Magnetohydrodynamical equilibria with current singularities and continuous rotational transform

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An impossible (or improbable) trinity

- In stellarators, we want
  - 3D MHD equilibria;
  - nested toroidal flux surfaces;
  - smoothness.
- This is generally impossible
  - due to pathologies at rational surfaces.

THE PHYSICS OF FLUIDS VOLUME 10, NUMBER 1 JANUARY 1967

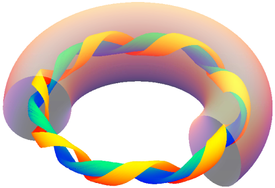
Toroidal Containment of a Plasma

HAROLD GRAD
Courant Institute of Mathematical Sciences, New York University, New York, New York
(Received 5 July 1966; final manuscript received 10 October 1966)

The question of plasma containment in a torus is much more complicated than in an open-ended mirror system. Serious questions arise of the nonexistence of flux surfaces, of noncontained particle drifts, and of nonexistence of self-consistent equilibria at small gyroradius.

Example: ideal perturbed equilibria

- Resonant magnetic perturbations (RMPs):
  - forced: external, amplitude prescribed;
  - spontaneous: instability, amplitude TBD.
- RDR's boundary-layer approach:
  - nonlinear, perturbed equilibrium;
  - consistently preserves magnetic flux;
  - current singularity at resonant surface;
  - applies to RMPs in general[1].



PC: R. Fridström

THE PHYSICS OF FLUIDS VOLUME 16, NUMBER 11 NOVEMBER 1973

Nonlinear properties of the internal $m = 1$ kink instability in the cylindrical tokamak

Marshall N. Rosenbluth
Institute for Advanced Study, Princeton, New Jersey 08540

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(Received 4 April 1973)

An analysis is presented, in a cylindrical approximation, of the nonlinear behavior of the $m = 1$ magnetohydrodynamic kink instability that occurs in a tokamak when the "safety factor" $q(r) = rB_z/RB_\theta(r)$ falls below unity on axis. A kinked neighboring equilibrium is found, which is accessible from the initial straight equilibrium in the sense of satisfying the flux-conservation constraints. Owing to the singular nature of the fundamental, all harmonics are excited in a singular region near where $q(r) = 1$.

[1] Boozer & Pomphrey, PoP 2010; Loizu & Helander, PoP 2017.

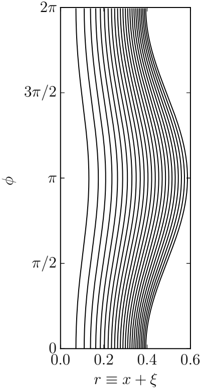
Contents of this talk

- A pedagogical (poor man's) version[2] of RDR's approach:
  - bare-bones, in 2D slab geometry.
- Rigorous application to externally forced RMP:
  - the Hahm–Kulsrud–Taylor problem;
  - includes 2nd-order correction from matching the $1/x$ term;
  - demonstrates the universality of RDR's approach.
- Direct, quantitative validation against numerical solution
  - from a flux-preserving Grad–Shafranov solver.
- Rotational transform stays invariant and continuous:
  - contrary to a recent claim.

[2] Zhou et al., arXiv:1810.08268 (2018), PoP in press.

2D slab reduction of a resonant layer

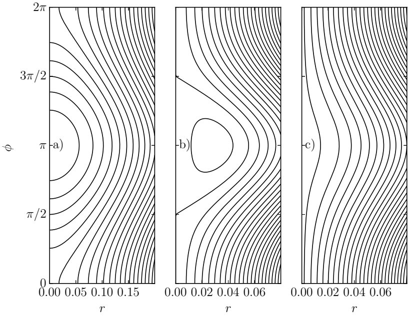
- 2D: single perturbation, helical symmetry.
- Slab: small perturbation, large aspect ratio.
- Coordinates (x, ϕ, ζ):
    - x: 'radial'; relative to resonant surface.
    - ϕ: 'tangential'; period $2\pi s$.
    - ζ: 'helical'; period $2\pi R$; ignorable.
- Incompressible: constant guide field $B_\zeta = B_{0\zeta}$.
- Initial equilibrium: $B_{0\phi} = j_0 x$. ($\psi_0 = j_0 x^2/2$.)
- The Hahm–Kulsrud–Taylor (HKT) problem[3]:
    - mirrored boundary displacement $\xi(x = \pm a, \phi) = \mp\delta \cos\phi$;
    - nice parity – only need to consider $x \in [0, a]$.



[3] Hahm & Kulsrud, PoF 1985.

Equilibria with ‘wrong’ topologies

- Multiple known equilibrium solutions:
  - (a) smooth with primary islands;
  - (b) singular with *residual* islands[4];
  - (c) singular with *more* nested surfaces[5].



- Force balance insufficient; need flux-preserving constraints.

[4] Boozer & Pomphrey, PoP 2010.
[5] Dewar et al., PoP 2013; Loizu et al., PoP 2015.

Flux-preserving constraints

- Solve for $\xi(x,\phi)$: radial displacement of flux surfaces.
  - x: unperturbed (flux) coordinate (Lagrangian labeling).
  - ϕ, $r \equiv x + \xi$: perturbed coordinates (Eulerian labeling).
- Tangential (ϕ) flux conservation: advection of ψ,

$$\psi[r(x,\phi),\phi] = \psi_0(x) = j_0 x^2/2. \tag{1}$$

- Helical (ζ) flux conservation: incompressibility,

$$\langle \xi(x,\phi) \rangle \equiv \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi}\, \xi(x,\phi) = 0. \tag{2}$$

- Also, force balance: Grad–Shafranov equation,

$$\nabla^2\psi = \left(\partial_r^2 + s^{-2}\partial_\phi^2\right)\psi = j_\zeta(\psi). \tag{3}$$

Outer region: linearization

- Assuming $|\partial_x \xi| \ll 1$ when $|x| \gg \delta$, linearize $\partial_r \psi$ in terms of ξ,

$$\partial_r \psi = \psi_0' / \partial_x r = \psi_0'(1 - \partial_x \xi) + O(\xi^2). \tag{4}$$

- Similarly, linearizing $\nabla^2 \psi$,

$$\nabla^2 \psi = \psi_0'' + \psi_0''' \xi - \left(\partial_x^2 + s^{-2} \partial_\phi^2\right) \left(\psi_0' \xi\right) + O(\xi^2). \tag{5}$$

- With $\xi \sim \cos\phi$, linear solution must satisfy

$$\left(\partial_x^2 + s^{-2} \partial_\phi^2\right) \left(\psi_0' \xi\right) = \psi_0''' \xi. \tag{6}$$

- This is equivalent to $\mathbf{F}(\boldsymbol{\xi}) = 0$.

Outer-region solution

- With initial condition $\psi_0 = j_0 x^2/2$,

$$\left(\partial_x^2 + s^{-2}\partial_\phi^2\right)(x\xi) = 0. \tag{7}$$

- Linear solution has two branches[6],

$$\xi(x,\phi) = x^{-1}[C\sinh(|x|/s) + D\cosh(x/s)]\cos\phi. \tag{8}$$

- Boundary condition at $x = \pm a$ gives one constraint,

$$C\sinh(a/s) + D\cosh(a/s) = -\delta a. \tag{9}$$

- Asymptote approaching the resonant surface $x = 0$,

$$\xi(x \to 0^\pm, \phi) = (\pm C/s + D/x)\cos\phi. \tag{10}$$

- Even with $D = 0$, ξ is discontinuous at $x = 0$. Unphysical.

[6] Zweibel & Li, ApJ 1987.

Inner-layer solution

- In the resonant layer ($|x| \ll \delta$), RDR dropped $s^{-2}\partial_\phi^2\psi \ll \partial_r^2\psi$,

$$\partial_r^2\psi = j_\zeta(\psi) \Rightarrow (\partial_r\psi)^2 = j_0^2[F(\psi) + g(\phi)]. \tag{11}$$

- With $\partial_r\psi\partial_x r = \partial_x\psi_0 = j_0 x$ and $f(x) = F(j_0x^2/2)$,

$$\partial_x r = |x|/\sqrt{f(x) + g(\phi)}. \tag{12}$$

- Integrate to get the displacement, using $\partial_x\xi = \partial_x r - 1$,

$$\xi(x,\phi) = h(\phi) + \int_0^x \mathrm{d}x' \left[\frac{|x'|}{\sqrt{f(x') + g(\phi)}} - 1\right]. \tag{13}$$

'Zonal' current singularity

- Tangential magnetic field is given by

$$B_\phi(x,\phi) = \partial_r \psi = \text{sgn}(x) j_0 \sqrt{f(x) + g(\phi)}. \qquad (14)$$

- Delta-function (surface) current at $x = 0$,

$$I'_\delta(\phi) = 2 j_0 \sqrt{f(0) + g(\phi)}. \qquad (15)$$

  Flux-surface average $\langle I'_\delta \rangle \neq 0$: 'zonal'.

- Incompressibility constraint $\langle \xi \rangle = 0$ requires

$$\langle \partial_x r \rangle = 1 \Rightarrow \left\langle [f(x) + g(\phi)]^{-1/2} \right\rangle = |x|^{-1}. \qquad (16)$$

- I'_δ must be zero somewhere[7]: $f(0) = -g_{\min}$. [We set $f(0) = 0$.]

[7] Boozer & Pomphrey, PoP 2010.

Invariance of rotational transform

- Field line flow in the slab system,

$$\frac{s\mathrm{d}\phi}{B_\phi} = \frac{R\mathrm{d}\zeta}{B_\zeta}. \tag{17}$$

- Increment in ζ gives rotational transform ι,

$$\Delta\zeta = \frac{sB_\zeta}{R}\int_0^{2\pi}\frac{\mathrm{d}\phi}{B_\phi} \Rightarrow \iota = \frac{2\pi}{\Delta\zeta} = \frac{R}{sB_\zeta}\left\langle B_\phi^{-1}\right\rangle^{-1}. \tag{18}$$

- Tangential (ϕ) flux conservation: $B_\phi = B_{0\phi}(x)/\partial_x r$,

$$\iota = \frac{R}{sB_\zeta}\left\langle B_\phi^{-1}\right\rangle^{-1} = \frac{RB_{0\phi}}{sB_\zeta}\left\langle \partial_x r\right\rangle^{-1}. \tag{19}$$

- Helical (ζ) flux conservation: $\langle \partial_x r\rangle = 1$ and $B_\zeta = B_{0\zeta}$,

$$\iota = \frac{RB_{0\phi}}{sB_\zeta} = \frac{RB_{0\phi}}{sB_{0\zeta}}. \tag{20}$$

Matching: the constant term

- $|x| \to \infty$ asymptote of the inner-layer solution,

$$\xi(x \to \pm\infty, \phi) = h(\phi) \pm \int_0^{\infty} \mathrm{d}x \left[\frac{x}{\sqrt{f(x) + g(\phi)}} - 1 \right] + \frac{g(\phi)}{2x}. \quad (21)$$

- $|x| \to 0$ asymptote of the outer-region solution,

$$\xi(x \to 0^{\pm}, \phi) = (\pm C/s + D/x) \cos\phi. \quad (22)$$

- Matching the constant term gives $h(\phi) = 0$, and

$$\int_0^{\infty} \mathrm{d}x \left[\frac{x}{\sqrt{f(x) + g(\phi)}} - 1 \right] = (C/s) \cos\phi. \quad (23)$$

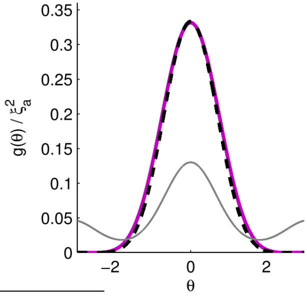
Approximate form of $g(\phi)$

- Using $\langle [f(x)+g(\phi)]^{-1/2}\rangle = |x|$, integral equation for $g(\phi)$:

$$\int_0^\infty \mathrm{d}f \left\langle (f+g)^{-1/2}\right\rangle^{-3} \left\langle (f+g)^{-3/2}\right\rangle \left[(f+g)^{-1/2} - \left\langle (f+g)^{-1/2}\right\rangle\right] = (2C/s)\cos\phi. \tag{24}$$

- The numerical solution of $g(\phi)$ can be well approximated by[8]

$$g(\phi) \approx 4C^2/(3s^2)\cos^8(\phi/2). \tag{25}$$



[8] Loizu & Helander, PoP 2017.

Matching: the $1/x$ term

- Unfortunately, $g(\phi)/2$ and $D\cos\phi$ cannot be matched exactly.
- Expand $g(\phi) = \sum \Gamma_m \cos(m\phi)$ and match the $m = 1$ component,

$$D = \Gamma_1/2 = \langle g(\phi)\cos\phi\rangle \approx 7C^2/(24s^2). \tag{26}$$

- Substituting into the boundary condition,

$$7C^2\cosh(a/s)/(24s^2) + C\sinh(a/s) + \delta a = 0. \tag{27}$$

- We keep one root: '2nd-order' solution,

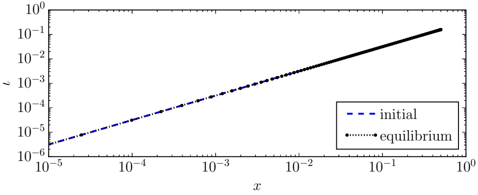
$$C = \frac{\sqrt{\sinh^2(a/s) - 7\delta a\cosh(a/s)/(6s^2)} - \sinh(a/s)}{7\cosh(a/s)/(12s^2)}. \tag{28}$$

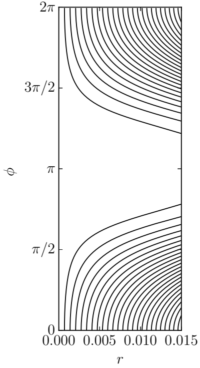
- Without matching the $1/x$ term: '1st-order' solution,

$$C = -\delta a/\sinh(a/s),\ D = 0. \tag{29}$$

Flux-preserving Grad–Shafranov solver

- Same hybrid Lagrangian-Eulerian labeling (x, ϕ).
- B_ζ solved self-consistently from flux conservation[9].
- Incompressibility approximated with strong B_ζ.
- Verified against a fully-Lagrangian method[10].
- Rotational transform invariant and continuous.

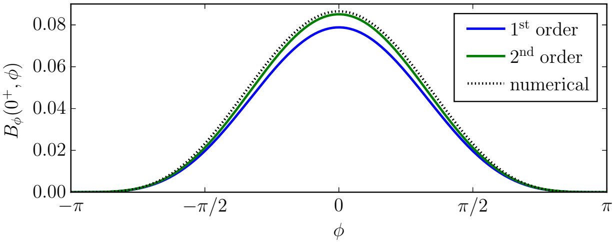




[9] Huang et al., ApJL 2009.
[10] Zhou et al., PRE 2016.

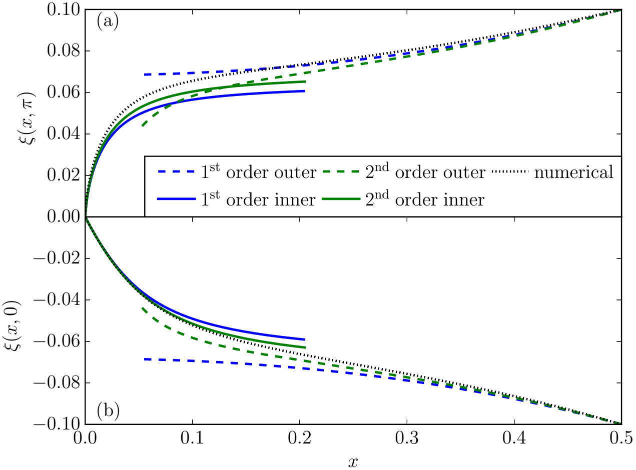
Comparison: structure of current singularity

- 2nd-order solution shows better agreement than 1st-order.



Comparison: radial displacement

- Agreement is slightly better at $\phi = 0$ than at $\phi = \pi$.



Summary and outlook

- A pedagogical version of RDR's approach:
  - bare-bones, in 2D slab geometry;
  - demonstration of universality;
  - includes 2nd-order correction from matching the $1/x$ term;
  - direct, quantitative validation against numerical solution;
  - rotational transform stays invariant and continuous.
- Further validation in more realistic geometry: SPEC? VMEC?
- Generalization to 3D line-tied geometry (the Parker problem)?