Designing Stellarator Coils

Differentiating the coil geometry with respect to the plasma boundary

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1) A brief review of designing stellarator coils is presented.

2) Some recent research is described.

3) Outstanding research topics are suggested.

For simplicity, restrict attention to vacuum fields. Vacuum field in given volume defined by boundary conditions.

1. Given volume $\mathcal{V}$, with closed boundary, $\mathcal{S} \equiv \partial\mathcal{V}$.
2. Vacuum fields satisfy $\nabla \times \mathbf{B} = 0$, which suggests $\mathbf{B} = \nabla\Phi$.
3. Given a suitable boundary condition, e.g. $\mathbf{B} \cdot \mathbf{n}$ on $\mathcal{S}$.
4. Divergence-free fields, $\nabla \cdot \mathbf{B} = 0$, implies constraint of net flux $\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$.
5. Toroidal flux $\Psi \equiv \oint_{\mathcal{L}} \mathbf{A} \cdot d\mathbf{l}$, (require one loop integral per "hole").
6. In $\mathcal{V}$, solution to $\nabla \cdot \nabla\Phi = 0$ is unique.

Task: design coils that give required $\mathbf{B} \cdot \mathbf{n}$ on given "target surface", $\bar{\mathbf{x}}(\theta, \zeta)$.

<begin physics>

The Biot-Savart law gives "vacuum" magnetic field.

<end physics> ! everything to follow is mathematics.

1. Volume currents : $\mathbf{B}(\bar{\mathbf{x}}) = \int_{\mathcal{V}} \frac{\mathbf{j}(\mathbf{x}) \times \mathbf{r}}{r^3} \, dv,$? ? ?

2. Surface currents : $\mathbf{B}(\bar{\mathbf{x}}) = \int_{\mathcal{S}} \frac{\mathbf{K}(\mathbf{x}) \times \mathbf{r}}{r^3} \, ds,$ NESCOIL, REGCOIL

3. Line currents : $\mathbf{B}(\bar{\mathbf{x}}) = \sum_i I_i \int_{\mathcal{L}_i} \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3} \, dl,$ ONSET, COILOPT++, FOCUS

$\mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x},$

$\mathbf{j}$ is the volume current density,

$\mathbf{K}$ is the surface current density,

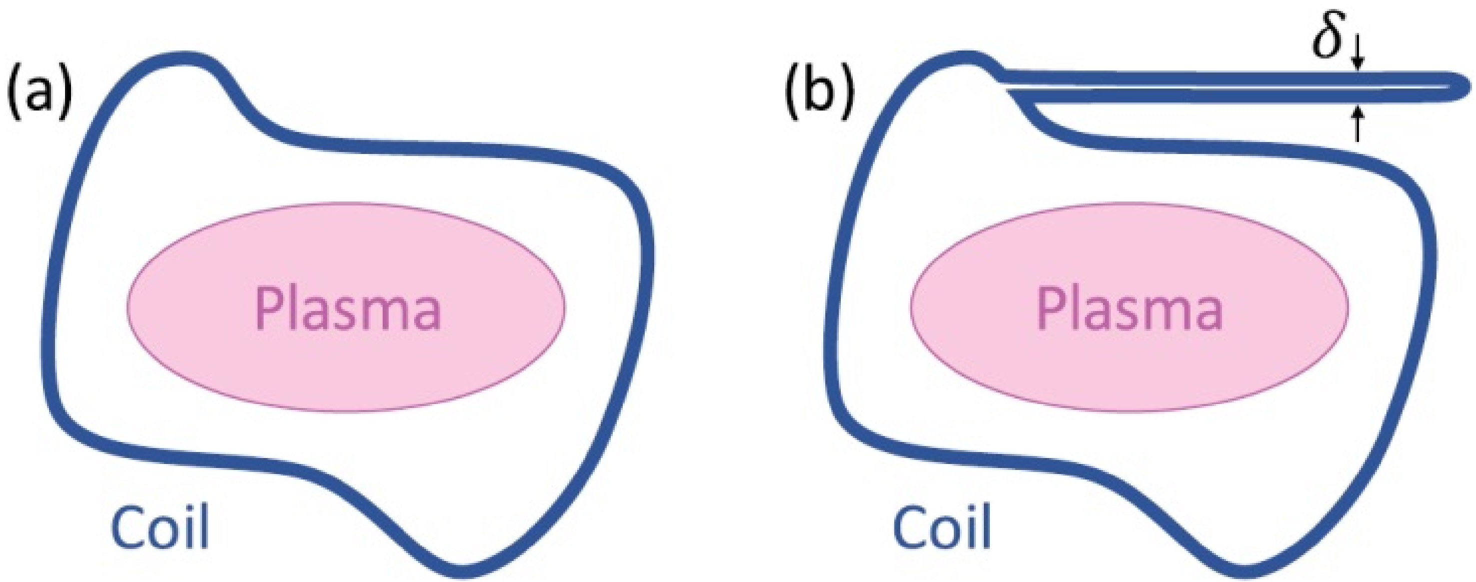
I_i is the current through the i-th filament, $\mathbf{x}_i(l)$.

Coil design is important because $\sum\{\text{coil "complexity", tolerances}\} \equiv \$\$\$$

"The largest driver of the (NCSX) project cost growth were the **accuracy requirements** required for fabrication and assembly of the Stellarator core"
[R. L. Strykowsky, T. Brown *et al.,* 23rd IEEE/NPSS Symposium on Fusion Engineering, (2009)]

"The magnetic field coils are among the **most expensive** components in any magnetic fusion system, and non-planar coils of a stellarator are particularly **difficult to design and fabricate** due to the need for precise three-dimensional shaping."
"Any advances in stellarator coil design an potentially have a **significant impact on the cost and feasibility of fusion energy**"
[M. Landreman, Nucl. Fusion **57**, 046003 (2017)]

This is a “real world”, mathematically challenging problem. The optimal coil geometry is “ill-posed”.



[M. Landreman, Nucl. Fusion **57**, 046003 (2017)]

This picture is easy for humans to understand, but it requires some care to explain to computers which coil geometry is preferred.

Additionally, with "finite coils", cannot generally get *exactly* the required field.
So, instead, minimize an error functional.

1. Minimize **quadratic-flux functional**, [P. Merkel, Nucl. Fusion **27**, 867 (1987)]

$$\mathcal{F} \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 \, ds,$$

   where $B_n \equiv \mathbf{B} \cdot \mathbf{n}$, where $\mathbf{n}$ is normal to prescribed target surface, $\bar{\mathbf{x}}(\theta, \zeta)$.

2. Most coil optimization algorithms seek to minimize $\mathcal{F}$.
3. Simple to include plasma currents, $\mathbf{B} \equiv \mathbf{B}_{Plasma} + \mathbf{B}_{Coils}$, but here $\mathbf{B}_P = 0$.
4. Simple to include an additional factor, e.g., $\oint_{\mathcal{S}} \frac{1}{2} w_{m,n} |B_{m,n}^n| ds$,
   i.e., recognize some "error fields" are more important to control than others;
5. Next question: how to represent the external currents = coils?

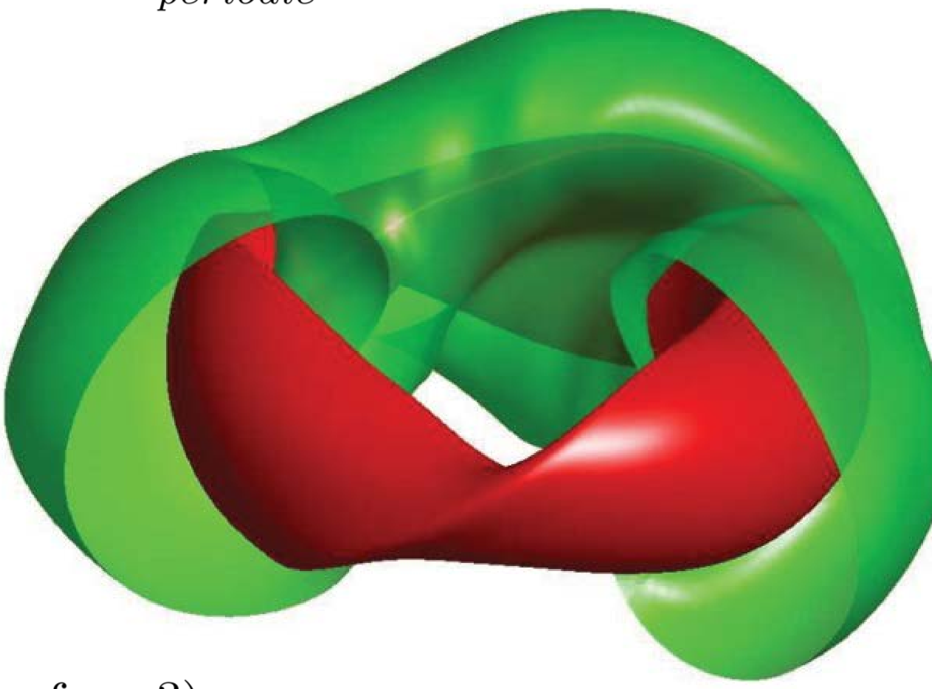
NESCOIL: Introduce external "winding surface" and continuous "current potential".

[P. Merkel, Nucl. Fusion **27**, 867 (1987)]

1. Introduce toroidal "winding" surface, $\mathbf{x}(\theta, \zeta)$, outside target surface.

2. Continuous surface current potential, $\Phi(\theta, \zeta)$, divergence-free surface current given by $\mathbf{K} \equiv \mathbf{n} \times \nabla\Phi$, $\quad \Phi = \underbrace{I\,\theta + G\,\zeta}_{currents} \; + \; \underbrace{\tilde{\Phi}(\theta, \zeta)}_{periodic}$.

3. The magnetic field is
$$\mathbf{B}(\bar{\mathbf{x}}) = \int_{\mathcal{S}} \frac{\mathbf{K}(\mathbf{x}) \times \mathbf{r}}{r^3} ds,$$
where $\mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}$.



4. The winding surface determines the current potential.
(Where does the winding surface come from?)

NESCOIL: Solving for the current potential on a given winding surface reduces to a linear equation.

1. $\Phi \;=\; I\,\theta + G\,\zeta + \sum_{m,n}^{M,N} \phi_{m,n} \sin(m\theta - n\zeta) \;=\; I\,\theta + G\,\zeta + \sum_i \phi_i \sin(m_i\theta - n_i\zeta).$

2. Pack degrees-of-freedom into vector, $\boldsymbol{\phi} \equiv \{\phi_i\}$.

3. $\mathcal{F} = \oint_{\mathcal{S}} \frac{1}{2} B_n^2 \, ds$ is a quadratic function of $\boldsymbol{\phi}$,

$$\mathcal{F} = \int_{\mathcal{S}} \frac{1}{2} B_n^2 \, ds = \frac{1}{2}\, \boldsymbol{\phi}^T \cdot \mathcal{A} \cdot \boldsymbol{\phi} + \mathcal{B} \cdot \boldsymbol{\phi} + \mathcal{C}. \tag{1}$$

4. Easy to find the minimum, $\mathcal{A} \cdot \boldsymbol{\phi} + \mathcal{B} = 0$.

5. NESCOIL is fast and widely used.

6. But, it is ill-conditioned. As M, N increase, solution does not converge.

NESCOIL: High order Fourier harmonics on target surface require large Fourier harmonics on winding

1. The Fourier harmonics of the magnetic field decay with distance, d, and high (m, n) harmonics decay more rapidly than small.
2. To create given $\epsilon_{m,n}$ on target surface, require $\mathcal{O}(\epsilon_{m,n} d^m)$ on winding surface.
3. Consequently, NESCOIL does not converge when the Fourier resolution, M and N, increases.
4. [This slide needs more work.]

REGCOIL = REG-ularized nes-COIL: Include "smoothing" term into target function.

[M. Landreman, Nucl. Fusion **57**, 046003 (2017)]

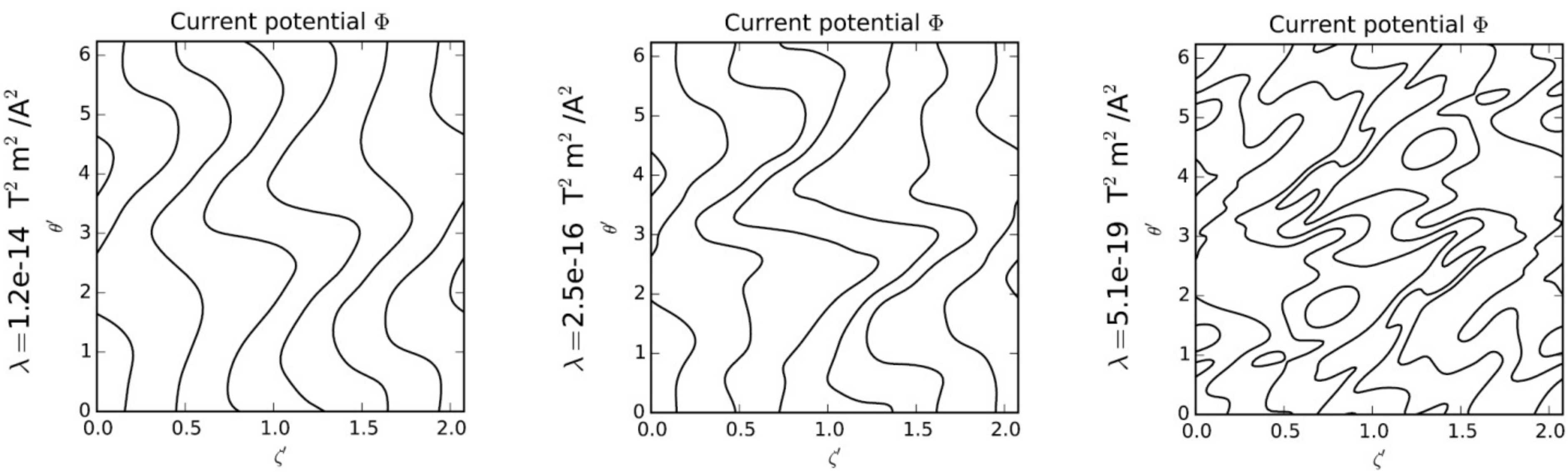
1. Minimize $\mathcal{F} \equiv \underbrace{\oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds}_{\text{target surface}} + \lambda \underbrace{\oint_{\mathcal{W}} \frac{1}{2} K^2 ds}_{\text{winding surface}},$ recall $\mathbf{K} = \mathbf{n} \times \nabla \Phi$,

2. λ is a "regularization" parameter,

3. $\mathcal{F}$ remains a quadratic function of $\boldsymbol{\phi}$.

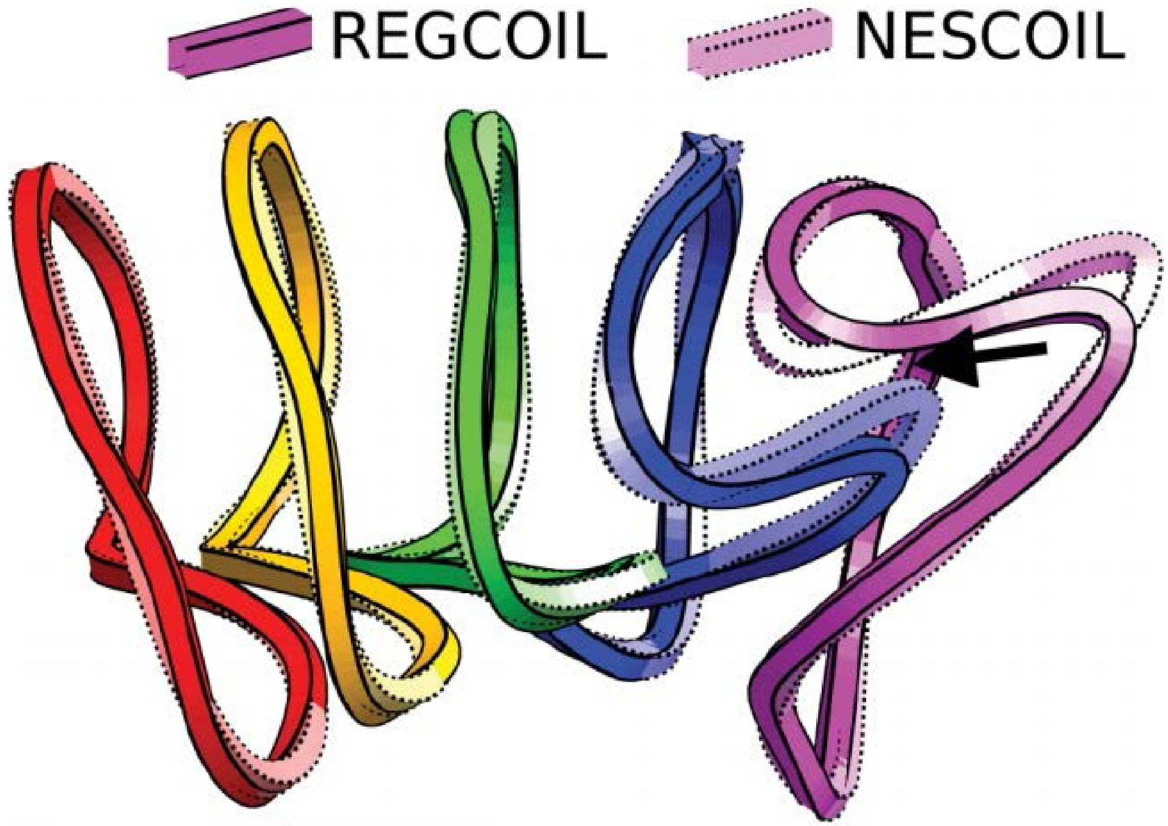
4. REGCOIL gives smoother Φ than NESCOIL. Discrete coils are contours of Φ.

increasing regularization ⟷ decreasing regularization (NESCOIL limit)



REGCOIL can control smoothing.

$$\mathcal{F} \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 ds \; + \; \lambda \oint_{\mathcal{W}} \frac{1}{2} K^2 ds, \qquad \lambda \text{ is a "regularization" parameter.}$$



But relevant engineering quantities,

such as maximum radius of curvature, coil-coil separation, diagnostic access, . . .,

not directly controllable using current potential methods.

COILOPT: Coils represented as filaments on a prescribed winding surface.

1. Steepest-descent methods optimize the coil geometry
   [D. J. Strickler, L. A. Berry and S. P. Hirshman, Fusion Sci. Tech. **41**, 107 (2002)]

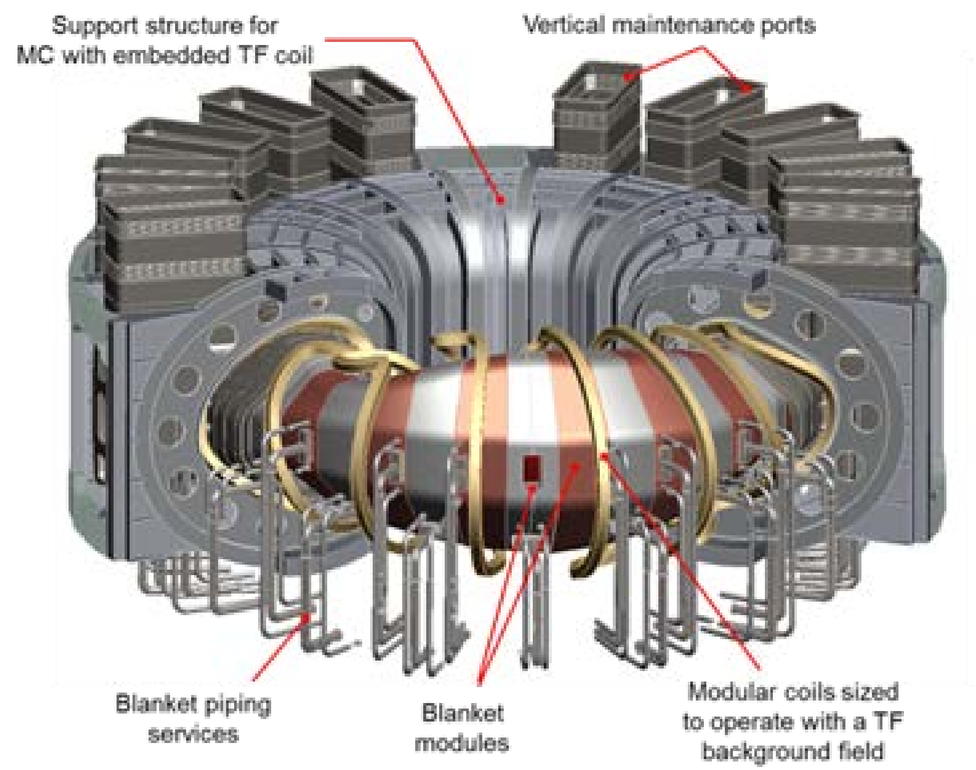
2. Winding surface, $\mathbf{x}(\theta, \zeta)$, and coils, $\theta(l)$ and $\zeta(l)$, represented using Fourier harmonics.

3. Re-written in C++ as COILOPT^{++} , [J. Breslau, unpublished (2012)]

4. Option to use cubic B-splines to optimize local coil features,
   e.g., straight coils on outer midplane

5. Target function, $\chi^2 \equiv \sum_i \omega_i \chi_i^2$, includes:
   coil-coil separation;
   coil length, curvature and torsion;
   inter-coil fields and forces; . .

6. Global nonlinear optimization methods:
   differential evolution,
   particle swarm,
   simulated annealing,
   Levenberg-Marquardt, . .



But why do we need a winding surface?
FOCUS: 1st idea
Coils = closed 1D curves free to move in 3D space.
(A penalty on the length provides regularization.)

1. Introduce $\mathbf{x}_i(l),\ i = 1, \ldots, N_C,$ to represent closed current-carrying curves.
2. The magnetic field is $\mathbf{B}_i(\bar{\mathbf{x}}) = I_i \oint_i \frac{\mathbf{x}_i' \times \mathbf{r}}{r^3}\, dl.$
3. For simplicity, set $I_i = 1$. (Trivial solutions avoided, ignore toroidal flux constraint.)
4. Penalized quadratic-flux functional,

$$\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2\, ds + \omega L, \quad \text{where} \;\; L[\mathbf{x}_i] = \sum_i \oint |\mathbf{x}_i'|\, dl.$$

5. $\mathcal{F}$ depends on the geometry of (i) the target surface, and (ii) the coil "lines".
6. In the following, we present the variation of $\mathcal{F}$
   i. w.r.t. variations in the geometry of the surface, and
   ii. w.r.t. variations in the geometry of the lines.

FOCUS: 2nd idea
Compute gradients analytically.
(Variational calculus of line integrals gives the gradient of the penalized quadratic flux w.r.t. coil variations.)

1. Variations in the curve induce variations in the field, $\delta \mathbf{B}(\bar{\mathbf{x}}) = \oint_i (\delta \mathbf{x}_i \times \mathbf{x}_i') \cdot \mathbf{R}_i \, dl$, where $\mathbf{R} = \dfrac{3\,\mathbf{r}\,\mathbf{r}}{r^5} - \dfrac{\mathbf{I}}{r^3}$, and $\mathbf{I}$ is the "idemfactor", e.g. $\mathbf{I} = \mathbf{i}\,\mathbf{i} + \mathbf{j}\,\mathbf{j} + \mathbf{k}\,\mathbf{k}$.

    i. Tangential variations do not change $\mathbf{B}$.

3. Variation in the length, $\delta L_i = -\oint (\delta \mathbf{x}_i \times \mathbf{x}_i') \cdot \boldsymbol{\kappa}_i$, where $\underbrace{\boldsymbol{\kappa} \equiv \dfrac{\mathbf{x}' \times \mathbf{x}''}{(\mathbf{x}' \cdot \mathbf{x}')^{3/2}}}_{\text{curvature}}$

3. The first variation of the penalized quadratic-flux, $\mathcal{F}[\mathbf{x}_i, \bar{\mathbf{x}}] \equiv \oint_{\mathcal{S}} \frac{1}{2} B_n^2 \, ds + \omega L$, is

$$\delta \mathcal{F} = \oint_i \delta \mathbf{x}_i \cdot \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}} dl, \;\; \text{where} \;\; \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}} \equiv \mathbf{x}_i' \times \left(\oint_{\mathcal{S}} \mathbf{R}_{i,n} \, B_n \, ds + \omega \, \boldsymbol{\kappa}_i \right).$$

Steepest Descent can be used to find (local) minima. Continuous evolution gives implicit linking constraint.

1. "Slow motion" steepest-descent algorithm is easy to implement,

$$\frac{\partial \mathbf{x}_i}{\partial \tau} = - \left. \frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right|_{\bar{\mathbf{x}}}, \qquad \frac{\partial \mathcal{F}}{\partial \tau} = - \oint_i \left(\frac{\delta \mathcal{F}}{\delta \mathbf{x}_i} \right)^2 dl \leq 0.$$

2. Coils cannot continuously pass through surface, as this would produce infinities; so the descent algorithm preserves

   $$\text{the Gauss linking integral } = \frac{1}{4\pi} \oint_i \oint_a \frac{\mathbf{x}_i - \mathbf{x}_a}{|\mathbf{x}_i - \mathbf{x}_a|^3} \cdot d\mathbf{x}_i \times d\mathbf{x}_a,$$

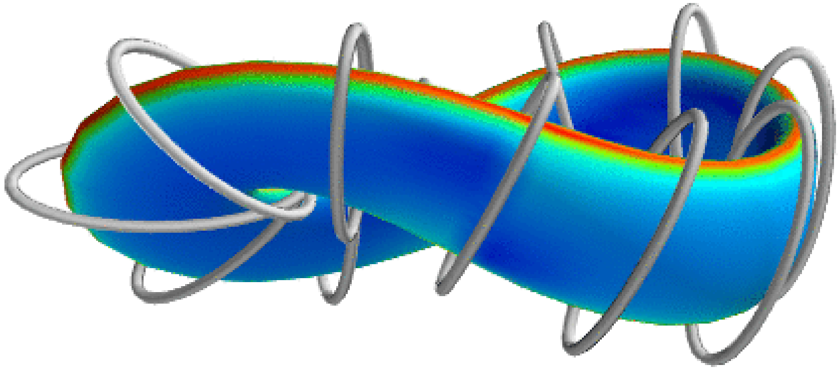
   and thereby avoids the trivial solution that the coils are removed to infinity.

3. The choice of numerical method introduces a theoretical constraint.

Flexible Optimized Coils Using Space (FOCUS) curves
"New method to design stellarator coils without the winding surface",
Caoxiang Zhu, Stuart R. Hudson *et al.,* Nucl. Fusion **58**, 016008 (2017)

FOCUS users = { PPPL, U. Wisconsin, NIFS, Zhejiang U.}

Steepest Descent can be used to find (local) minima.
Continuous evolution yields implicit linking constraint.



FOCUS: 3rd idea
Second derivatives allow fast algorithms / sensitivity

1. Let $\mathbf{c} \equiv \{\mathbf{x}_{i,n}\}$, degrees-of-freedom that parameterize coil geometry.

   For example, $\mathbf{x}_i(l) = x_i(l)\mathbf{i} + y_i(l)\mathbf{j} + z_i(l)\mathbf{z}$ where

$$x_i(l) = \sum_n \left[x^c_{i,n}\cos(nl) + x^s_{i,n}\sin(nl)\right] \tag{1}$$

$$y_i(l) = \sum_n \left[y^c_{i,n}\cos(nl) + y^s_{i,n}\sin(nl)\right] \tag{2}$$

$$z_i(l) = \sum_n \left[z^c_{i,n}\cos(nl) + z^s_{i,n}\sin(nl)\right] \tag{3}$$

2. $\mathcal{F}(\mathbf{c}+\delta\mathbf{c}) \approx \mathcal{F}(\mathbf{c}) + \nabla_{\mathbf{c}}\mathcal{F}\cdot\delta\mathbf{c} + \dfrac{1}{2}\delta\mathbf{c}^T\cdot\nabla^2_{\mathbf{cc}}\mathcal{F}\cdot\delta\mathbf{c}$

3. Inverting Hessian allows Newton method.
   [Caoxiang Zhu, Stuart R. Hudson *et al.*, Plasma Phys. Control. Fusion **60**, 065008 (2018)]

4. Eigenvalues of Hessian describe sensitivity to coil placement errors.
   [Caoxiang Zhu, Stuart R. Hudson *et al.*, Plasma Phys. Control. Fusion **60**, 054016 (2018)]

5. A piecewise-linear representation has been implemented and is presently being tested.

FOCUS: 4th idea
The quadratic-flux is an analytic function of surface.

1. The variation in $\mathcal{F}$ resulting from variations, $\delta \mathbf{x}_i$ and $\delta \bar{\mathbf{x}}$, in the geometry of the i-th coil and the surface is

$$\delta^2 \mathcal{F} \equiv \oint_i \delta \mathbf{x}_i \cdot \oint_{\mathcal{S}} \frac{\delta^2 F}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} \cdot \delta \bar{\mathbf{x}} \;\; ds \, dl, \tag{1}$$

$$\frac{\delta^2 \mathcal{F}}{\delta \mathbf{x}_i \delta \bar{\mathbf{x}}} = \mathbf{x}_i' \times (\mathbf{R}_S \cdot \nabla B_n + \mathbf{B}_S \cdot \nabla \mathbf{R}_n + B_n \, \mathbf{R} \cdot \mathbf{H}) \, \mathbf{n}, \tag{2}$$

where

i. $\mathbf{B}_S \equiv \mathbf{B} - B_n \mathbf{n}$ is projection of $\mathbf{B}$ in the tangent plane to $\bar{\mathbf{x}}$, and $\mathbf{R}_S \equiv \mathbf{R} - \mathbf{R}_n \mathbf{n}$,

ii. the mean curvature can be written $\mathbf{H} \equiv -\mathbf{n} \, (\nabla \cdot \mathbf{n})$,

iii. the calculus of variations of the quadratic-flux w.r.t. surface variations was presented by Dewar *et al.* [Phys. Lett. A **194**, 49 (1994)].

3. Shape of optimal coils changes as the surface changes to preserve $\nabla_{\mathbf{c}} \mathcal{F} = 0$,

$$\nabla_{\mathbf{c}} \mathcal{F}(\mathbf{c} + \delta \mathbf{c}, \mathbf{s} + \delta \mathbf{s}) \approx \nabla^2_{\mathbf{cc}} \mathcal{F} \cdot \delta \mathbf{c} + \nabla^2_{\mathbf{cs}} \mathcal{F} \cdot \delta \mathbf{s} = 0, \tag{3}$$

from this derive derivative of coil geometry w.r.t. target surface,

$$\frac{\partial \mathbf{c}}{\partial \mathbf{s}} = - \left(\nabla^2_{\mathbf{cc}} \mathcal{F} \right)^{-1} \cdot \nabla^2_{\mathbf{cs}} \mathcal{F}. \tag{4}$$

Ongoing Research:
Can the target surface be varied to simplify the coils under the constraint of conserved plasma properties?

1. Introduce a measure of coil complexity, $\mathcal{C}(\mathbf{c})$, that we wish to minimize, e.g. integrated squared torsion

$$\mathcal{C} \equiv \oint \frac{1}{2}\tau^2 dl, \quad \text{where} \quad \tau \equiv \frac{\mathbf{x}' \cdot \mathbf{x}'' \times \mathbf{x}'''}{|\mathbf{x}' \times \mathbf{x}''|^2}. \tag{1}$$

   $\mathcal{C}$ quantifies the "non-planar-ness" of the coils.

2. Introduce a plasma property, $\mathcal{P}(\bar{\mathbf{x}})$, that we wish to constrain.

3. Can minimize coil complexity subject to constrained plasma properties, i.e. extremize

$$\mathcal{G}[\bar{\mathbf{x}}] \equiv \mathcal{C}[\mathbf{x}_i[\bar{\mathbf{x}}]] + \lambda \left(\mathcal{P}[\bar{\mathbf{x}}] - \mathcal{P}_0 \right), \tag{2}$$

   where λ is a Lagrange multiplier.

5. Solutions satisfy $\dfrac{\delta \mathbf{x}_i}{\delta \bar{\mathbf{x}}} \cdot \dfrac{\delta \mathcal{C}}{\delta \mathbf{x}_i} + \lambda \dfrac{\delta \mathcal{P}}{\delta \bar{\mathbf{x}}} = 0.$

Example: rotational-transform on axis depends on "ellipticity" of the boundary and torsion of axis.

1. Rotational-transform on axis, ι_a, can be produced in vacuum

   i. by shaping the boundary (e.g., rotating ellipse),

   ii. by shaping the magnetic axis (through torsion),

   iii. or by both, $\iota_a = \dfrac{(\epsilon-1)^2}{\epsilon^2+1}\dfrac{N}{2} + \dfrac{2\epsilon}{\epsilon^2+1}\bar{\tau}$. [C. Mercier, Nucl. Fusion **4**, 213 (1964)]

   There is freedom to change boundary at $\iota_a = const.$, and this freedom can be used to simplify the coils.

2. Construct family of curves, $\mathbf{x}_a(\zeta)$, with $Length = 2\pi$, with constrained integrated torsion by extremizing $F \equiv \int \kappa^2 \, d\zeta + \mu \left(\int \tau \, d\zeta - \tau_0 \right)$

   ["Non-planar elasticae as optimal curves for the magnetic axis of stellarators", D. Pfefferlé, L. Gunderson *et al.*, in preparation]

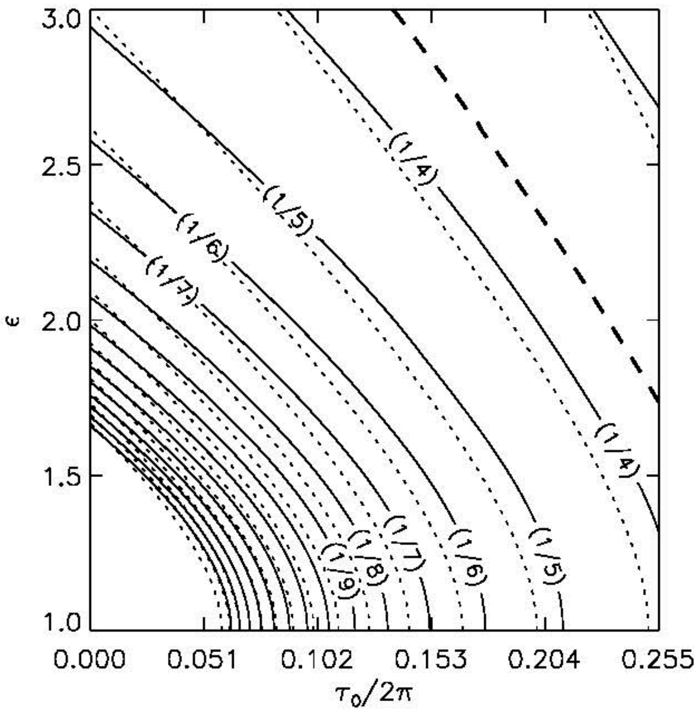
4. Construct 2-parameter (τ_0, ϵ) family of surfaces

$$\bar{\mathbf{x}}(\theta,\zeta) \equiv \mathbf{x}_a(\zeta) + \rho \,(\, \epsilon^{1/2} \cos\theta \, \bar{\mathbf{n}} + \epsilon^{-1/2} \sin\theta \, \bar{\mathbf{b}} \,), \quad \text{choose } \rho = 0.2.$$

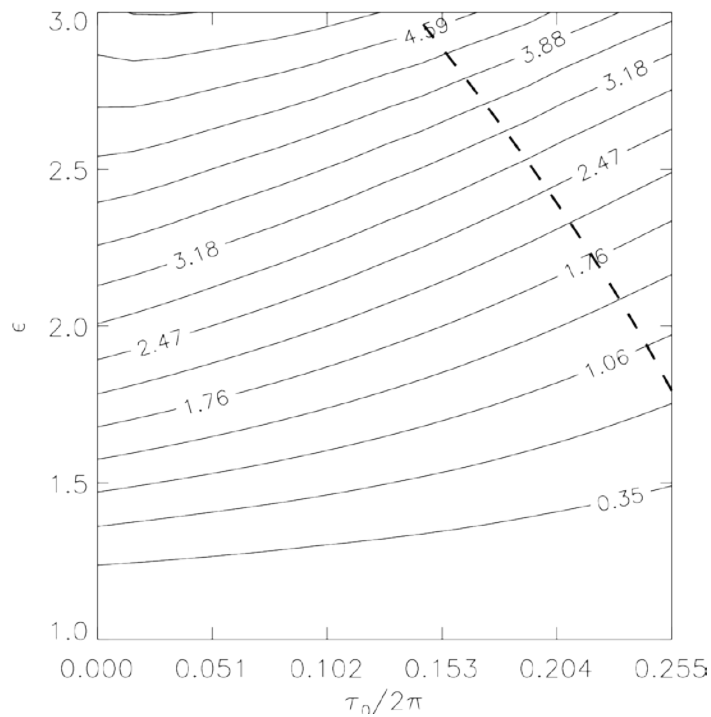
5. Coils constructed using FOCUS. Can compute $\iota_a(\tau_0,\epsilon)$ and $\mathcal{C}(\tau_0,\epsilon)$ numerically.

Rotational-transform and coil complexity depend on the ellipticity of the boundary and torsion of the axis.

contours of $\iota_a(\tau_0, \epsilon)$



contours of $\mathcal{C}(\tau_0, \epsilon)$



1. Using $N_C = 18$ coils, each with current $I_i = 1$, length penalty $\omega = 2$, . . .
2. Solid line from numerical evaluation; dotted line from Mercier's expression.
3. Dashed line shows noble irrational $\iota_a = \dfrac{1 + 1\gamma}{4 + 3\gamma} \approx 0.276, \qquad \gamma = (1 + \sqrt{5})/2.$

A circular cross-section with axis torsion gives simpler coils than a rotating ellipse with circular magnetic axis

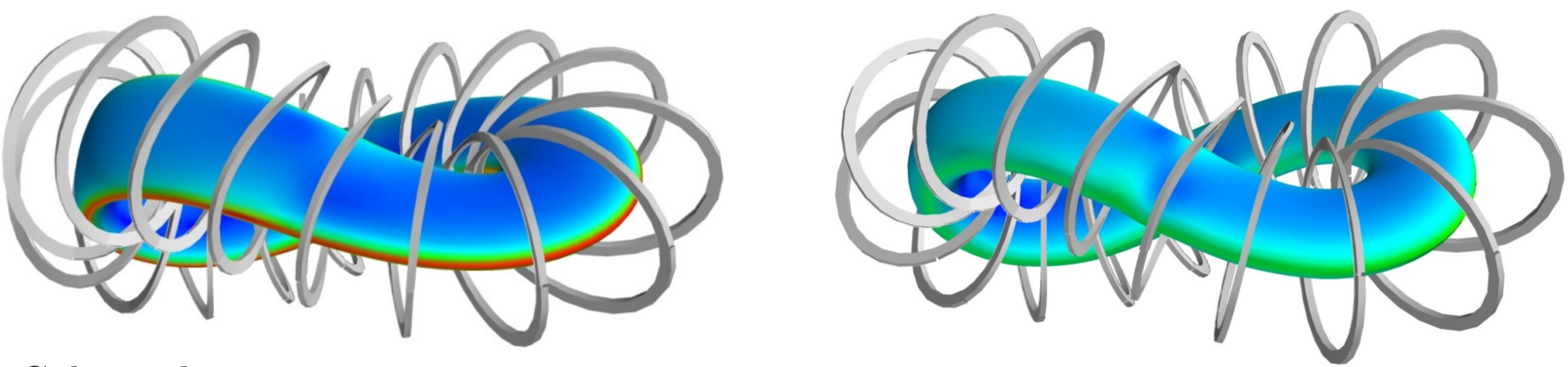
1. "Simple" in this case means more planar.

2. The following have

   i. the same rotational-transform on axis, $\iota\!\!\!{-}_a \approx 0.276$, and good flux surfaces,

   ii. total volume $= 0.799m^3$, 18 coils, $N_{FP} \equiv$ field-periods $= 1$,

   iii average length and complexity of the coils is $\langle L \rangle = 2.92m$ and $\langle \mathcal{C} \rangle = 4.87m^{-1}$, and $\langle L \rangle = 2.74m$ and $\langle \mathcal{C} \rangle = 0.674m^{-1}$.



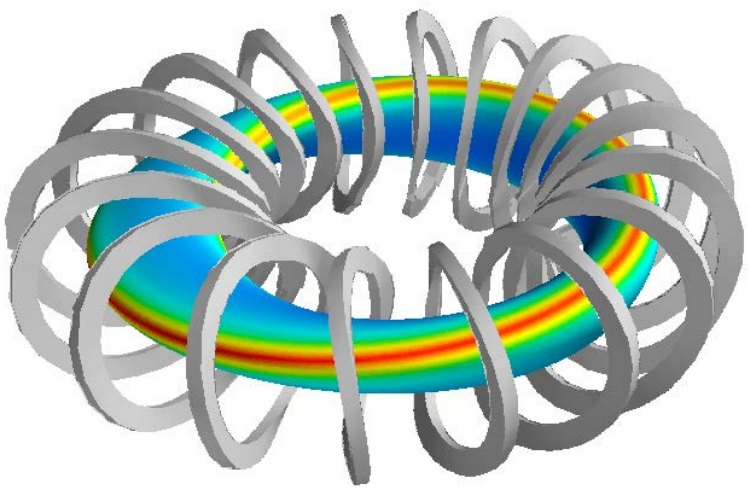
3. Color indicates mean curvature.

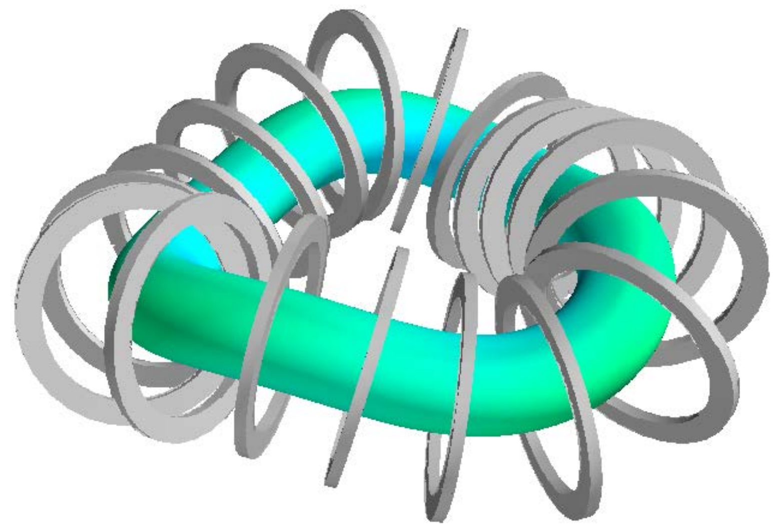
4. Future work: how does shaping affect integrability, quasisymmetry, omnigenity, ... ?

Another example:
one purely elliptical, the other purely torsion;
the same rotational-transform, one with simpler coils.

1. The following have

    i. the same rotational-transform on axis, $\iota \approx 0.101$, and good flux surfaces





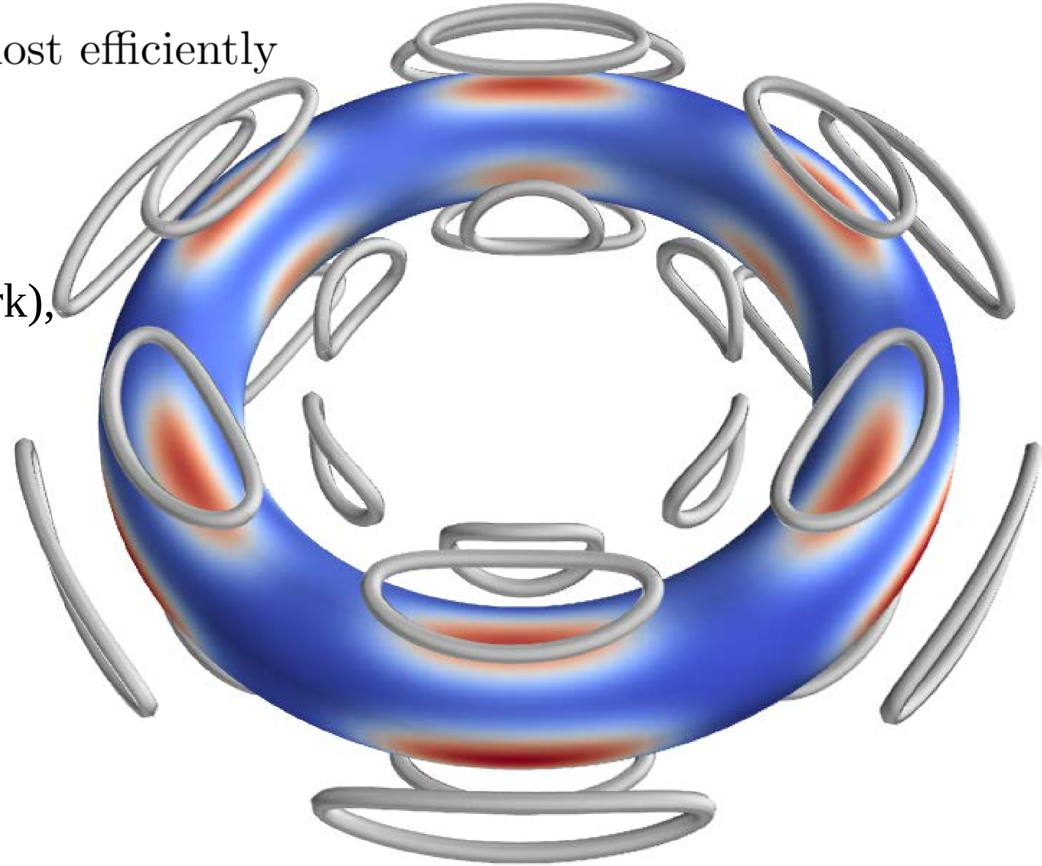
We can use stellarator codes for tokamaks: given required normal field on boundary (from IPEC), what is the optimal geometry of the trim coils?

1. B_n on the plasma boundary that most efficiently couples/controls internal ideal MHD instabilities is computed using IPEC
   [Jong-Kyu Park & Nikolas Logan, Phys. Plasmas **24**, 032505 (2017)]

2. The geometry of the trim coils that most efficiently produces required B_n is determined by REGCOIL & FOCUS .

3. Ongoing research involving:

i. Nikolas Logan & 박종규 (Jong-Kyu Park),
ii. 祝曹祥 & Stuart Hudson, and
iii. 孙恬恬 (Tiantian Sun),
Ph.D. student, Durham University, U.K.



A lot we don't know about designing vacuum fields! Any insights on any of the following will have an immediate impact on the stellarator community.

1. How can we optimize over the coil topology? What about other *plasma* topologies, i.e. knots? What other linking arrangements are possible? Open mathematical question: how many ways can a coil link a knot?
2. Can the REGCOIL regularization be changed? Are there regularizations that are related to engineering constraints, such as length of the coils $\equiv$ length of contours of Φ ?
3. How do we define "flexibility"? What does it mean, mathematically, that one coil set is more flexible than another? How many "different" vacuum fields can be produced by varying the coil currents?
4. What is the best proxy for $cost? Planar-ness? Convexity? (Convex coils can be wound under tension.)
5. How do we bring the insights regarding tolerances to coil misplacement (eigenvalues of the FOCUS Hessian, "sensitivity") into the optimization?
6. We have not yet even incorporated "finite-build", i.e., non-zero thick coils, into the optimization. Recent advances in superconductors need to be included.
7. Can perfectly integrable 3D vacuum fields be constructed?
8. How is integrability related to rotational-transform, shear?
9. How do integrability and quasi-symmetry vary with coil cost?
10. How do we design "divertors", i.e., how to create large magnetic islands near the edge?
11. What about the fractal structure of 3D fields? Do we need to compute the flux across cantori?
12. The BIG question: how do we balance plasma performance against cost of the coils? This talk has said nothing about the effects of positive plasma pressure. To investigate plasma effects on the equilibrium, we need a well-posed equilibrium model/code = MRxMHD/SPEC.

Some relevant papers

1) Quadratic flux minimizing surfaces,
   R. L. Dewar, S. R. Hudson & P. F. Price, Phys. Lett. A **194**, 49 (1994)

2) FOCUS, Caoxiang Zhu, Stuart R. Hudson *et al*., Nucl. Fusion **58**, 016008 (2017)

3) Invited Talk, International Stellarator Heliotron Workshop (ISHW), 2017, C. Zhu *et al.*

4) FOCUS (Newton method)
   Caoxiang Zhu, Stuart R. Hudson *et al*., Plasma Phys. Control. Fusion **60**, 065008 (2018)

5) FOCUS (Eigenvalue sensitivity)
   Caoxiang Zhu, Stuart R. Hudson *et al*., Plasma Phys. Control. Fusion **60**, 054016 (2018)

6) Selected Oral, International Sherwood Fusion Theory Conference, 2018, C. Zhu *et al*.

7) Differentiating the coil geometry w.r.t. the target surface
   S. R. Hudson, C. Zhu *et al*., Phys. Lett A, under review (2018)

Variations in line integrals with respect to variations in the line: length

$$L \equiv \oint (\mathbf{x}' \cdot \mathbf{x}')^{1/2}\, dl \tag{1}$$

$$\delta L = \oint (\mathbf{x}' \cdot \mathbf{x}')^{-1/2} (\mathbf{x}' \cdot \delta\mathbf{x}')\, dl \tag{2}$$

$$= \oint \delta\mathbf{x} \cdot \mathbf{x}' (\mathbf{x}' \cdot \mathbf{x}')^{-3/2} \mathbf{x}' \cdot \mathbf{x}''\, dl - \oint \delta\mathbf{x} \cdot \mathbf{x}'' (\mathbf{x}' \cdot \mathbf{x}')^{-1/2}\, dl \tag{3}$$

Correct, but not transparent. Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change length?

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Use $(\delta\mathbf{x} \times \mathbf{x}') \cdot (\mathbf{x}' \times \mathbf{x}'') = (\delta\mathbf{x} \cdot \mathbf{x}') \cdot (\mathbf{x}' \cdot \mathbf{x}'') - (\delta\mathbf{x} \cdot \mathbf{x}'') \cdot (\mathbf{x}' \cdot \mathbf{x}')$.

$$\delta L = -\oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \boldsymbol{\kappa}, \quad \text{where} \quad \underbrace{\boldsymbol{\kappa} \equiv \frac{\mathbf{x}' \times \mathbf{x}''}{(\mathbf{x}' \cdot \mathbf{x}')^{3/2}}}_{\text{curvature}} \tag{4}$$

Variations in line integrals with respect to variations in the line: magnetic field

$$\mathbf{B} = \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\, dl, \quad \text{where} \;\; \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \;\; r \equiv \sqrt{\mathbf{r} \cdot \mathbf{r}}, \;\; \mathbf{x}' \equiv \partial_l \mathbf{x} \tag{1}$$

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$$\delta\mathbf{B} = \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl \qquad - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3 \oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl$$

Variations in line integrals with respect to variations in the line: magnetic field

$$
\begin{aligned}
\mathbf{B} &= \oint \frac{(\mathbf{x}' \times \mathbf{r})}{r^3}\, dl, \quad \text{where } \mathbf{r} \equiv \bar{\mathbf{x}} - \mathbf{x}, \quad r \equiv \sqrt{\mathbf{r}\cdot\mathbf{r}}, \quad \mathbf{x}' \equiv \partial_l \mathbf{x} \qquad (1) \\
\delta\mathbf{B} &= \oint \frac{(\delta\mathbf{x}' \times \mathbf{r})}{r^3}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl \\
&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r}\cdot\mathbf{x}')}{r^5}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r}\cdot\delta\mathbf{x})}{r^5}\, dl
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&= \oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r} \cdot \mathbf{x}')}{r^5}\, dl - \oint \frac{(\mathbf{x}' \times \delta\mathbf{x})}{r^3}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r} \cdot \delta\mathbf{x})}{r^5}\, dl \\
&= 2\oint \frac{(\delta\mathbf{x} \times \mathbf{x}')}{r^3}\, dl - 3\oint \frac{(\delta\mathbf{x} \times \mathbf{r})(\mathbf{r} \cdot \mathbf{x}')}{r^5}\, dl + 3\oint \frac{(\mathbf{x}' \times \mathbf{r})(\mathbf{r} \cdot \delta\mathbf{x})}{r^5}\, dl
\end{aligned} \tag{2}$$

Correct, but not "transparent". Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change $\mathbf{B}$?

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Correct, but not "transparent". Do tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, change $\mathbf{B}$?

Use $\mathbf{r} \times [\mathbf{r} \times (\delta\mathbf{x} \times \mathbf{x}')] = \mathbf{r} \times [\delta\mathbf{x}\,(\mathbf{r}\cdot\mathbf{x}') - \mathbf{x}'\,(\mathbf{r}\cdot\delta\mathbf{x})] = (\mathbf{r} \times \delta\mathbf{x})\,(\mathbf{r}\cdot\mathbf{x}') - (\mathbf{r} \times \mathbf{x}')\,(\mathbf{r}\cdot\delta\mathbf{x})$

$$\delta\mathbf{B} = \oint \left[\frac{(\delta\mathbf{x} \times \mathbf{x}' \cdot \mathbf{r})\, 3\,\mathbf{r}}{r^5} - \frac{\delta\mathbf{x} \times \mathbf{x}'}{r^3} \right] dl \tag{3}$$

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common factor

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$$= \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \left(\frac{\mathbf{r}\, 3\, \mathbf{r}}{r^5} - \frac{\mathbf{I}}{r^3}\right) dl, \quad \text{where } \mathbf{v}\cdot\mathbf{I} = \mathbf{I}\cdot\mathbf{v} = \mathbf{v}, \tag{4}$$

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$$\delta\mathbf{B} = \oint (\delta\mathbf{x} \times \mathbf{x}') \cdot \mathbf{R}\, dl \tag{5}$$

This is concise, and shows that tangential variations, $\delta\mathbf{x} \times \mathbf{x}' = 0$, do not alter the field.

Variations of surface integrals with changes in the surface: surface area and mean curvature

1. Parametrized surface, $\mathbf{x}(\theta, \zeta)$, tangent vectors $\mathbf{x}_\theta \equiv \dfrac{\partial \mathbf{x}}{\partial \theta}$ and $\mathbf{x}_\zeta \equiv \dfrac{\partial \mathbf{x}}{\partial \zeta}$,

$$\text{normal} \ \ \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \ \ \mathrm{d(area)} \ \ ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

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2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce $\quad \delta\mathbf{x}_\theta \equiv \partial_\theta \delta\mathbf{x}, \quad \delta\mathbf{x}_\zeta \equiv \partial_\zeta \delta\mathbf{x}$

$$\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \;=\; \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) \tag{2}$$

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   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \rightarrow \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned} \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2) \\ &= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3) \end{aligned}$$

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$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

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&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta && (5)
\end{aligned}$$

Variations of surface integrals with changes in the surface: surface area and mean curvature

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   where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \to \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2}\, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n}\cdot(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta|\, d\theta d\zeta &= -\int \delta\mathbf{x}\cdot(\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n})\, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x}\cdot(\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n})\, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x}\cdot(\mathbf{x}_\zeta\partial_\theta - \mathbf{x}_\theta\partial_\zeta) \times \mathbf{n}\; d\theta d\zeta && (6)
\end{aligned}$$

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$$\text{normal} \ \ \mathbf{n} \equiv \frac{\mathbf{x}_\theta \times \mathbf{x}_\zeta}{|\mathbf{x}_\theta \times \mathbf{x}_\zeta|}, \ \ \text{d(area)} \ \ ds \equiv |\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta. \tag{1}$$

where $|\mathbf{x}_\theta \times \mathbf{x}_\zeta| = [(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{1/2}$.

2. Variations $\mathbf{x}(\theta,\zeta) \rightarrow \mathbf{x}(\theta,\zeta) + \delta\mathbf{x}(\theta,\zeta)$ induce

$$\begin{aligned}
\delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| &= \frac{1}{2}[(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\mathbf{x}_\theta \times \mathbf{x}_\zeta)]^{-1/2} \, 2\,(\mathbf{x}_\theta \times \mathbf{x}_\zeta).(\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta + \mathbf{x}_\theta \times \delta\mathbf{x}_\zeta) && (2)\\
&= \mathbf{n} \cdot (\delta\mathbf{x}_\theta \times \mathbf{x}_\zeta - \delta\mathbf{x}_\zeta \times \mathbf{x}_\theta) && (3)\\
&= \delta\mathbf{x}_\theta \cdot (\mathbf{x}_\zeta \times \mathbf{n}) - \delta\mathbf{x}_\zeta \cdot (\mathbf{x}_\theta \times \mathbf{n}) && (4)\\
\int \delta|\mathbf{x}_\theta \times \mathbf{x}_\zeta| \, d\theta d\zeta &= -\int \delta\mathbf{x} \cdot (\mathbf{x}_{\zeta\theta} \times \mathbf{n} + \mathbf{x}_\zeta \partial_\theta \times \mathbf{n}) \, d\theta d\zeta \\
&\quad + \int \delta\mathbf{x} \cdot (\mathbf{x}_{\theta\zeta} \times \mathbf{n} + \mathbf{x}_\theta \partial_\zeta \times \mathbf{n}) \, d\theta d\zeta && (5)\\
&= -\int \delta\mathbf{x} \cdot (\mathbf{x}_\zeta \partial_\theta - \mathbf{x}_\theta \partial_\zeta) \times \mathbf{n} \, d\theta d\zeta && (6)\\
&= -\int \delta\mathbf{x} \cdot \mathbf{n} \, (\nabla \cdot \mathbf{n}) ds, \quad \text{and only normal variations matter.} && (7)
\end{aligned}$$