Current Singularity Formation in Line-tied Magnetic Fields: the Parker Problem

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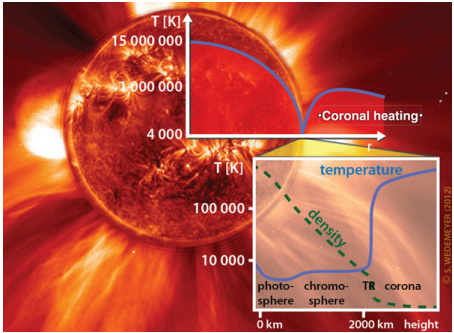
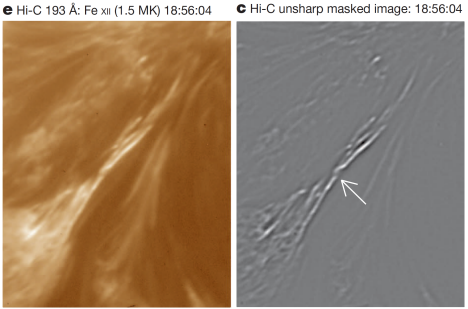
Coronal heating by nano-flares

- 'Conflicting' features of solar corona:
  - nearly perfect conductivity ($S \gtrsim 10^{12}$);
  - very high temperature ($T \gtrsim 10^6$ K).



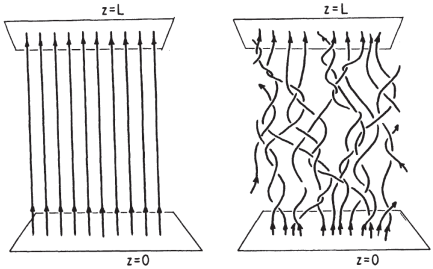
- A possible mechanism[1]:
  - current singularities tend to form;
  - then, magnetic reconnection.
- Observational evidence exists[2].



[1] E. N. Parker, Astrophys. J. 174, 499 (1972).
[2] J. W. Cirtain et al., Nature 493, 501 (2013).

The Parker problem

- Ideal, mathematical abstraction[3]:
  - can genuine current singularities emerge in 3D line-tied plasmas?
- Practical relevance: tendency matters;
  - if not singular, how thin?
- Line-tied geometry: no closed field lines;
  - unlike toroidal fusion plasmas.
- Challenge: to preserve magnetic topology;
  - both analytically and numerically.
- Still controversial today.



(a) (b)

[3] E. N. Parker, *Spontaneous Current Sheets in Magnetic Fields: With Applications to Stellar X-rays* (Oxford University Press, New York, 1994).

Highlights

- Lagrangian labeling: the favorable description;
  - built-in frozen-in equation;
  - magnetic topology automatically preserved.
- Discretization: ideal MHD on a moving mesh;
  - **no artificial reconnection**, unlike Eulerian methods;
  - variational, with discrete exterior calculus.
- Current singularity formation in 2D: **conclusively** confirmed;
  - the Hahm–Kulsrud–Taylor (HKT) problem;
  - analytical and numerical solutions agree.
- Extension to 3D line-tied geometry: inconclusive;
  - smooth linear solution;
  - smooth nonlinear solution for short systems;
  - scaling **suggests** finite-length singularity.

Ideal MHD in Lagrangian labeling

- Dynamical variable: fluid configuration map $\mathbf{x}(\mathbf{x}_0, t)$. [$x_{ij} \doteq \partial x_i/\partial x_{0j}$, $J \doteq \det(x_{ij})$.]
- Advection equations are formally solved:

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \quad \Leftrightarrow \quad \rho \, \mathrm{d}^3 x = \rho_0 \, \mathrm{d}^3 x_0 \quad \Leftrightarrow \quad \rho = \rho_0 / J, \tag{1a}$$

$$\partial_t (p/\rho^\gamma) + \mathbf{v} \cdot \nabla (p/\rho^\gamma) = 0 \quad \Leftrightarrow \quad p/\rho^\gamma = p_0/\rho_0^\gamma \quad \Leftrightarrow \quad p = p_0/J^\gamma, \tag{1b}$$

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \quad \Leftrightarrow \quad B_i \, \mathrm{d}S_i = B_{0i} \, \mathrm{d}S_{0i} \quad \Leftrightarrow \quad B_i = x_{ij} B_{0j}/J. \tag{1c}$$

- Then, **built into** the momentum equation:

$$\rho_0 \ddot{x}_i = B_{0j} \frac{\partial}{\partial x_{0j}} \left(\frac{x_{ik} B_{0k}}{J} \right) - \frac{\partial J}{\partial x_{ij}} \frac{\partial}{\partial x_{0j}} \left(\frac{p_0}{J^\gamma} + \frac{x_{kl} x_{km} B_{0l} B_{0m}}{2J^2} \right). \tag{2}$$

- A field theory, with Lagrangian[4]:

$$L[\mathbf{x}] = \int \left[\frac{1}{2} \rho_0 \dot{x}^2 - \frac{p_0}{(\gamma - 1) J^{\gamma - 1}} - \frac{x_{ij} x_{ik} B_{0j} B_{0k}}{2J} \right] \mathrm{d}^3 x_0. \tag{3}$$

[4] W. A. Newcomb, Nucl. Fusion 1962, Suppl. 2, 451 (1962)

Current singularity formation: problem statement

- Customary to study equilibria. (Dynamics can be complicated.)
- Perfectly-conducting plasma: magnetic topology preserved.
- Eulerian equilibrium equation,

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \nabla p, \tag{4}$$

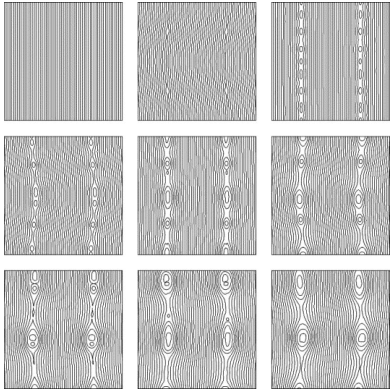
  is underdetermined:
  - difficult to attach the topological constraint;
  - existence of singularities does not imply formation.
- Lagrangian equilibrium equation has the constraint built-in:

$$B_{0j} \frac{\partial}{\partial x_{0j}} \left(\frac{x_{ik} B_{0k}}{J} \right) = \frac{\partial J}{\partial x_{ij}} \frac{\partial}{\partial x_{0j}} \left(\frac{p_0}{J^\gamma} + \frac{x_{kl} x_{km} B_{0l} B_{0m}}{2J^2} \right). \tag{5}$$

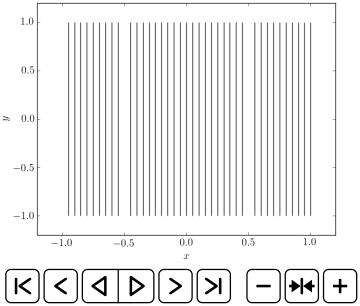
- **Objective**: find solutions with current singularities to Eq. (5), given smooth initial and boundary conditions.

Ideal MHD on a moving mesh

- Eulerian ideal MHD simulations:
  - artificial reconnection[5].



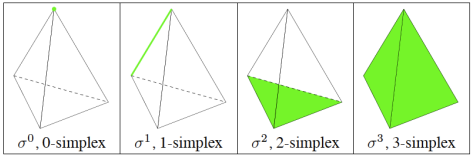
- Discretization in Lagrangian labeling:
  - moving mesh simulates fluid motion;
  - no artificial reconnection.



- Can be done directly[6], but we shall...

[5] T. A. Gardiner and J. M. Stone, J. Comput. Phys. 205, 509 (2005).
[6] I. J. D. Craig and A. D. Sneyd, Astrophys. J. 311, 451 (1986).

Variational integration for ideal MHD



- Discrete exterior calculus (DEC)[7]:
  - discrete manifold;
  - discrete differential forms;
  - discrete Stokes' theorem guarantees $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot \mathbf{j} = 0$.

- Discretize the Lagrangian:
  - conservative many-body form[8],

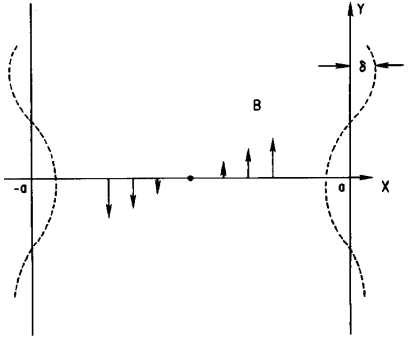
$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{\sigma_0^0} \frac{1}{2} M(\sigma_0^0)\dot{x}^2 - W(\mathbf{x}); \quad (6)$$

  - discrete equilibrium, $\partial W / \partial \mathbf{x} = 0$.
- Add friction for equilibration.
- (Structure-preserving numerical methods in plasma physics[9]: guiding center, PIC...)

[7] M. Desbrun et al., arXiv:math/0508341 (2005).
[8] Y. Zhou et al., Phys. Plasmas 21, 102109 (2014).
[9] P. J. Morrison, Phys. Plasmas 24, 055502 (2017); J. Xiao et al., Plasma Sci. Technol. 20, 110501 (2018).

The Hahm–Kulsrud–Taylor (HKT) problem

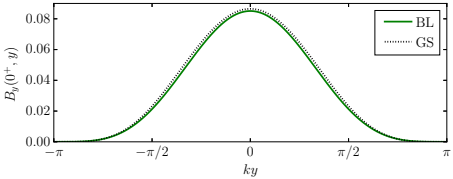
- Relevance to toroidal fusion plasmas:
  - resonant magnetic perturbations;
  - 3D equilibria with nested surfaces.
- 2D incompressible plasma, $B_{0y} = x_0$.
- Mirrored sinusoidal boundary displacement[10]: $\xi(\pm a, y) = \mp\delta \cos ky$.
- Multiple known equilibrium solutions[11]:
  - some contain current singularities;
  - magnetic topology not preserved.

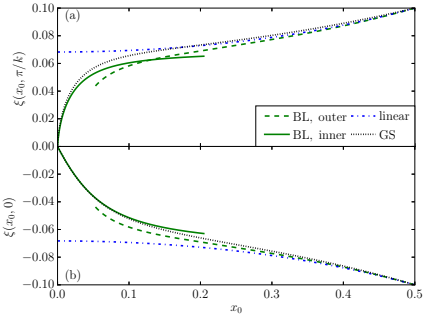


[10] T. S. Hahm and R. M. Kulsrud, Phys. Fluids 28, 2412 (1985).
[11] R. L. Dewar et al., Phys. Plasmas 20, 082103 (2013); J. Loizu et al., Phys. Plasmas 22, 022501 (2015).

Topologically constrained equilibrium solution

- ξ: Lagrangian displacement.
  - Linear solution: unphysical, discontinuous displacement.
  - Boundary-layer (BL) solution[12]: nonlinear, continuous displacement.
  - Numerical solution: flux-preserving Grad–Shafranov (GS) solver[13].



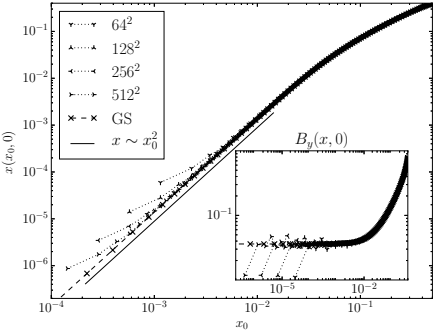


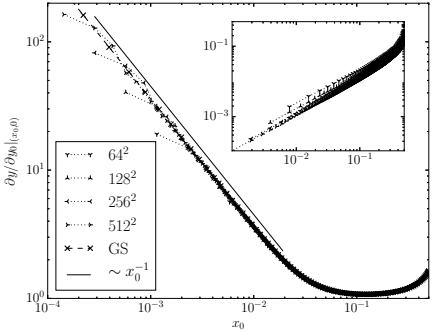
- Discontinuous $\langle B_y \rangle$ but $\iota \sim \langle B_y^{-1} \rangle^{-1}$:
  - stays continuous, despite a recent claim[14].

[12] Y. Zhou et al., arXiv:1810.08268 (2018); M. N. Rosenbluth et al., Phys. Fluids 16, 1894 (1973).
[13] Y.-M. Huang et al., Astrophys. J. 699, L144 (2009).
[14] J. Loizu and P. Helander, Phys. Plasmas 24, 040701 (2017).

Fully Lagrangian solution converges to GS solution

- Normal (x) direction, at $y_0 = 0$:
  - $x \sim x_0^2$, $B_y = B_{0y}/(\partial x/\partial x_0) \sim \mathrm{sgn}(x)$;
  - plasma infinitely compressed[15].
- $J = 1$ requires tangential (y) direction:
  - $\partial y/\partial y_0 \sim x_0^{-1}$, non-differentiable;
  - plasma infinitely stretched.



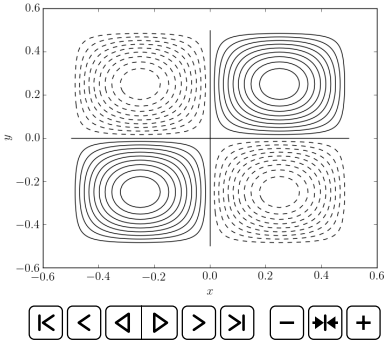


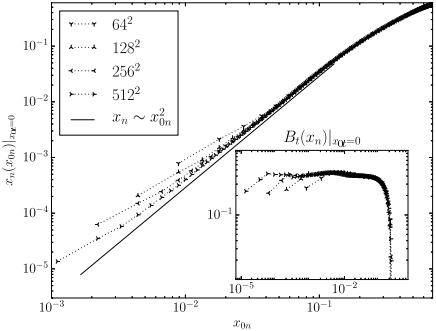
- GS solver: limited applicability. Lagrangian method: more complex topology, 3D.

[15] Y. Zhou et al., Phys. Rev. E 93, 023205 (2016).

General mechanism for current singularities in 2D

- Coalescence instability of magnetic islands[16]:
  - different drive, more complex topology;
  - same quadratic mapping as in HKT[17].

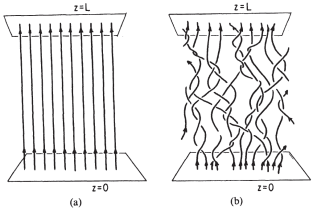




- Recipe: compress a sheared field.
- How about 3D line-tied geometry?

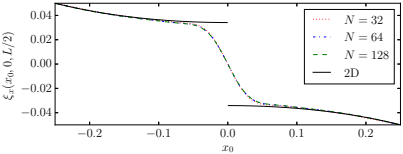
[16] D. W. Longcope and H. R. Strauss, Phys. Fluids B 5, 2858 (1993).
[17] Y. Zhou, Ph.D. thesis, Princeton University, arXiv:1708.08523 (2017).

Line-tied geometry: smooth linear solution

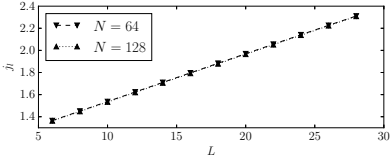


- Parker's original setup:
  - uniform $\mathbf{B}_0 = \hat{z}$;
  - prescribed footpoint motion.
- Our formula:
  - known 'susceptible' $\mathbf{B}_0$ in 2D;
  - no-slip footpoints;
  - **focus**: effect of line-tying.

- Line-tied HKT: boundary perturbation $\xi(\pm a, y, z) = \mp\delta \cos ky \sin(\pi z/L)$.
- Given L, linear solution is smooth[18].



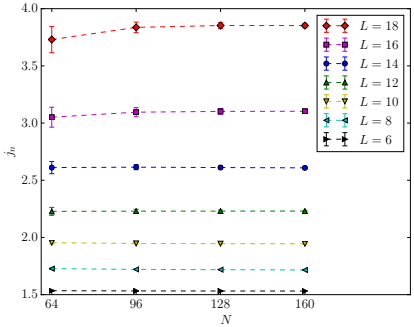
- Current density j_l scales linearly with L.

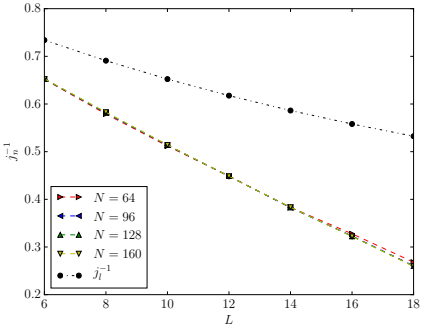


[18] Consistent with E. G. Zweibel and H.-S. Li, Astrophys. J. 312, 423 (1987).

Nonlinear solutions: convergence and scaling

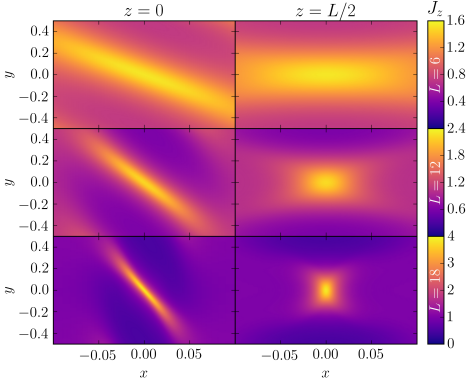
- For short systems, nonlinear solutions are smooth and well-resolved[19].
- $j_n \sim (L_n - L)^{-1}$, **suggests** finite-length singularity:
  - stronger than proposed exponential scaling[20];
  - echoes existing analytical theory[21].





[19] Y. Zhou et al., Astrophys. J. 852, 3 (2018).
[20] D. W. Longcope and H. R. Strauss, Astrophys. J. 437, 851 (1994).
[21] C. S. Ng and A. Bhattacharjee, Phys. Plasmas 5, 4028 (1998).

Longer systems: stronger shear tear up the mesh



Current density at footpoint and mid-plane

- Reduced MHD equilibrium equation,

$$\mathbf{B} \cdot \nabla j_z = 0. \tag{7}$$

Singularity threaded through all z[22].

- In-plane field ($z = z_0, B_{0z} = 1, J = 1$),

$$\mathbf{B}_\perp = \frac{\partial \mathbf{x}_\perp}{\partial \mathbf{x}_{0\perp}} \cdot \mathbf{B}_{0\perp} + \frac{\partial \mathbf{x}_\perp}{\partial z_0}. \tag{8}$$

- At the footpoints ($\mathbf{x}_\perp = \mathbf{x}_{0\perp}$ at $z = 0, L$),

$$j_z = j_{0z} + \hat{z} \cdot \nabla_\perp \times \frac{\partial \mathbf{x}_\perp}{\partial z_0}. \tag{9}$$

Only the second term can be singular.

- Strong shear: an inherent feature.

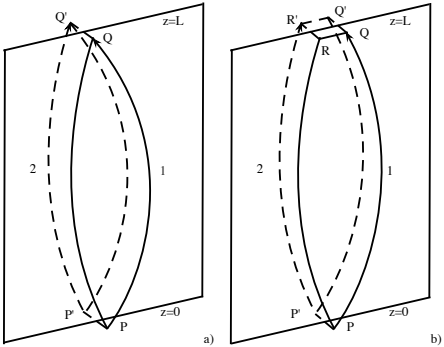
[22] A. A. van Ballegooijen, Astrophys. J. 298, 421 (1985).

Summary and outlook

- Lagrangian labeling: the favorable description.
  - Built-in frozen-in equation.
  - Guaranteed invariance of magnetic topology.
- Numerical method: ideal MHD on a moving mesh.
  - No artificial reconnection.
  - Variational discretization; with discrete exterior calculus.
  - Re-meshing? Eulerian?
- Current singularity formation in 2D: conclusively confirmed.
  - Analytical and numerical solutions agree.
  - General mechanism in 2D.
  - Dynamics? Time scale?
- 3D line-tied geometry: inconclusive.
  - Smooth nonlinear solution for short systems.
  - Scaling suggests finite-length singularity.
  - Better discretization more robust against strong shear?
  - Generalize RDR's boundary-layer approach?

On the impossibility of current singulairities

- An existing proof[23] has an oversight.



- (a) Alleged contradiction:

$$\int_{1+2} B(l)\,[1-\mathbf{b}_2(l)\cdot\mathbf{b}_1(l)]\;\mathrm{d}l = 0. \quad (10)$$

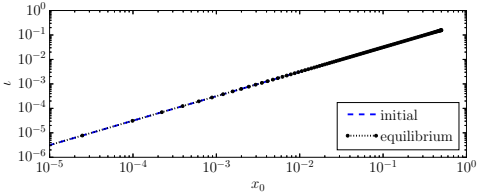
- (b) Accounting for the 'hole':

$$\int_{1+2} B(l)\,[1-\mathbf{b}_2(l)\cdot\mathbf{b}_1(l)]\;\mathrm{d}l = \oint_{\partial\Omega} \mathbf{B}\cdot\mathrm{d}\mathbf{l} = \int_{\Omega} \mathbf{j}\cdot\mathrm{d}\mathbf{S}. \quad (11)$$

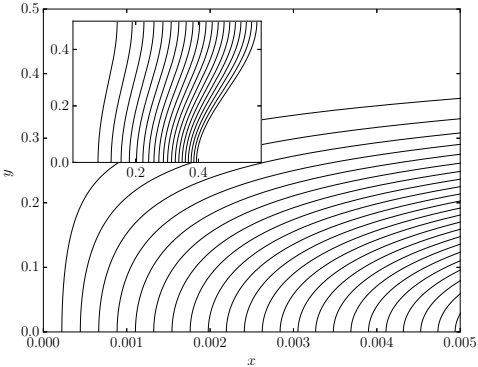
- Magnetic shear keeps possibility alive.

[23] S. C. Cowley, D. W. Longcope, and R. N. Sudan, Phys. Rep. 283, 227 (1997).

Discontinuous $\langle B_y \rangle$ but continuous rotational transform



- Rotational transform $\iota \sim \langle B_y^{-1} \rangle^{-1}$:
  - stays invariant and continuous;
  - contrary to a recent claim.



Equilibrium solution: flux surfaces

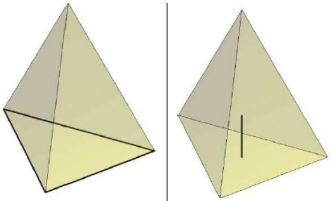
Discrete Lagrangians and advection equations

- In Eulerian labeling (σ^k, t): fixed mesh,

$$L(\mathbf{v}, \rho, p, \mathbf{B}) = \sum_{\sigma^3} \left[\frac{\langle \rho, \sigma^3 \rangle}{8} \sum_{\sigma^0 \prec \sigma^3} \langle v^2, \sigma^0 \rangle - \frac{\langle p, \sigma^3 \rangle}{\gamma - 1} - \sum_{\sigma^2 \prec \sigma^3} \frac{|\star \sigma^2|}{2|\sigma^2|} \langle B, \sigma^2 \rangle^2 \right]. \tag{12}$$

- In Lagrangian labeling (σ_0^k, t): moving mesh,

$$L(\mathbf{x}, \dot{\mathbf{x}}) = \sum_{\sigma_0^3} \left[\frac{\langle \rho_0, \sigma_0^3 \rangle}{8} \sum_{\sigma_0^0 \prec \sigma_0^3} \dot{x}^2 - \frac{\langle p_0, \sigma_0^3 \rangle}{(\gamma - 1) J^{\gamma - 1}} - \sum_{\sigma_0^2 \prec \sigma_0^3} \frac{|\star \sigma^2|}{2|\sigma^2|} \langle B_0, \sigma_0^2 \rangle^2 \right]. \tag{13}$$



σ^2, 2-simplex $\quad$ $\star\sigma^2$, 1-cell

- Discrete advection equations:

$$\langle \rho, \sigma^3 \rangle = \langle \rho_0, \sigma_0^3 \rangle, \tag{14a}$$

$$\frac{\langle p, \sigma^3 \rangle |\sigma^3|^{\gamma - 1}}{\langle \rho, \sigma^3 \rangle^\gamma} = \frac{\langle p_0, \sigma_0^3 \rangle |\sigma_0^3|^{\gamma - 1}}{\langle \rho_0, \sigma_0^3 \rangle^\gamma}, \tag{14b}$$

$$\langle B, \sigma^2 \rangle = \langle B_0, \sigma_0^2 \rangle. \tag{14c}$$