi}_h$, using constraint that in single poloidal transit of flux surface $\tilde{\theta}_h$ increases (or decreases) by $2M\pi$, get, to leading order,

$$q_h \approx \mathfrak{T}_0 B^\phi ac \ln(1 / \sqrt{\tilde{\psi}_h}) / (2\pi). \quad (16)$$

$$\tilde{\theta}_h \approx \pi \ln \left(\frac{\sqrt{c^2 \tilde{\psi}_h + \tilde{\theta}_{h0}^2} + \tilde{\theta}_{h0}}{c\sqrt{\tilde{\psi}_h}} \right) / \ln(1 / \sqrt{\tilde{\psi}_h}). \quad (17)$$

Comparison with narrow island calculation

$$\tilde{\theta}_h \approx \pi \ln \left(\frac{\sqrt{c^2 \tilde{\psi}_h + \tilde{\theta}_{h0}^2} + \tilde{\theta}_{h0}}{c \sqrt{\tilde{\psi}_h}} \right) / \ln(1 / \sqrt{\tilde{\psi}_h}).$$

For narrow island calculation, $a = \sqrt{2 / i'_{h0}}$, $c = \sqrt{2 / \varepsilon}$, $\tilde{\psi}_h = 2\varepsilon(\rho^2 - 1)$, $\mathfrak{I}_0 B^\phi = 1$, and $\theta_h = \pi / M - \tilde{\theta}_h$. Making further use of the assumptions $\rho^2 - 1 \ll 1$ and $0 \leq \pi - \theta_{h0} \ll 1$, we find that:

- **The expressions for the magnetic coordinates obtained by the two methods are equal to leading order.**

Physical Interpretation

- Cross-field particle drifts generally lead to local charge accumulation.
 - Charge neutralized by currents along field lines — the Pfirsch-Schlüter (PS) currents.
 - In MHD, current due to cross-field drifts is \mathbf{j}_\perp , charge accumulation is $\nabla \cdot \mathbf{j}_\perp$.
- X-line has closed trajectory \Rightarrow accumulates net charge if $\langle \nabla \cdot \mathbf{j}_\perp \rangle \neq 0$.
 - Charge cannot be neutralized by flow along field line.
- Field lines on flux surfaces approaching separatrix linger increasingly near X-line.
 - Current must flow over increasingly large distance to neutralize the charge \Rightarrow current density becomes increasingly large.
- If configuration is stellarator symmetric about a point on the X-line, $\nabla \cdot \mathbf{j}_\perp$ antisymmetric about that point.
 - Get equal and opposite charge accumulation at diametrically opposed points about the symmetry point.
 - Current along X-line can neutralize the charge.
 - PS currents remain bounded on flux surfaces approaching the separatrix.

Summary

- It is not correct to use parallel component of MHD force balance equation , $\mathbf{B} \cdot \nabla p = 0$, near magnetic islands.
- Region near magnetic islands can be handled by a closed subset of MHD equilibrium equations that depend only on perpendicular force balance, and are decoupled from parallel force balance, Eqs. (2) – (4).
- Pressure distribution determined by transport.
 - Cannot use flux surface averaged transport equations near small island, or near separatrix of large island.
- Have calculated the equilibrium pressure-driven current for an analytically tractable magnetic field.
- More general calculation of pressure driven current in neighborhood of magnetic island X-line, and for rotational transform near separatrix, also presented.
- For non-stellarator-symmetric configurations, the pressure driven current goes like $1/\iota_h$ at the X-line, and ι_h goes to zero logarithmically.
- Integrable singularity \rightarrow less sensitive to details of cutoff at X-line.
- In the special case where an MHD equilibrium is stellarator-symmetric about a point on the X-line, the singular component of the current vanishes.

Reference:

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Extra Material

Island Stability

Generalized Rutherford equation:

$$\frac{0.8}{r} \frac{dw}{dt} = \frac{1}{\tau_R} \left[\Delta'(w)r + D_{nc} \frac{w}{w^2 + w_d^2} - \frac{D_{pol}}{w^3} + D_R \frac{4.6}{w} + D_R f(w) \right]$$

$D_{NC} \propto \beta_p$, $D_R \propto \beta \Rightarrow D_{NC}$ term dominates conventional D_R term in conventional tokamaks

Hegna: D_{NC} and conventional D_R terms comparable in spherical torus.

$f(w) \propto w^{1/2}$ for large w ?

Absence of Singular Current for Simply Nested Flux Surfaces that are Quasi-symmetric

- In equilibrium magnetic field that has nested flux surfaces, pressure driven equilibrium current has $1/x$ singularity at rational surfaces, except at quasi-symmetric surfaces.
- Equilibrium magnetic field with nested flux surfaces, \mathbf{B} , can be written in terms of Boozer coordinates as

$$\mathbf{B} = \nabla \psi_t \times \nabla \theta + \nabla \zeta \times \nabla \psi_p, \quad (\text{C.1})$$

and

$$\mathbf{B} = I(\psi_t) \nabla \theta + g(\psi_t) \nabla \zeta + \gamma(\psi_t, \theta, \zeta) \nabla \psi_t. \quad (\text{C.2})$$

- Dotting Eq. (C.1) with $\nabla \zeta$ gives $\mathbf{B} \cdot \nabla \zeta = \nabla \psi_t \times \nabla \theta \cdot \nabla \zeta$. Dotting Eq. (C.1) with Eq. (C.2) gives $B^2 = \nabla \psi_t \times \nabla \theta \cdot \nabla \zeta (g + \iota I)$. It follows that

$$\nabla p \times \mathbf{B} \cdot \nabla (1/B^2) / \mathbf{B} \cdot \nabla \zeta = p'(\psi_t) \left(I \frac{\partial}{\partial \zeta} - g \frac{\partial}{\partial \theta} \right) (1/B^2),$$

giving

$$\left(j_{\parallel} / B \right)_{nm} = \frac{dp}{d\psi_t} \frac{mg + nI}{\iota m - n} \left(1/B^2 \right)_{nm}, \text{ if } n \neq 0 \text{ or } m \neq 0. \quad (\text{C3})$$

- The $n = 0, m = 0$ component of j_{\parallel} / B is determined by the specified profile of net current.

Quasi-symmetric Flux Surface (continued)

$$\left(j_{\parallel} / B \right)_{nm} = \frac{dp}{d\psi_t} \frac{mg + nI}{m - n} \left(1 / B^2 \right)_{nm}, \text{ if } n \neq 0 \text{ or } m \neq 0. \quad (\text{C3})$$

- Quasisymmetric flux surface: $1 / B^2$ is axisymmetric or helically symmetric when expressed in Boozer coordinates.
 - Resonant Fourier components of $1 / B^2$ vanish at quasisymmetric flux surface, except possibly at surfaces where helical pitch of field line matches that of symmetry direction, as is also possible in a helical configuration.
- A large aspect ratio, analytic equilibrium calculation finds that quasisymmetry can only be satisfied exactly at one flux surface, unless the configuration is strictly symmetric. Deviation from quasisymmetry of the other surfaces is third order in the inverse aspect ratio, so that resonant Fourier components are generally small at those flux surfaces, relative to those of a non-quasi-symmetric configuration.