Velocimetry and the aperture problem for 2D incompressible flows

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The inference of velocity fields from 2D movies evolving conserved scalars (optical flow) is fundamentally ambiguous due to the well-known “aperture problem”: velocities along isocontours of the scalar are not visible. This may even corrupt the inference of velocity fields averaged at scales longer than the typical length scale of features in the scalar field, as in the barber-pole effect. However, for divergence-free flows, a stream-function formulation allows us to show that the "invisible velocity" vanishes in the surface average over any closed scalar isocontour. This error-free averaged velocity may be used as an “anchor” for a more reliable inference of the larger-scale velocity field, or to test model-based optical-flow schemes. We have also used the stream-function formulation to derive a new method of optical flow for divergence-free flows. We discuss the new algorithm, including details of discretization, boundary conditions, and image preprocessing that can significantly affect its performance. A simple implementation of the new method is shown to work well for a number of synthetic movies, and is also applied to a GPI movie of edge turbulence in NSTX.

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Overview

- Background: velocimetry and the aperture problem
- Properties of the "invisible velocity" when $\nabla \cdot \boldsymbol{v} = 0$
- Averaging procedures to eliminate the invisible velocity
- A forward-problem formulation for 2D velocimetry with $\nabla \cdot \boldsymbol{v} = 0$
- Conclusions

This work is part of a collaboration with Ahmed Diallo, Stewart Zweben, and Santanu Banerjee, doing experimental investigation of the L-H transition on NSTX with gas puff imaging (GPI).

Objective: infer 2D velocities from time-dependent movies

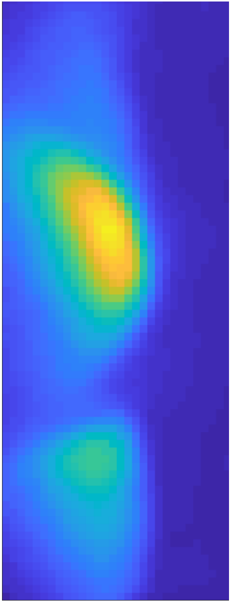
- Given a 2D movie $n(x, y, t)$ (e.g. GPI, right), we wish to infer $\boldsymbol{v}(x, y, t)$.
- Assume the imaged scalar (n) obeys a 2D continuity equation:

$$\partial_t n + \boldsymbol{v} \cdot \nabla n = 0$$

- GPI does not exactly follow this, but we neglect such errors for this talk.
- Many other applications: computer vision, cardiac flow (via X-ray), satellite cloud imaging, turbulent flow with tracer dyes,...
- Unfortunately this is underdetermined:

$$\frac{\boldsymbol{v} \cdot \nabla n}{|\nabla n|} = \frac{\partial_t n}{|\nabla n|} \text{ but } \frac{\boldsymbol{v} \cdot \hat{z} \times \nabla n}{|\nabla n|} = ?,$$

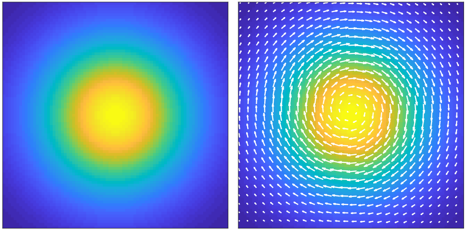
called the "aperture problem"
- Goal: use $\nabla \cdot \boldsymbol{v} = 0$ to disambiguate



Unfortunately, even the $\nabla \cdot \boldsymbol{v} = 0$ problem is underdetermined.

Consider this stationary movie$\Longrightarrow$

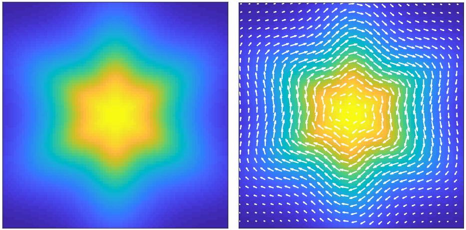
- Consistent with $\boldsymbol{v} = 0$
- Also consistent with $\boldsymbol{v} = v(r)\hat{\theta}$



But what if we break θ-symmetry?

Consider this stationary movie$\Rightarrow$

- Consistent with $\boldsymbol{v} = 0$ (left)
- Also consistent with wavy $\boldsymbol{v}$ (right)



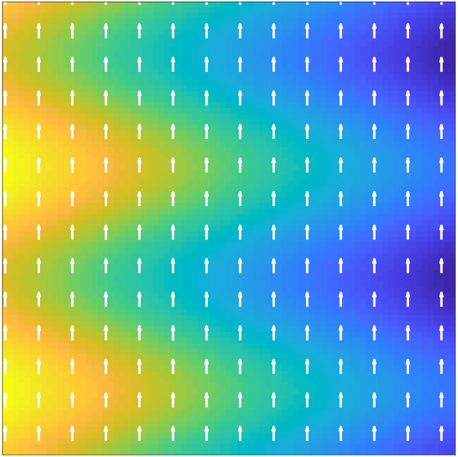
The resulting error can survive even in the mean.

Consider a simple drift wave

$$n(x, y, t) = n_0 - n_0' x + n_s \sin(ky - \omega t)$$

It looks just like rigid advection along $\hat{y}$

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{rig}} \doteq (\omega/k)\hat{y}$$



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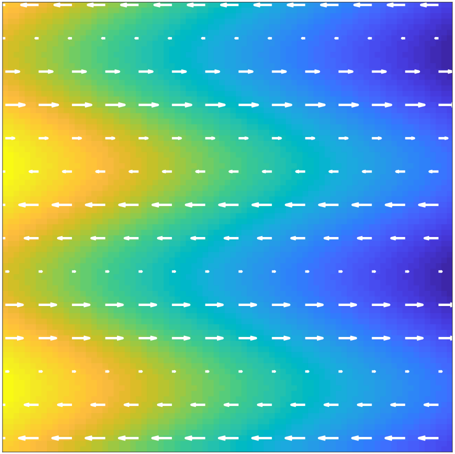
$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{rig}} \doteq (\omega/k)\hat{y}$$

But the real velocity field is

$$\boldsymbol{v} = \boldsymbol{v}_{\mathrm{dw}} \doteq -(n_s\omega/n_0')\cos(ky - \omega t)\hat{x}$$

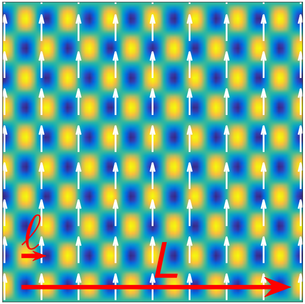


This is just the aperture effect, since $(\boldsymbol{v}_{\mathrm{rig}} - \boldsymbol{v}_{\mathrm{dw}}) \perp \nabla n$, and it's common in plasmas due to electron adiabatic response.

Why can we see both components of $\boldsymbol{v}$ in everyday life?

Our everyday experience usually satisfies an ordering, roughly:†

- If n varies on length ℓ
- and $\boldsymbol{v}$ varies on length L
- then $\ell \ll L$



If this is true, then we may see all of $\boldsymbol{v}$:
Consider two points $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ separated by a short length $|\boldsymbol{x}_2 - \boldsymbol{x}_1| \sim \ell$, then:

$$\boldsymbol{v}(\boldsymbol{x}_1) \cdot (\nabla n)(\boldsymbol{x}_1) = (\partial_t n)(\boldsymbol{x}_1)$$
$$\boldsymbol{v}(\boldsymbol{x}_1) \cdot (\nabla n)(\boldsymbol{x}_2) \approx \boldsymbol{v}(\boldsymbol{x}_2) \cdot (\nabla n)(\boldsymbol{x}_2) = (\partial_t n)(\boldsymbol{x}_2)$$

Since $(\nabla n)(\boldsymbol{x}_2) \not\approx (\nabla n)(\boldsymbol{x}_1)$, we know both components of $\boldsymbol{v}(\boldsymbol{x}_1)$ to $O(\ell/L)$.
This ordering underlies most velocimetry.
However, this requires $\boldsymbol{v}$ *really* varies slowly relative to n, not just that we are interested in the part of $\boldsymbol{v}$ that varies slowly relative to n (c.f. drift wave).

†Rigid-body motion is a bit more complicated than this, but is similarly constrained.

The form of the "invisible velocity" is constrained by $\nabla \cdot \boldsymbol{v} = 0$, even for strongly sheared flows.

- Incompressibility implies that we may use a potential formulation:

$$\boldsymbol{v} = \hat{z} \times \nabla\phi, \text{ thus}$$

$$\partial_t n = -\hat{z} \times \nabla\phi \cdot \nabla n = \hat{z} \times \nabla n \cdot \nabla\phi = \{n, \phi\}$$

- The aperture problem is due to null space (kernel) of $(\hat{z} \times \nabla n \cdot \nabla)$, that is, due to the space of functions Φ such that $\hat{z} \times \nabla n \cdot \nabla\Phi = 0$.
- By inspection, this null space consists of all functions Φ that are constant along isocontours of n.

That is, if we find *any* "inferred potential" ϕ_{inf} such that $\hat{z} \times \nabla n \cdot \nabla\phi_{\text{inf}} = \partial_t n$, then it differs from the true ϕ by a Φ satisfying

$$\hat{z} \times \nabla n \cdot \nabla(\phi_{\text{inf}} - \phi) = \hat{z} \times \nabla n \cdot \nabla\Phi = 0,$$

thus a $\Phi = \phi_{\text{inf}} - \phi$ that is constant along isocontours of n.

We may use the form of Φ to eliminate the "invisible velocity" under well-chosen averaging.

Consider the line-averaged velocity across some curve C connecting $\boldsymbol{x}^{\mathrm{s}}$ to $\boldsymbol{x}^{\mathrm{e}}$. Using the potential formulation, we have

$$\mathrm{d}\ell\, v_\perp = (\hat{z} \times \mathrm{d}\boldsymbol{\ell}) \cdot \boldsymbol{v} = (\hat{z} \times \mathrm{d}\boldsymbol{\ell}) \cdot (\hat{z} \times \nabla\phi) = \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi, \text{ thus}$$

$$\int_C \mathrm{d}\ell\, v_\perp = \int_C \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi = \phi(\boldsymbol{x}^{\mathrm{e}}) - \phi(\boldsymbol{x}^{\mathrm{s}}).$$

This means that for $\boldsymbol{x}^{\mathrm{s}}$ and $\boldsymbol{x}^{\mathrm{e}}$ on the same density isocontour, for any divergence-free inferred velocity $\boldsymbol{v}_{\mathrm{inf}}$ we have

$$\begin{aligned}\int_C \mathrm{d}\ell\, v_{\perp,\mathrm{inf}} = \int_C \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi_{\mathrm{inf}} = \int_C \mathrm{d}\boldsymbol{\ell} \cdot \nabla(\phi + \Phi) \\ = \int_C \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi + \Phi(\boldsymbol{x}^{\mathrm{e}}) - \Phi(\boldsymbol{x}^{\mathrm{s}}) = \int_C \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi = \int_C \mathrm{d}\ell\, v_\perp,\end{aligned}$$

since $\Phi(\boldsymbol{x}^{\mathrm{e}}) = \Phi(\boldsymbol{x}^{\mathrm{s}})$.

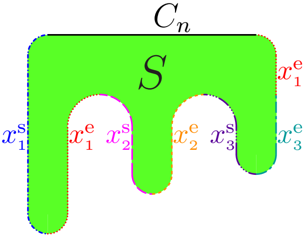
This means that our line-averaged $v_{\perp,\mathrm{inf}}$ is accurate, given:

- $\boldsymbol{x}^{\mathrm{s}}$ and $\boldsymbol{x}^{\mathrm{e}}$ lay on the same density isocontour (not just same value of n)
- $\boldsymbol{v}$ and $\boldsymbol{v}_{\mathrm{inf}}$ (or more precisely $\boldsymbol{v}_{\mathrm{inf}} - \boldsymbol{v}$) are divergence-free

We may eliminate both components of the invisible velocity by averaging over the area inside a density isocontour.

Our previous result has the special cases

$$\int_{x^{\rm s}}^{x^{\rm e}} \mathrm{d}x'\, v_{y,\mathrm{inf}}(x', y) = \int_{x^{\rm s}}^{x^{\rm e}} \mathrm{d}x'\, v_y$$
$$\int_{y^{\rm s}}^{y^{\rm e}} \mathrm{d}y'\, v_{x,\mathrm{inf}}(x, y') = \int_{y^{\rm s}}^{y^{\rm e}} \mathrm{d}y'\, v_x$$



Let S be a simply-connected area bounded by a density isocontour C_n, e.g.⇑

$$\int_S \mathrm{d}A\, v_{y,\mathrm{inf}} = \int_{y^{\rm s}}^{y^{\rm e}} \mathrm{d}y' \sum_{j=1}^{N(y')} \int_{x_j^{\rm s}(y')}^{x_j^{\rm e}(y')} \mathrm{d}x'\, \partial_x(\phi + \Phi) = \int_S \mathrm{d}A\, v_y,$$

because $\Phi(x_j^{\rm e}(y'), y') = \Phi(x_j^{\rm s}(y'), y')$ for all j and y'. Similarly

$$\int_S \mathrm{d}A\, v_{x,\mathrm{inf}} = \int_S \mathrm{d}A\, v_x, \text{ thus } \int_S \mathrm{d}A\, \boldsymbol{v}_{\mathrm{inf}} = \int_S \mathrm{d}A\, \boldsymbol{v}.$$

The averaging result has some easy generalizations.

Suppose the averaging result holds for both an outer region $S_{\rm out}$ and an inner region $S_{\rm in} \subset S_{\rm out}$, then

$$\int_{S_{\rm out}-S_{\rm in}} \mathrm{d}A\, \boldsymbol{v}_{\mathsf{inf}} = \int_{S_{\rm out}-S_{\rm in}} \mathrm{d}A\, \boldsymbol{v}.$$

Let $S_{\rm in}$ be bounded by an isocontour $C_{\rm in}$ with $n = n_{\rm in}$ and $S_{\rm out}$ bounded by a $C_{\rm out}$ with $n = n_{\rm out}$, then differentiate this relation with respect to $n_{\rm out}$ to get

$$\int_{C_{\rm in}} \frac{\mathrm{d}\ell}{|\nabla n|} \boldsymbol{v}_{\mathsf{inf}} = \int_{C_{\rm in}} \frac{\mathrm{d}\ell}{|\nabla n|}\, \boldsymbol{v}.$$

Since $n = n_{\rm in}$ is a constant on $C_{\rm in}$, we may also conclude

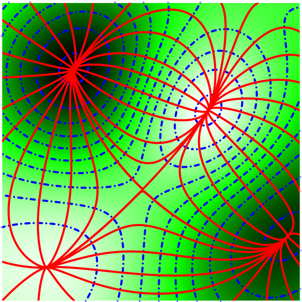
$$\int_{C_{\rm in}} \frac{\mathrm{d}\ell}{|\nabla n|} n\boldsymbol{v}_{\mathsf{inf}} = \int_{C_{\rm in}} \frac{\mathrm{d}\ell}{|\nabla n|}\, n\boldsymbol{v} \Rightarrow \int_S \mathrm{d}A\, n\boldsymbol{v}_{\mathsf{inf}} = \int_S \mathrm{d}A\, n\boldsymbol{v},$$

so the appropriate area integral of density flux is also accurate. The same result holds with $n \to N$, where N is any function that is constant on isocontours of n.

Digression: Irrotational flows tend to be well-defined.

For this slide only, assume irrotational flow

$$\nabla \times \boldsymbol{v} = 0 \Rightarrow \boldsymbol{v} = -\nabla\psi, \text{ thus}$$
$$(\nabla n) \cdot \nabla\psi = \partial_t n.$$



In this case, the invisible velocity corresponds to all Ψ s.t. $(\nabla n) \cdot \nabla\Psi = 0,$

thus Ψ constant along the "gradient curves" traced out by ∇n, e.g. red lines in this sketch$\Rightarrow$

One then has many analogous averaging results for example the averaged inferred velocity is accurate,

$$\int_S \mathrm{d}A\, \boldsymbol{v}_{\mathrm{inf}} = \int_S \mathrm{d}A\, \boldsymbol{v},$$

for an area S bounded by a closed contour made of gradient curves, BUT:

- the gradient lines often connect all points in an image, meaning that the inferred irrotational flow is pointwise accurate (without averaging),
- for irrotational flows, the term $n\nabla \cdot \boldsymbol{v}$ should be retained
    - but the additional term $n\nabla^2\psi$ is easy to manage

To infer a consistent divergence-free velocity, we recast the problem as advection-diffusion of ϕ.

Multiply the continuity equation by a constant c and add a dissipation term:

$$\boldsymbol{v}_n \cdot \nabla\phi = S_\phi - \mathcal{D}\phi, \text{ with } \boldsymbol{v}_n \doteq c\hat{z} \times \nabla n,\ S_\phi \doteq c\partial_t n,$$

and $\mathcal{D}$ a positive linear operator.

- A steady-state advection/diffusion (of ϕ), with a source term.
- $\hat{z} \times \nabla n \cdot \nabla\phi = \{n, \phi\}$ can (should) be discretized with Arakawa bracket.
- Symmetrizing over $c = \pm 1$ often reduces artifacts.

For comparison, Amini (*Computer Vision* 1994) used a variational approach to get the higher-order equation

$$\{n, \{n, \phi\}\} = \{n, \partial_t n\} + \mathcal{D}\phi,$$

roughly $\hat{z} \times \nabla n \cdot \nabla$ of my equation (except the dissipation/regularization $\mathcal{D}$).

We can mitigate the effects of incompatible data.

Let $\mathrm{d}\boldsymbol{\ell}$ point along a density isocontour, then solve by characteristics

$$\int \mathrm{d}\boldsymbol{\ell} \cdot \nabla\phi = \int \mathrm{d}\ell \frac{\hat{z} \times \nabla n}{|\nabla n|} \cdot \nabla\phi = \int \mathrm{d}\ell \frac{\partial_t n}{|\nabla n|}$$

For a closed density isocontour, single-valued ϕ requires: $\oint_{C_n} \mathrm{d}\ell \frac{\partial_t n}{|\nabla n|} = 0$

Data that fails to satisfy this has a portion of $\partial_t n$ (call it $\overline{\partial_t n}$) that is constant along density isocontour, for which the algorithm reduces to

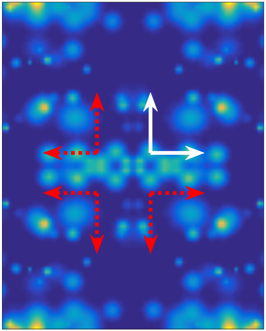
$$\mathcal{D}\overline{\phi} = c\overline{\partial_t n}, \text{ thus } \overline{\phi} \propto \mathcal{D}^{-1} c\overline{\partial_t n}$$

becomes large for small $|\mathcal{D}|$.

To mitigate the effects of this:

- Average results of calculations with $c = \pm 1$, canceling $\overline{\phi}$
  - but nondissipative equation is invariant to c
- Subtract the isocontour average from $\partial_t n$ before velocimetry
  - Can use $\overline{\partial_t n} \approx \nu \bar{S}$ for $\bar{S}$ solving $\{n, \bar{S}\} = -\nu \bar{S} + \partial_t n$

The differential constraint $\nabla \cdot \boldsymbol{v} = 0$ means we must consider boundary conditions for ϕ.



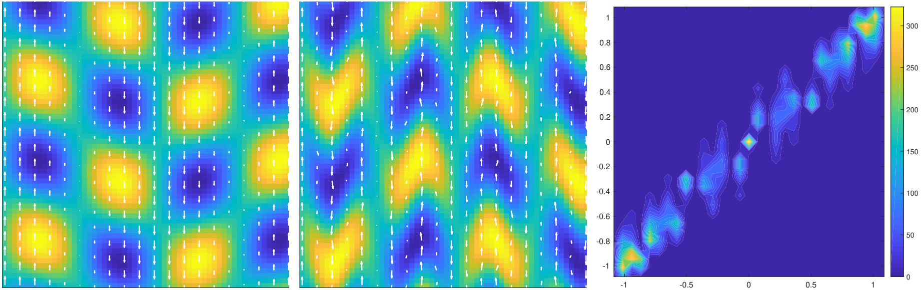
Many typical choices have implications, spelled out here for example x boundaries:

- Periodic implies $\int \mathrm{d}x \, v_{y,\mathrm{inf}} = 0$.
- Dirichlet forces $\int \mathrm{d}x \, v_{y,\mathrm{inf}} = \phi(L_x, y) - \phi(0, y)$
  - also sets $v_{x,\mathrm{inf}} = -\partial_y \phi$ at x boundaries
- Neumann $\partial_x \phi = 0$ sets $v_{y,\mathrm{inf}} = 0$ at boundary
  - but $v_{x,\mathrm{inf}}$ is free

The best general-purpose appears to be periodicity on an extended domain:

- Original domain in upper-right quadrant $(x > 0, \ y > 0)$
- Other quadrants: $n(x, y) = n(|x|, |y|)$, $S_\phi(x, y) = (\mathrm{sign} x)(\mathrm{sign} y) S_\phi(|x|, |y|)$
- Nondissipative equation unchanged, except sign flip in lower-right and upper-left quadrants. (Note: dissipation has no sign flip.)
- Guarantees well-behaved countours, sets tangential velocity to zero.
- Actually only requires doubling domain size, with "flip-periodic" BCs.

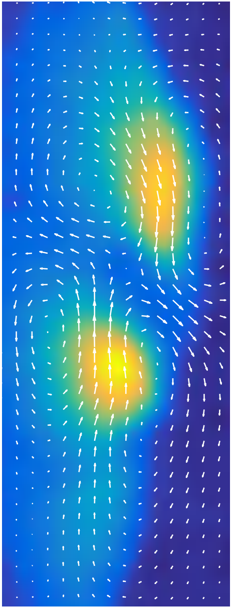
Simple tests demonstrate basic algorithm performance.



- Subject simple sinuosoidal pattern to horizontally-sheared vertical flow
  - Left: $\boldsymbol{v} = (\cos x)\,\hat{y}$, thus $\ell < L$ (regular ordering, but marginal)
  - Center: $\boldsymbol{v} = (\cos 5x)\,\hat{y}$, thus $\ell > L$ (opposite standard ordering)
- Algorithm still captures v_y pretty well (inferred vs actual v_y, right)
- For real-life applications, many other challenges:
  - boundary conditions, incompatible or underresolved data,...
  - tests of more realistic cases are ongoing
- Fundamentally, the 'hidden velocity' is still there
  - other solutions equally valid, but proper averaging removes the difference

We are starting to apply the algorithm to NSTX GPI data.

- NSTX gas-puff-imaging (GPI) diagnostic
  - view along $\boldsymbol{B}$ to see perp dynamics
- Emission a complicated function
  - fluctuations taken roughly $\propto n$
  - perp velocity approx divergence-free
- In terms of GPI, averaging result roughly:
  - blob translation accurate, spin may not be
  - result only holds if $\nabla \cdot \boldsymbol{v}_{\text{inf}} = 0$
- NSTX GPI used to study L-H transition
  - 17 shots from 2010 campaign
  - evaluate zonal flows and energy transfer
  - challenging problem: error analysis ongoing
- Algorithm may be used for any other 2D movies with roughly incompressible 2D flows
  - Simple, lightweight Matlab implementation



Conclusions

- Velocities can be inferred from 2D movies via a continuity equation
  - but the problem is underdetermined without additional constraints
- Assumption of incompressible flow $\nabla \cdot \boldsymbol{v} = 0$ mostly disambiguates
  - but certain invisible flows are still permitted
  - such errors can survive even under total spatial averaging
  - if ∇n changes direction more rapidly in space than $\boldsymbol{v}$ varies, then can disambiguate (this assumption underlies classic “optical flow”)
- Even for strongly-sheared flows, the invisible velocity has restricted form: $\int_S \mathrm{d}A\, \boldsymbol{v}_{\text{inf}} = \int_S \mathrm{d}A\, \boldsymbol{v}$, when area S is bounded by a density isocontour
  - For this, need both inferred ($\boldsymbol{v}_{\text{inf}}$) and actual ($\boldsymbol{v}$) velocities incompressible
  - Density flux and generalizions are also accurate under this average
  - Other averaging regions may be easily constructed
- To exploit this result, do velocimetry enforcing $\nabla \cdot \boldsymbol{v} = 0$
  - Solve as inhomogeneous advection-diffusion equation for stream function ϕ
    - Or use (higher-order) variational formulation
  - Best boundary conditions are periodicity on a symmetric extension
  - Algorithm is implemented in Matlab, and is being tested and applied to NSTX GPI data.