Realisability of discontinuous MHD equilibria

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Introduction

• 3D MHD equilibria
• Discontinuous MHD models: Multi-region relaxed MHD (MRxMHD)

New equilibrium model

• Construction and properties
• Linear stability
• Comparison of limits and implications

Next steps and on-going work
Why does existence of 3D MHD equilibria matter?

- Whether tokamaks or stellarators, magnetically confined fusion devices rely on achieving steady-state or quasi-steady-state operating conditions.

- We want to find equilibria, assuming they exist.

- So, do 3D MHD equilibria exist? **Not necessarily.**
Non-existence of smooth 3D MHD equilibria (Grad 1967)

MHD equilibrium equations:

\[ \nabla p = J \times B \quad \nabla \times B = \mu_0 J \quad \nabla \cdot B = 0 \]

Do smooth 3D MHD equilibria (with globally non-uniform pressure) exist? Possibly not.

- Assume continuously flux surfaces and writing \( J = \nabla p \times \nabla \xi \).
- Force balance yields \( \nabla p = (\nabla p \cdot B) \nabla \xi - (\nabla \xi \cdot B) \nabla p \).
- Which is a ‘magnetic differential equation’ (i.e. of the form \( B \cdot \nabla r = s \)).
- A non-trivial continuous solution for \( \xi \) exists except in the neighbourhood of every rational \( m/n \), where \( \xi \) is singular and yields infinite \( J \) and \( \int J \cdot dS \). Not desirable.
- This means \( J_\parallel = \nabla p \cdot (\nabla \xi \times B) \) is also singular unless \( \nabla p = 0 \).
- But, a continuous non-uniform pressure profile with \( \nabla p = 0 \) near every rational surface leads to strange pressure gradients.

“The function \( p \) is continuous but its derivative is pathological.”

[Grad, Phys. Fluids 10, 137 (1967)]
MHD equilibrium equations:

\[ \nabla p = J \times B \quad \nabla \times B = \mu_0 J \quad \nabla \cdot B = 0 \]

Do non-smooth 3D MHD equilibria exist? Yes.

• Proof by Bruno and Laurence for globally non-uniform stepped pressure in toroidal domain.

• Pressure jumps occur at highly irrational surfaces which are most robust to perturbations away from axisymmetry (KAM invariant tori).

• Discontinuities satisfy the jump conditions: \[ [p + B^2/2] = 0 \text{ and } B \cdot \hat{n} = 0. \]

Discontinuous MHD models are not a new idea

- Non-smooth 3D MHD equilibria have been rigorously demonstrated to exist.
- How do we put these mathematical solutions in a physical context?

**Sharp boundary MHD**

- Constant pressure plasma + surface current + vacuum
- Early work: *Biermann et al., Z. Naturforsch. 12a (1957), Kaiser and Salat, PoP 1.2 (1994)*

**Multi-region Relaxed Magnetohydrodynamics (MRxMHD)**

- Developed by R. L. Dewar and collaborators.
Multi-Region Relaxed MHD is based on extremising energy

- Minimisation of **energy** with discretised plasma volume subject to constraints.
- Multi-Region Relaxed MHD $i$th volume helicity: $\mathcal{K}_i \equiv \int_{V_i} A \cdot B \, dv$

$$F = \sum_i \left[ \int_{V_i} \left( \frac{p}{\gamma - 1} + \frac{B^2}{2} \right) \, dv - \frac{\mu_i}{2} (\mathcal{K}_i - \mathcal{K}_{0,i}) \right]$$

- Nested Taylor-relaxed volumes, each satisfying $\nabla \times B = \mu_i B$.
- Existence of ‘special’ ideal interfaces which can support pressure jumps.
- Pressure is constant in each volume and globally non-uniform.
Stepped Pressure Equilibrium Code (SPEC)

- Numerical implementation of MRxMHD in the Stepped Pressure Equilibrium Code (SPEC) developed by Stuart Hudson (PPPL) [Hudson et al., PoP 19 (2012)].

Example SPEC reconstruction of DIII-D equilibrium with RMP field applied:

Pressure profile taken from STELLOPT reconstruction and discretised for SPEC input with $N = 32$ volumes.

Poincaré plot of SPEC computed equilibrium

Large islands at $q = 2$ surface corresponding to significant flattening of pressure profile.
Open questions of discontinuous MHD models

MRxMHD and its numerical implementation have many promising features:

- Solid mathematical foundation.
- Natural extension of Taylor states.
- Does not assume continuously nested flux surfaces but allows globally non-uniform pressure.
- Accommodates equilibria with islands, chaotic field regions and some flux surfaces.

But, are these states real?

- Discontinuous pressure profiles are unintuitive. Is there a parameter regime, within the validity of single fluid MHD, where this is a physically a valid model?
- If realisable, the dynamical mechanism by which such states form is not (yet) understood.
- Is there a contradiction with studies of current sheet dynamics which show thin current sheets cannot be reached due to ideal instability? [Loureiro et al., PoP 14 (2007)]
Despite this, there may still be some hope for discontinuous MHD models:

- All natural systems appear to evolve to states which minimise energy.
- Ideal MHD constraints yield, loosely speaking, ‘maximally constrained’ equilibria.
- By contrast, MRxMHD has only a discrete set of constraints.
- This should produce states which are more energetically favourable.

Defining “physically realisable”:

- States must exist for some non-trivial timescale.
- Ideally stable and sufficiently stable to fast growing resistive instabilities.
- Ultimately, can be achieved in experiments or observed nature within the ‘resolution’ of single fluid MHD.
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• Linear stability
• Comparison of limits and implications

Next steps and on-going work
A new (continuous) cylindrical equilibrium model

• To study continuous and discontinuous equilibria within a single framework, we develop a new continuous equilibrium model with alternating regions of constant and non-uniform pressure.

• Discontinuous states accessible as a configuration limit.

• Allows for direct comparison of properties.

• Equilibria can be constructed for an arbitrary number of volumes.

• To investigate properties we take the simplest non-trivial case: 4 volumes.

[Wright et al., PoP to appear]
Construction of cylindrical equilibria with radially localised pressure gradients

- 1D cylinder with radius \( r_4 = 1 \) and axial period \( L = 2\pi R \).

**When \( \nabla p \neq 0 \):**
- Equilibrium: \( J \times B = \nabla p \).
- In 3D, this region supports continuously nested flux surfaces.
- \( q \) is prescribed: constant and irrational, i.e. no resonances.

**When \( \nabla p = 0 \):**
- Equilibrium: \( \nabla \times B = \mu_i B \).
- In 3D, this region supports chaotic fields.
Construction of cylindrical equilibria with radially localised pressure gradients

- Pressure profile is prescribed.
- In the ideal regions we prescribe $q$.
- The equilibrium is completed by matching quantities at the internal boundaries.
- Continuity of $p$, $\nabla p$, $B_z$ and $q$, which means no singular current sheets in equilibrium.

Example equilibrium $q$-profile.

Example (prescribed) equilibrium pressure profile.
Example of 4-volume cylindrical equilibrium

<table>
<thead>
<tr>
<th>Parameter values:</th>
<th>$r_2 = 0.5\ m$</th>
<th>$r_4 = 1\ m$</th>
<th>$p_2 = 0.7 \times 10^4\ Pa$</th>
<th>$q_1 = \varphi$</th>
<th>$q_{\text{edge}} = \varphi^3$</th>
<th>$R = 10\ m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_1 = 0.25\ m$</td>
<td>$r_3 = 0.75\ m$</td>
<td>$p_0 = 1 \times 10^4\ Pa$</td>
<td>$p_4 = 0.5 \times 10^4\ Pa$</td>
<td>$q_3 = \varphi^2$</td>
<td>$\varphi = (1 + \sqrt{2})/5$</td>
<td>$B_0 = 1\ T$</td>
</tr>
</tbody>
</table>
Ideal stability for tokamak-like parameters

• Recall, an important criterion for realisability is ideal stability.

**Internal modes:**
• Resonant surfaces \( r = r_s \) occur whenever \( q = m/n \).
• Stability for \( m \to \infty \) given by Suydam’s criterion:

\[
\frac{r^2 B_z^2}{8} \left( \frac{1}{q} \frac{dq}{dr} \right)^2 + \frac{dp}{dr} > 0
\]

at every resonant surface.

• The model satisfies \( \frac{B_\theta}{r B_z} \sim \epsilon, \frac{q}{1} \sim 1 \) and \( \frac{\mu_0 p}{B_z^2} \sim \epsilon \) or \( \epsilon^2 \) where \( \epsilon = a/R \).

• In this regime, we expect ideal stability for all \( m \geq 1 \) if \( q(r = 0) > 1 \).

[Freidberg, Rev. Mod. Phys 54.3 (1982)].

**External modes:**
• The choice of plasma-wall boundary (i.e. no vacuum region) precludes external modes.
• While this leads to a jump in the parallel current density at the plasma edge, it greatly simplifies the identification of the ideally stable equilibria and suffices as a first approximation.
Parameter ordering is important for ideal stability

- We do not expect this model to be realisable for arbitrary parameter regimes, but we would like it to be realisable for physically relevant parameters (namely tokamak-like).
- Parameter ordering critical for our ideal stability analysis and conclusions.
- To demonstrate this, we examined a higher pressure non-tokamak-like case and performed linear M3D-C1 calculations.

M3D-C1 input profiles:

Ordering: \( \frac{B_\theta}{r B_z} \sim \epsilon^2 \), \( q \sim 1 \) and \( \frac{\mu_0 p}{B_z^2} \gg \epsilon \) where \( \epsilon = 0.1 \).

(Tokamak ordering: \( \frac{B_\theta}{r B_z} \sim \epsilon \), \( q \sim 1 \) and \( \frac{\mu_0 p}{B_z^2} \sim \epsilon \) or \( \epsilon^2 \).)
Demonstrating that parameter ordering is important for ideal stability

- Using M3D-C1 we evolve the equilibrium according to the linearised resistive MHD equations (with constant resistivity).
- A mode with $m = 3$, $n = 1$ structure is unstable.
- Appears ideal and not localised to resonant surface.

Growth rate independent of $\eta$ indicates ideal instability.

\[ \gamma \sim 0.03 \tau_A^{-1} \]

Perturbed component of pressure at final time with $m = 3$, $n = 1$ structure.

Perturbed radial component of $B$ shows non-localisation of displacement.

Internal boundary at $r = r_s$

\[ \eta_0 = 10^{-8} \]

\[ \eta_0 = 10^{-7} \]

\[ \eta_0 = 10^{-6} \]
• Tearing stability determined by $\Delta' = \left( \lim_{r \to r_s^+} \psi' - \lim_{r \to r_s^-} \psi' \right) / \psi(r_s)$.
• $\psi$ is the solution to the ideal Newcomb equation:

$$\frac{d}{dr} \left( \frac{r^3}{k^2 r^2 + m^2} \frac{d\psi}{dr} \right) - \left[ \frac{g}{F^2} + F^{-1} \frac{d}{dr} \left( \frac{r^3}{k^2 r^2 + m^2} \frac{dF}{dr} \right) \right] \psi = 0.$$  

$$F = k B_z + \left( \frac{m}{r} \right) B_\theta,$$

$$g = \frac{(m^2 - 1) r F^2}{k^2 r^2 + m^2} + \frac{k^2 r^2}{k^2 r^2 + m^2} \left[ 2 \mu_0 \frac{dp}{dr} + r F^2 + \frac{2F (k r B_z - m B_\theta)}{k^2 r^2 + m^2} \right].$$

• And $k = -n/R$.

• Subject to the boundary conditions:

$$\psi(0) = 0, \text{ if } m \neq 1,$$

$$\psi'(0) = 0, \text{ if } m = 1,$$

$$\psi(r_s) = 1,$$

$$\psi(r_4 = 1) = 0.$$

• We find $\Delta'$ numerically in Mathematica by solving the boundary value problem.
Tearing stability for \( q = 2, 3, 4 \) resonances

For the example 4-volume equilibrium:
- \( \Delta' \) for a range of \( m \) and \( n \) corresponding to \( q = 2, 3, 4 \) resonances, i.e. \( n = mq_{res} \).
- Since \( \Delta' < 0 \), the system is tearing stable for values considered.
- \( \Delta' \) becomes more negative with large \( m \), suggests greatest susceptibility to low \( m \) (i.e. long wavelength) modes.
Discontinuous pressure and $q$ limits

- Define $\delta_2 = |r_2 - r_1|/|r_2|$ and $\delta_4 = |r_4 - r_3|/|r_4 - r_2|$ so that $\delta_2, \delta_4 \to 1$ is the discontinuous $p$ limit (i.e. MRxMHD-like) and $\delta_2, \delta_4 \to 0$ is the discontinuous $q$ limit.

Illustrative examples of typical $p$ and $q$ profiles:
Tearing stable ($\Delta' < 0$) for the $q = 2$ surface and increased stability in the discontinuous $p$-limit.

Field reversal limit:
- There exist threshold values of $\delta_2$ and $\delta_4$ below which $B_z$ must change sign in order to satisfy internal matching conditions.
- There is a maximum difference in $q$ which can be sustained across the relaxed region.
- In this work, the field reversal limit corresponds to $\delta_2 \approx 0.21$ and $\delta_4 \approx 0.4$. 
Identifying the destabilising mechanisms

- Two destabilising effects in equilibrium: discontinuities in $J$ and $\nabla p < 0$.
- Discontinuous $J_\parallel$ a characteristic feature of the model due to limited smoothness of pressure.
- No resonances where $\nabla p \neq 0$ so any $\nabla p < 0$ effects must be non-resonant.
- We investigate the effect of these destabilising mechanisms on higher $m$ modes by studying the $m = 13, n = 5$ mode for a range of different configurations.

Equilibrium profiles:

\[ \delta_2 = 0.214 \text{ and } \delta_4 = 0.5 \] (towards discontinuous $q$ limit).

\[ \delta_2 = 0.98 \text{ and } \delta_4 = 0.95 \] (towards discontinuous $p$ limit).
Comparison of tearing stability properties for different configurations (higher $m$)

- Perform a scan in $\delta_2 \in [0.214, 0.98]$ and $\delta_4 \in [0.5, 0.95]$ for $m = 13$, $n = 5$ mode which is a (relatively) low order rational close to $q_3 = \varphi^2 \approx 2.61$. 
What explains the behaviour?

- $\Delta'$ is not a smooth (or continuous) function of $\delta_2$ suggests qualitative change in system properties at critical values.

- If we view this sequence as a dynamical process, the system would not be able to cross the critical values where $\Delta'$ seemingly becomes divergent (viz. threshold current sheet thickness).

In practice, we would expect to see greater accumulation of pressure gradients.

So, what physics missing?
Achieving greater pressure gradient localisation by adding more physics

While the analysis shows regions of configuration space ($\delta_2 - \delta_4$) where 1D equilibria appear stable, i.e. could be realised, this space seems smaller than anticipated.

**Option 1: Suppress instabilities**

- Identify mechanisms to suppress instabilities, e.g. toroidal curvature.
- Discontinuous $J_\parallel$ is characteristic and known to drive instabilities.

**Option 2: Symmetry breaking**

- Allow spontaneous symmetry breaking (i.e. island formation).
- If saturated island width remains within relaxed regions ($\nabla p = 0$) then localisation of $\nabla p \neq 0$ is preserved.
- In the absence of symmetry, this would correspond to equilibria with both continuously nested flux surfaces ($\nabla p \neq 0$) and chaotic field regions ($\nabla p = 0$).
On-going and future work

In progress:

- Utilising this framework to compare more sophisticated continuous and discontinuous MHD models.
- Goal: Benchmark between SPEC and M3D-C1.
- Nonlinear M3D-C1 calculations initialised with 4-volume equilibria underway.
- Absence of fast instabilities \( (t \sim O(100\tau_A)) \) in preliminary results suggests robustness of steep pressure gradients for some tokamak-relevant parameters.

Additional on-going work:

- Investigate drivers of instability and role of non-resonant effects.
- Explore saturated island states.
- Develop self-consistent model for the evolution of SPEC inputs and a priori framework for constraints.
Resolving apparent divergence

• When $\nabla p = 0$, we can derive an analytic solution for $\psi$ which yields:

$$
\Delta'_{VC} = \frac{(\chi_R - \chi_L)[x_s^2(\gamma - 1)^2 + 4m^2\gamma]}{\pi r_s^3 \sqrt{\mu^2 - k^2} [\gamma J_{m-1}(x_s) + J_{m+1}(x_s)]},
$$

where $x_s = r_s \sqrt{\mu^2 - k^2}$, $\gamma = (\mu + k)/(\mu - k)$ and $\chi_L$ and $\chi_R$ denote $\chi$ in the regions $r < r_s$ and $r > r_s$.

• Singularity whenever $\mu^2 = k^2$ or $[\gamma J_{m-1}(x_s) + J_{m+1}(x_s)] = 0$ which implies:

$$
\gamma = \begin{cases} 
\frac{I_{m+1}(t)}{I_{m-1}(t)}, & \mu^2 < k^2 \\
\frac{J_{m+1}(t)}{J_{m-1}(t)}, & \mu^2 > k^2
\end{cases}
$$

• Commonly we have $\mu^2 < k^2$ where the condition is never satisfied, i.e. $\Delta'$ is not singular.
Verification of numerical $\Delta'$ calculation

- When $\nabla p = 0$, we can derive an analytic solution for $\psi$ which yields:

$$
\Delta'_{\text{VC}} = \frac{(\chi_R - \chi_L)[x_s^2(\gamma - 1)^2 + 4m^2\gamma]}{\pi r_s^3 \sqrt{\mu^2 - k^2[\gamma J_{m-1}(x_s) + J_{m+1}(x_s)]}},
$$

- We compared this against the numerical solutions for $\Delta'$. Shows good agreement. Differences may be due to patching of coefficients.

Example: for $p_0 = 10^6, p_2 = 7 \times 10^5, p_4 = 2 \times 10^5$

![Graphs showing comparison of numerical and analytic solutions for $\Delta'$.](image)
M3D-C1 mesh