An adjoint approach for the shape gradients of 3D MHD equilibria

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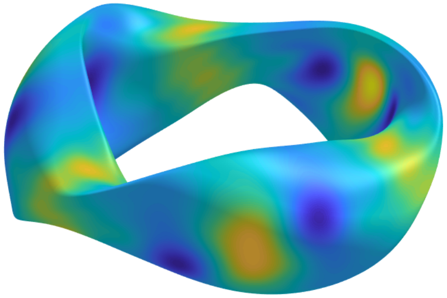


PPPL Research Seminar
November 25, 2019

Outline

- **Introduction**
  - Stellarator shape optimization
  - Adjoint methods
- Shape gradients for MHD equilibria
- Perturbed equilibrium approach
- Conclusions

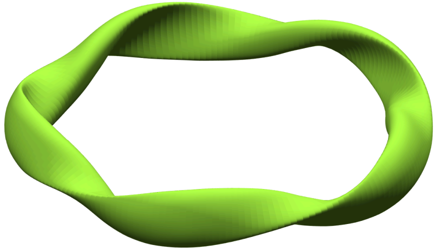
Stellarators require shape optimization (I)

Traditional two-step optimization

1. **MHD equilibrium optimization**
   (e.g. STELLOPT[1], ROSE[2])

   How to design boundary for optimal confinement?

MHD force balance



[1]D.A. Spong et al, *Nuclear Fusion*, 41 (2001).
[2]M. Drevlak et al, *Nuclear Fusion*, 59 (2019).

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Traditional two-step optimization

1. **MHD equilibrium optimization**
   (e.g. STELLOPT[1], ROSE[2])

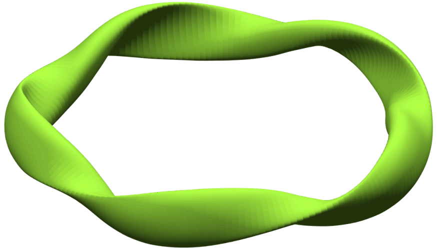
   How to design boundary for optimal confinement?

2. **Coil design**
   (e.g. REGCOIL[3], FOCUS[4])

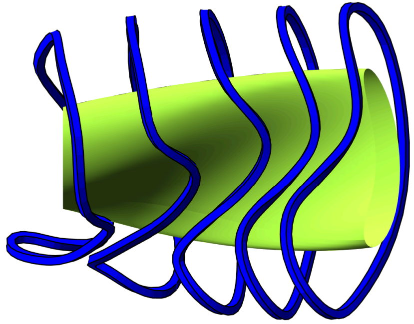
How to design feasible coils to obtain desired plasma boundary?

How sensitive is a figure of merit to coil displacements?

MHD force balance



Biot-Savart



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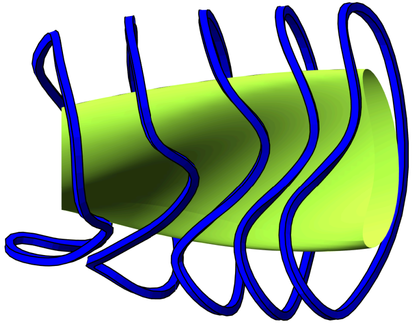
[3]M. Landreman, *Nuclear Fusion,* 57 (2017).
[4]C. Zhu et al, *Nuclear Fusion,* 58 (2017).

Stellarators require shape optimization (II)

Combined one-step optimization

MHD force balance



1. **MHD equilibrium direct optimization of coils**[1]

How to design coils for optimal confinement and engineering feasibility?

[1]D. Strickler et al, *IAEA FT/P2-06* (2003).

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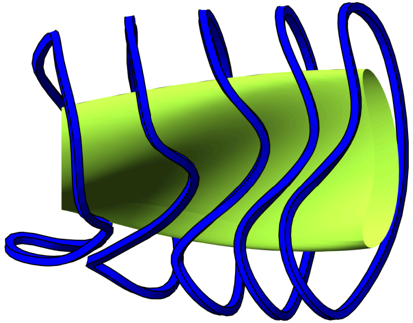
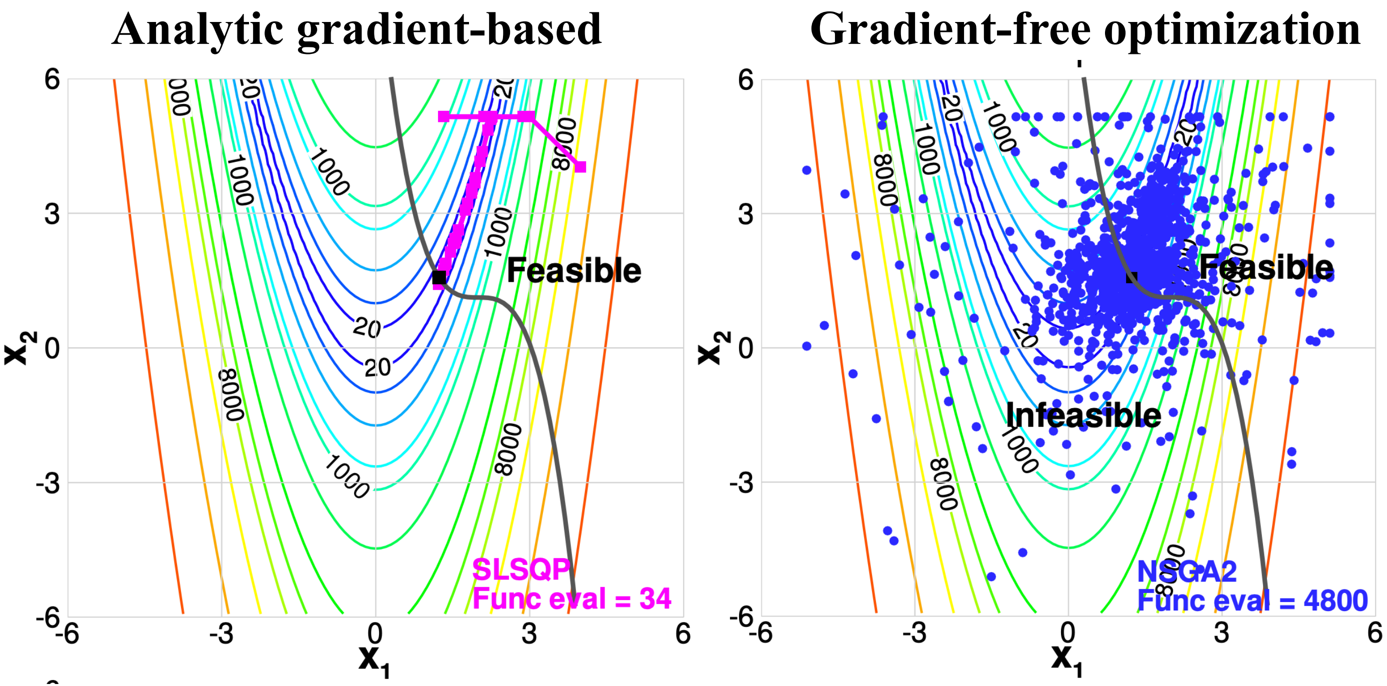


"The highest priority for technology is to better integrate the engineering design with the physics design at the earliest possible stage."
-Report from the National Stellarator Coordinating Committee[2]

[1]D. Strickler et al, *IAEA FT/P2-06* (2003).
[2]D.Gates et al, *J. Fusion Energy*, 37 (2018).

Analytic gradients are valuable in high-dimensional spaces (I)

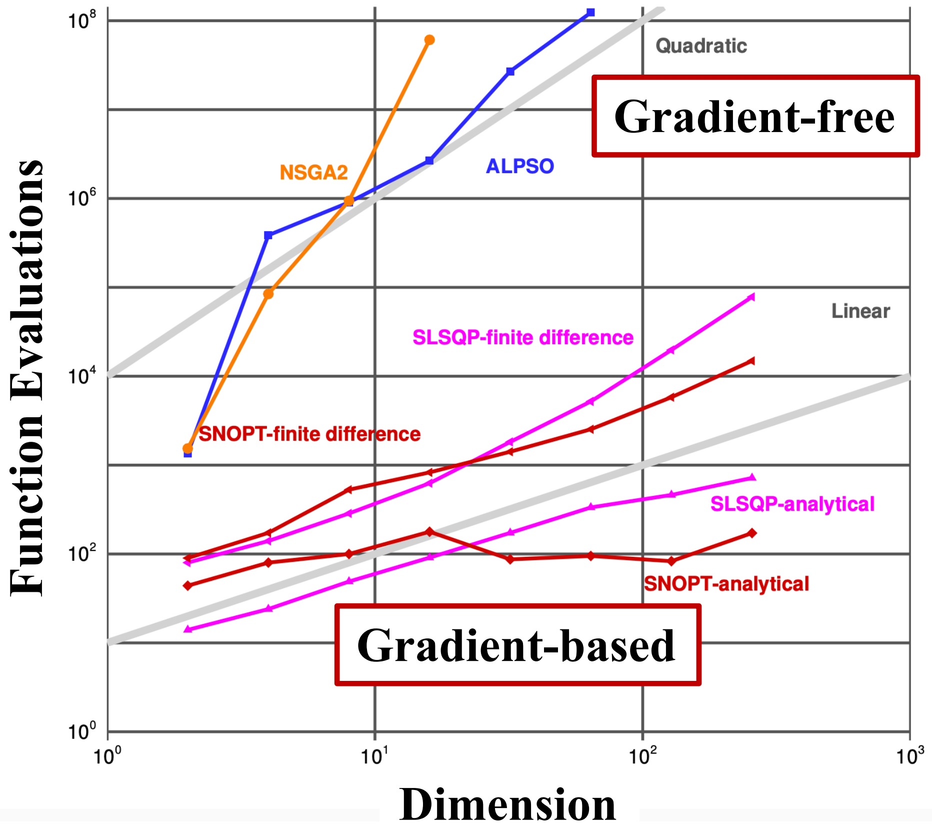


Minimization of 2D Rosenbrock function

Z. Lyu et al, *Proc. Inter. Conf. Comp. Fluid Dyn., 11* (2014).

Analytic gradients are valuable in high-dimensional spaces (II)

Minimization of ND Rosenbrock function



Z. Lyu et al, *Proc. Inter. Conf. Comp. Fluid Dyn., 11* (2014).

Adjoint method for analytic derivatives

- Figure of merit $f(\boldsymbol{x})$ s.t. $\boldsymbol{L}(\boldsymbol{x}) = 0$
- Goal: compute $\partial f(\boldsymbol{x})/\partial \Omega$ for $\Omega = \{\Omega_i\}_{i=1}^{N_\Omega}$

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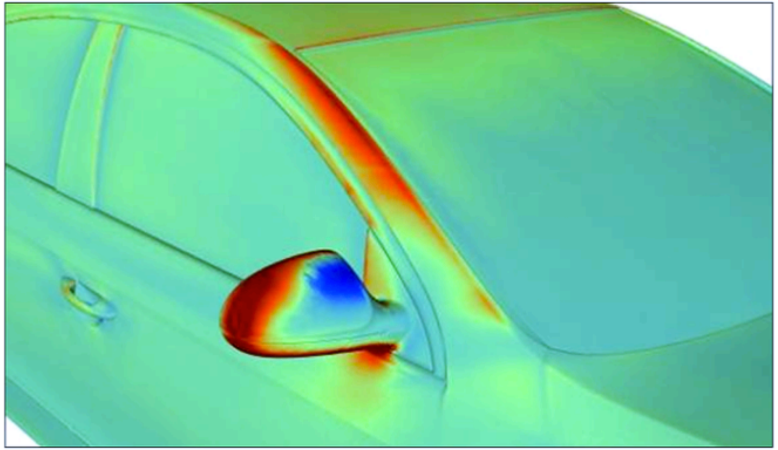
Adjoint methods widely used in computational fluid dynamics



Inward for smaller drag
Outward for smaller drag

C. Othmer, *J. Math. Industry,* 4 (2014).

Adjoint method for a linear system

- Goal: compute $\partial f(\boldsymbol{x})/\partial\Omega$ for $\Omega = \{\Omega_i\}_{i=1}^{N_\Omega}$ (**$\geq N_\Omega + 1$ solves with finite differences**)

$$f(\boldsymbol{x}) = \boldsymbol{c}^T\boldsymbol{x} \quad \text{s.t.} \quad \overleftrightarrow{\boldsymbol{A}}\boldsymbol{x} = \boldsymbol{b}$$

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- Compute perturbations of linear system

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- Get derivative with respect to all Ω_i with **2 solutions of linear system** ($\boldsymbol{x}$, $\boldsymbol{z}$)

$$\boxed{\frac{\partial f}{\partial \Omega_i} = \boldsymbol{z}^T\left(\frac{\partial \boldsymbol{b}}{\partial \Omega_i} - \frac{\partial \overleftrightarrow{\boldsymbol{A}}}{\partial \Omega_i}\boldsymbol{x}\right)}$$

Outline

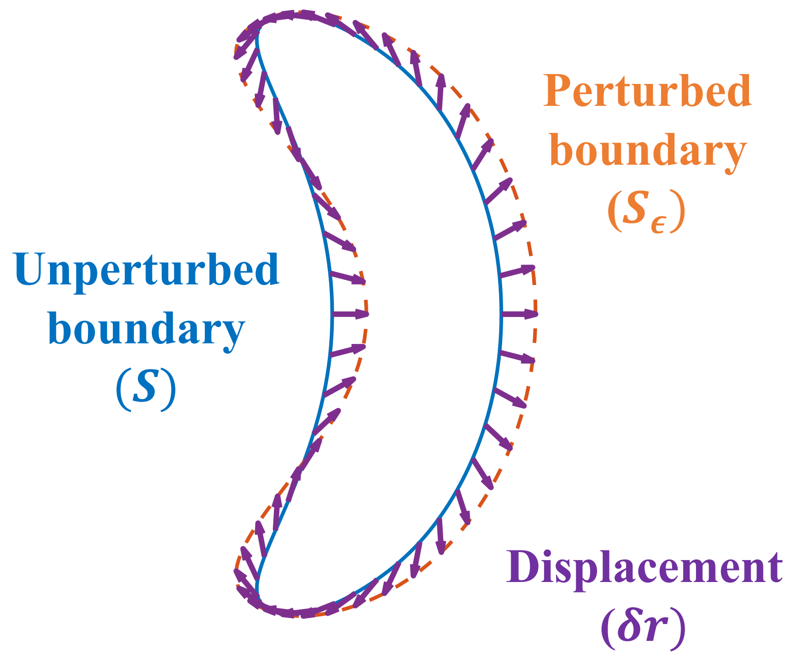
- Introduction
- **Shape gradients for MHD equilibria**
  - Introduction to shape gradients
  - Fixed-boundary relation
  - Free-boundary relation
- Perturbed equilibrium approach
- Conclusions

- ❑ Magnetic well
- ❑ Magnetic ripple
- ❑ Rotational transform
- ❑ Effective ripple ($\epsilon_{\text{eff}}^{3/2}$)
- ❑ Quasisymmetry

Describing derivatives with respect to plasma boundary

- $f(S)$ = physics objective depending on equilibrium field
- Surface is displaced by vector field δr

$$S_\epsilon = \{\boldsymbol{r}_0 + \epsilon\delta\boldsymbol{r} : \boldsymbol{r}_0 \in S\}$$



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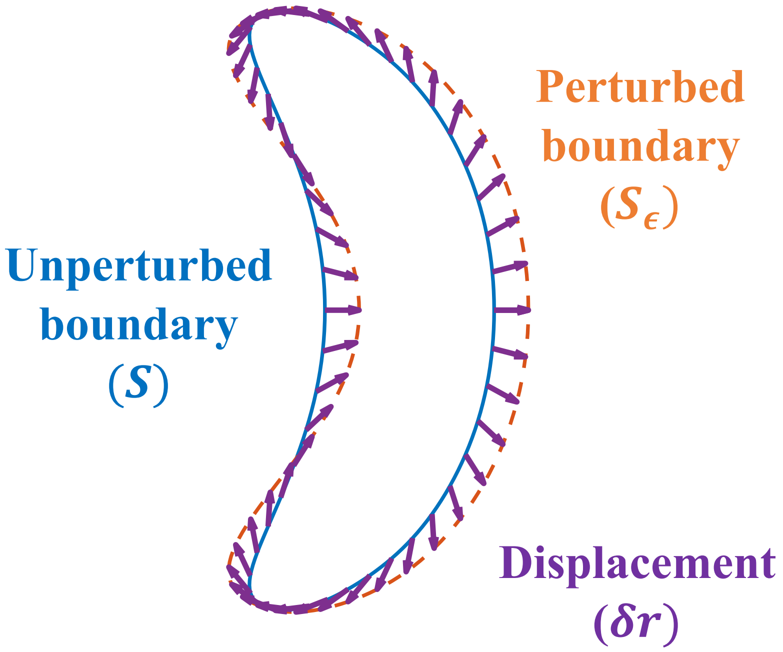

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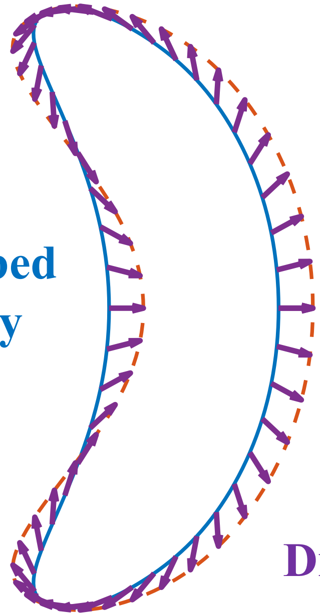
- Under assumption of smoothness
$$\delta f(\delta \boldsymbol{r}) = \int_S d^2x \, \delta \boldsymbol{r} \cdot \boldsymbol{n} \, \mathcal{G}$$
- For *any* $\delta \boldsymbol{r}$, shape gradient, $\mathcal{G}$, provides change to figure of merit, δf



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Why is the shape gradient ($\mathcal{G}$) useful?

- *Local* sensitivity information
- Quantifying engineering tolerances
- Gradient-based optimization

Computing MHD shape gradient directly is expensive

- S described by parameters $\{\Omega_i\}_1^{N_\Omega}$
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$$\mathcal{G} = \sum_j S_j \cos(m_j\theta - n_j\phi)$$

- Shape gradient computed from linear system

$$\frac{\partial f}{\partial \Omega_i} = \int_S d^2x \, S_j \cos(m_j\theta - n_j\phi) \frac{\partial \boldsymbol{r}}{\partial \Omega_i} \cdot \boldsymbol{n}$$

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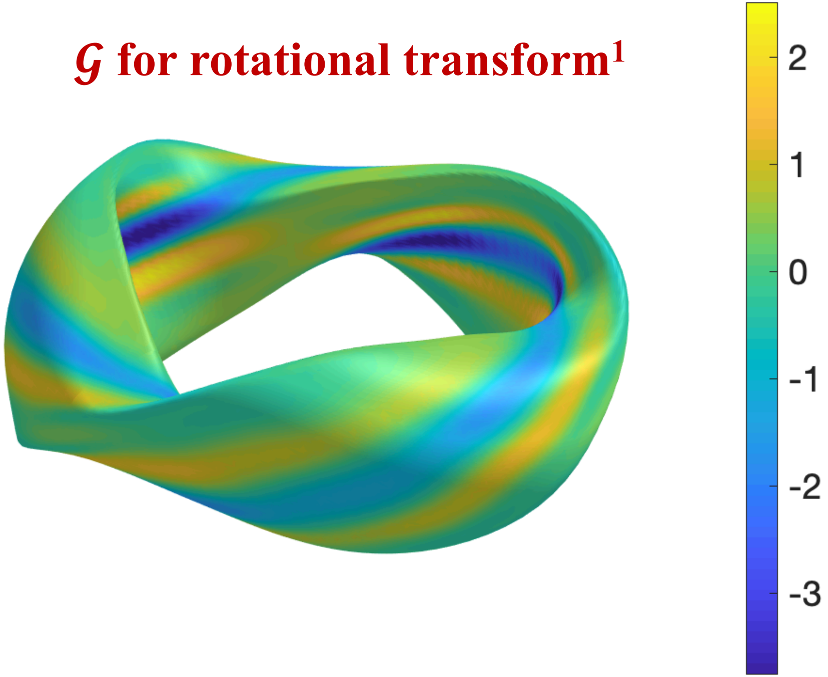
$\mathcal{G}$ for rotational transform[1]



[1]M. Landreman & E.J. Paul, *Nuclear Fusion,* 58 (2018).

Linearized MHD interpretation of shape derivatives

- MHD equilibrium with specified $p(\psi), \iota(\psi)$, and S_{plasma}

$$0 = \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} - \nabla p$$

Note: magnetic surfaces assumed (variational solution[1])

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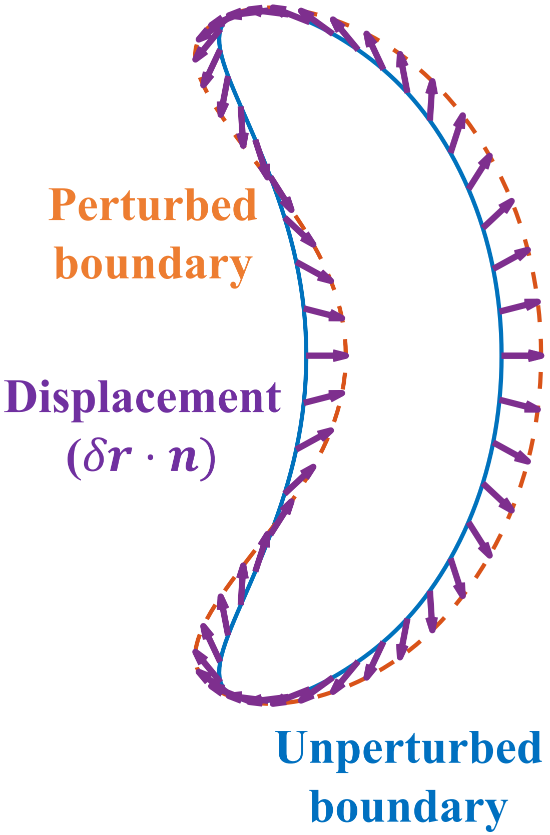

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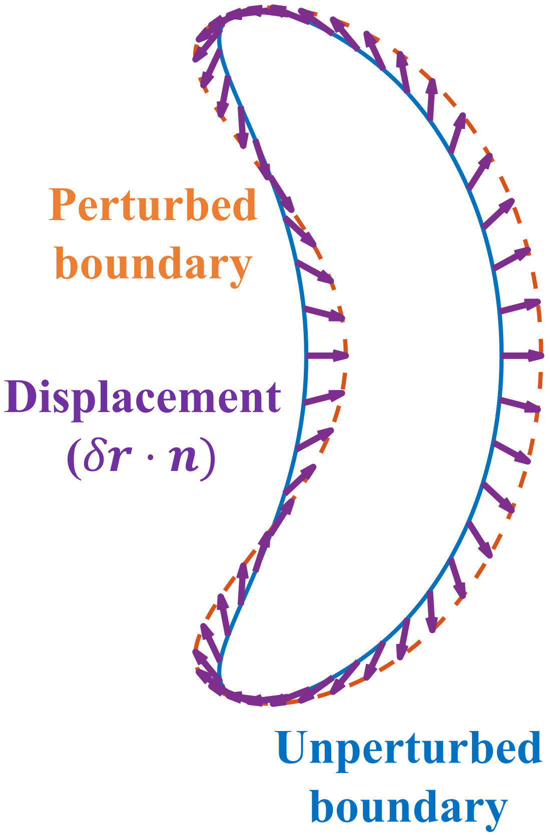
- Perturbed equilibrium with specified $\delta\boldsymbol{r}\cdot\boldsymbol{n}|_{S_{\text{plasma}}}$ satisfies

$$\boldsymbol{F}(\boldsymbol{\xi}_1) = \frac{(\nabla\times\boldsymbol{B})\times\delta\boldsymbol{B}_1 + \nabla\times(\delta\boldsymbol{B}_1)\times\boldsymbol{B}}{4\pi} - \boldsymbol{\nabla}\delta p(\boldsymbol{\xi}_1) = 0$$

$$\boldsymbol{\xi}_1\cdot\boldsymbol{n}\Big|_{S_{\text{plasma}}} = \delta\boldsymbol{r}\cdot\boldsymbol{n}\Big|_{S_{\text{plasma}}}$$



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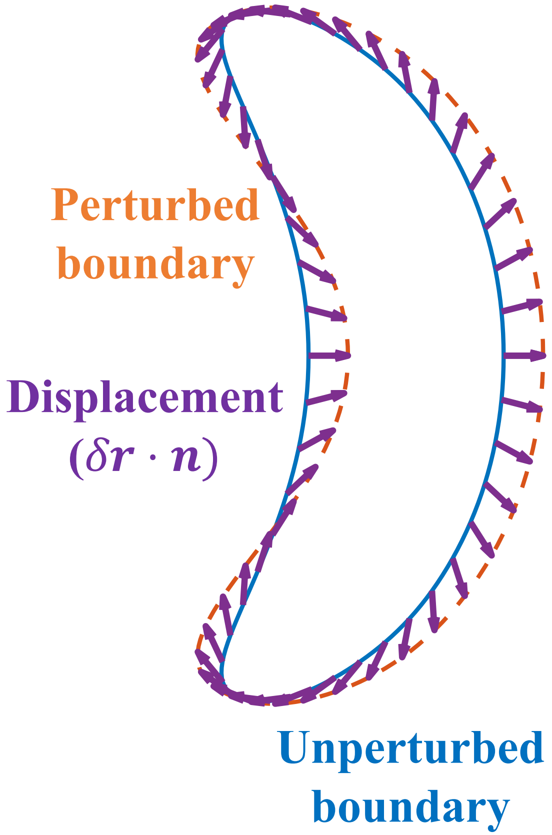
Linearized MHD interpretation of shape derivatives

Shape gradient calculation requires $\geq N_\Omega$ +1 solutions of

$$\boldsymbol{F}(\boldsymbol{\xi}_1) = 0$$

$$\boldsymbol{\xi}_1 \cdot \boldsymbol{n}\Big|_{S_{\text{plasma}}} = \delta\Omega_i \frac{\partial \boldsymbol{r}}{\partial \Omega_i} \cdot \boldsymbol{n}\Big|_{S_{\text{plasma}}}$$



Computing MHD shape gradient with adjoint approach[1]

Take advantage of self-adjointness of MHD force operator[2]

$$\int_{V_{\text{plasma}}} d^3x \left(-\boldsymbol{F}(\boldsymbol{\xi}_1)\cdot\boldsymbol{\xi}_2 + \boldsymbol{F}(\boldsymbol{\xi}_2)\cdot\boldsymbol{\xi}_1\right) + \frac{1}{4\pi}\int_{S_{\text{plasma}}} d^2x\, \boldsymbol{n}\cdot\left(\boldsymbol{\xi}_1 \delta\boldsymbol{B}_2\cdot\boldsymbol{B} - \boldsymbol{\xi}_2\delta\boldsymbol{B}_1\cdot\boldsymbol{B}\right) = 0$$

[1]T. Antonsen Jr., E.J. Paul, M. Landreman, *J. Plasma Phys.* 85 (2019).
[2]I.B. Bernstein et al, *Proc. Royal Society A,* 244 (1958).

Computing MHD shape gradient with adjoint approach[1]

Generalization: allow for $\boldsymbol{\delta\iota}$

$$\int_{V_{\text{plasma}}} d^3x\, \left(-\boldsymbol{F}(\boldsymbol{\xi}_1)\cdot\boldsymbol{\xi}_2 + \boldsymbol{F}(\boldsymbol{\xi}_2)\cdot\boldsymbol{\xi}_1\right) + \frac{1}{4\pi}\int_{S_{\text{plasma}}} d^2x\, \boldsymbol{n}\cdot\left(\boldsymbol{\xi}_1 \delta\boldsymbol{B}_2\cdot\boldsymbol{B} - \boldsymbol{\xi}_2\delta\boldsymbol{B}_1\cdot\boldsymbol{B}\right)$$
$$-\frac{2\pi}{c}\int_{V_{\text{plasma}}} d\psi\, \Big(\delta I_{T,2}(\psi)\delta\iota_1(\psi) - \delta I_{T,1}(\psi)\delta\iota_2(\psi)\Big) = 0$$

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1. Compute shape derivative for figure of merit

$$\delta f(\boldsymbol{\xi}_1) = \int_{V_{\text{plasma}}} d^3x\ \boldsymbol{\xi}_1\cdot\boldsymbol{A}_1 + \int_{S_{\text{plasma}}} d^2x\ \boldsymbol{n}\cdot\boldsymbol{\xi}_1 A_2$$

2. Adjoint displacement $\boldsymbol{\xi}_2$ satisfies

$$\boldsymbol{F}(\boldsymbol{\xi}_2) = -\boldsymbol{A}_1$$

$$\boldsymbol{\xi}_2\cdot\boldsymbol{n}|_{S_{\text{plasma}}} = 0$$

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Computing MHD shape gradient with adjoint approach[1]

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$\rightarrow$

$$\mathcal{G} = \left(\frac{\delta\boldsymbol{B}_2\cdot\boldsymbol{B}}{4\pi} + A_2\right)$$

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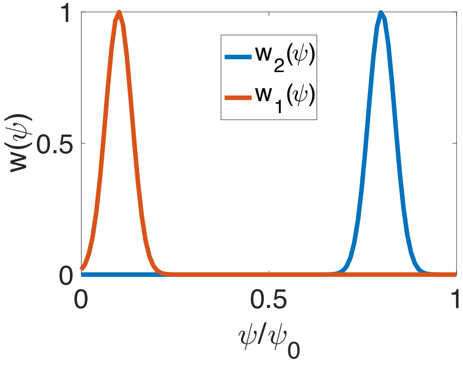
Magnetic well shape gradient requires pressure perturbation

$$f_W = \int_{V_{\text{plasma}}} d\psi \, (w_2(\psi) V'(\psi) - w_1(\psi) V'(\psi)) \approx V''(\psi)$$

- Proxy for interchange stability ($p'(\psi)V''(\psi) > 0$ favorable)



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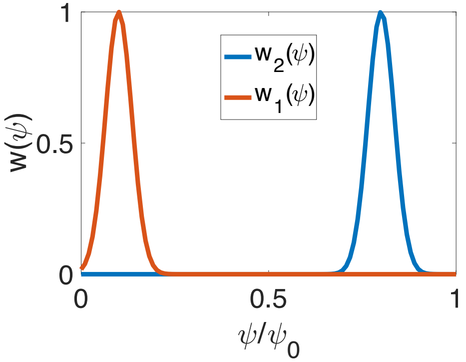
$$f_W = \int_{V_{\text{plasma}}} d\psi \left(w_2(\psi)\, V'(\psi) - w_1(\psi) V'(\psi)\right) \approx V''(\psi)$$

- Proxy for interchange stability ($p'(\psi)V''(\psi) > 0$ favorable)



$$\delta f_W(\boldsymbol{\xi}_1) = -\int_{V_{\text{plasma}}} d^3x\, \boldsymbol{\xi}_1 \cdot \nabla\big(w_2(\psi) - w_1(\psi)\big) + \int_{S_{\text{plasma}}} d^2x\, \boldsymbol{\xi}_1 \cdot \boldsymbol{n}\big(w_2(\psi) - w_1(\psi)\big)$$

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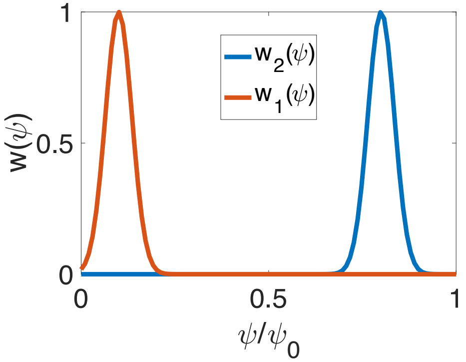

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Adjoint problem

$$\boldsymbol{F}(\boldsymbol{\xi}_2) = \nabla\big(w_2(\psi) - w_1(\psi)\big)$$

$$\boldsymbol{\xi}_2 \cdot \boldsymbol{n}\Big|_{S_{\text{plasma}}} = 0 \longrightarrow \mathcal{G}_W = \left(\frac{\delta \boldsymbol{B}_2 \cdot \boldsymbol{B}}{4\pi}\right)$$

$$\delta I_{T,2}(\psi) = 0$$

Magnetic well shape gradient computed with VMEC[1]

Linearization approximated with $\Delta_P \ll 1$

$$\boldsymbol{F}(\xi_2) = \nabla w(\psi)$$
$$\xi_2 \cdot \boldsymbol{n}\Big|_{S_{\text{plasma}}} = 0$$
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$$\approx$$

$$0 = \frac{(\nabla\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} - \nabla\big(p(\psi) + \Delta_P w(\psi)\big)$$
$S_{\text{plasma}}, I_T(\psi), p(\psi)$ prescribed

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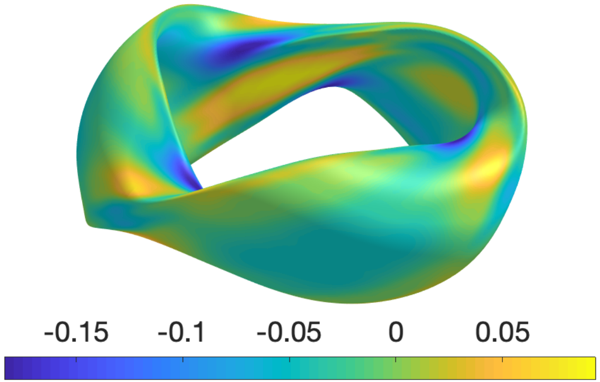
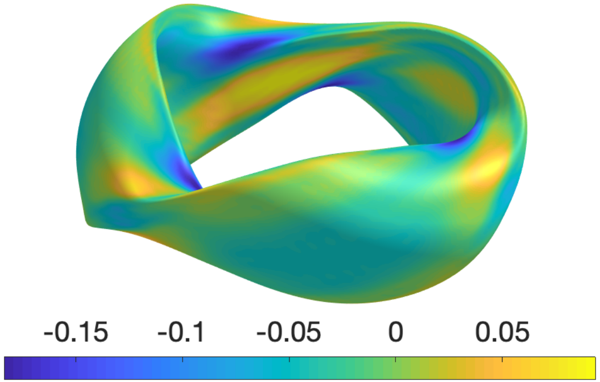
Finite difference



Adjoint



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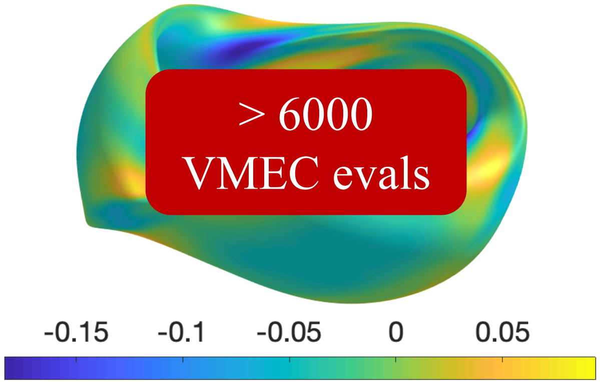

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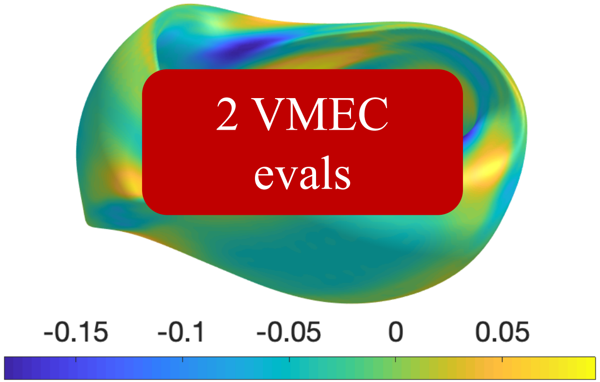

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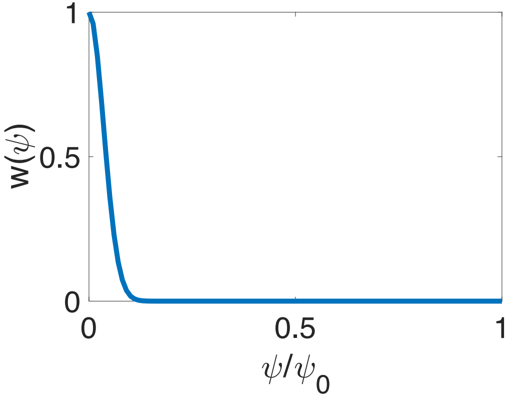
Magnetic ripple shape gradient requires anisotropic pressure

$$f_R = \int_{V_{\text{plasma}}} d^3x \underbrace{\frac{1}{2} w(\psi) \left(B - \overline{B}\right)^2}_{\tilde{f}_R} \qquad \overline{B} = \frac{\int_{V_{\text{plasma}}} d^3x \, w(\psi) B}{\int_{V_{\text{plasma}}} d^3x \, w(\psi)}$$

- Proxy for quasi-symmetry (guiding center confinement) near axis



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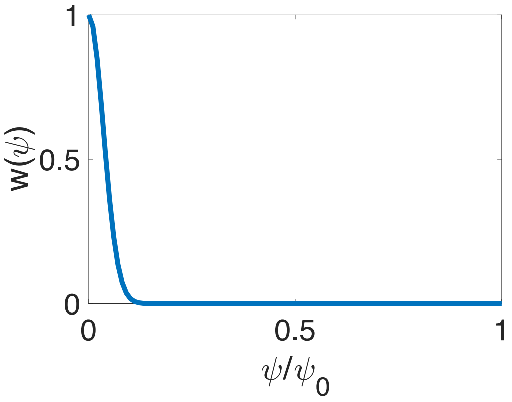

Adjoint Problem

$$\boldsymbol{F}(\boldsymbol{\xi}_2) = \nabla \cdot \boldsymbol{P}$$

$$\boldsymbol{\xi}_2 \cdot \boldsymbol{n}\Big|_{S_{\text{plasma}}} = 0$$

$$\boldsymbol{P} = p_{||} \boldsymbol{b}\boldsymbol{b} + p_{\perp} (\boldsymbol{I} - \boldsymbol{b}\boldsymbol{b})$$

$$p_{||} = \tilde{f}_R$$

$$p_{\perp} = p_{||} - B \frac{\partial p_{||}}{\partial B}$$

$$\longrightarrow \qquad \mathcal{G}_R = \frac{\delta \boldsymbol{B}_2 \cdot \boldsymbol{B}}{4\pi} + p_{\perp}$$

Variational principle for equilibria with anisotropic pressure

Equilibrium with anisotropic pressure

$$\frac{\boldsymbol{J}\times\boldsymbol{B}}{c} = \nabla\cdot\left(p_{||}(\psi,B)\boldsymbol{bb} + p_{\perp}(\psi,B)\left(\overleftrightarrow{\boldsymbol{I}} - \boldsymbol{bb}\right)\right)$$

$p_{\perp}(\psi,B)$ determined from parallel force balance

$$\frac{\partial p_{||}(\psi,B)}{\partial B} = \frac{p_{||}(\psi,B) - p_{\perp}(\psi,B)}{B}$$

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Stationary points of $W[B,p]$

$$W[B,p] = \int_{V_P} d^3x \, \frac{B^2}{8\pi} - p_{||}$$

Subject to:

1. Prescribed $p_{||}(\psi,B)$
2. Fixed $\iota(\psi)$
3. Magnetic surfaces

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- Solutions computed with ANIMEC[1] code
- Used for analysis of energetic particle contributions to equilibria

[1]W.A. Cooper et al, *Computer Phys. Comm.*, 72 (1992).

Magnetic ripple shape gradient computed with ANIMEC[1]

Linearization approximated with $\Delta_P \ll 1$

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$$\delta\iota_2(\psi) = 0$$

$\approx$

$$0 = \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} - \nabla\big(p(\psi)\big) - \Delta_P \nabla \cdot \boldsymbol{P}$$
$$S_{\text{plasma}},\ \iota(\psi),\ p(\psi),\ p_{||}(\psi, B) \quad \text{prescribed}$$

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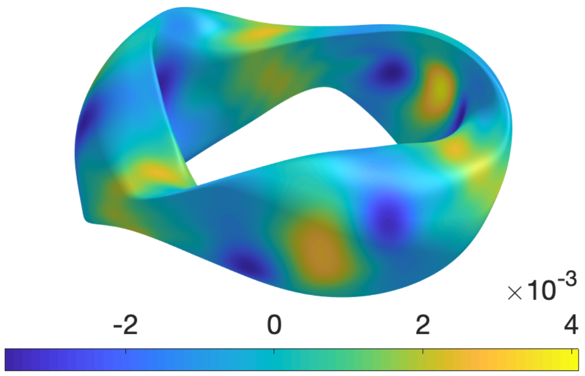
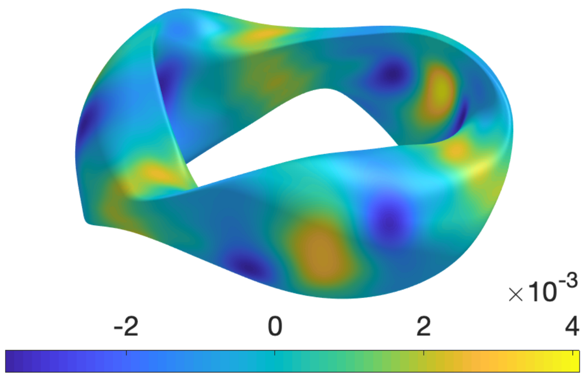
Finite difference



Adjoint



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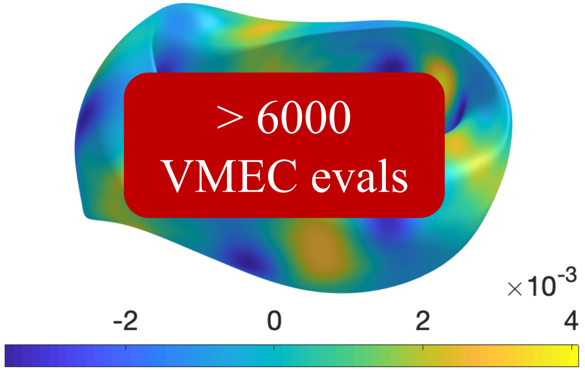

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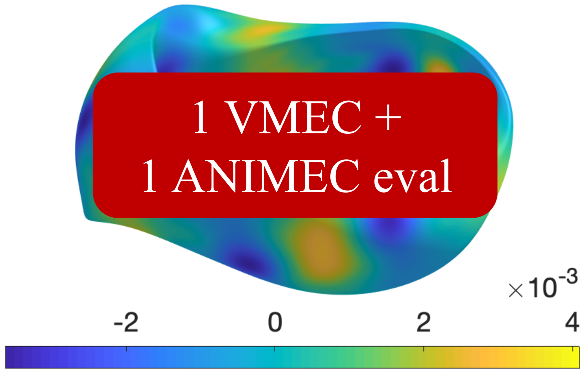

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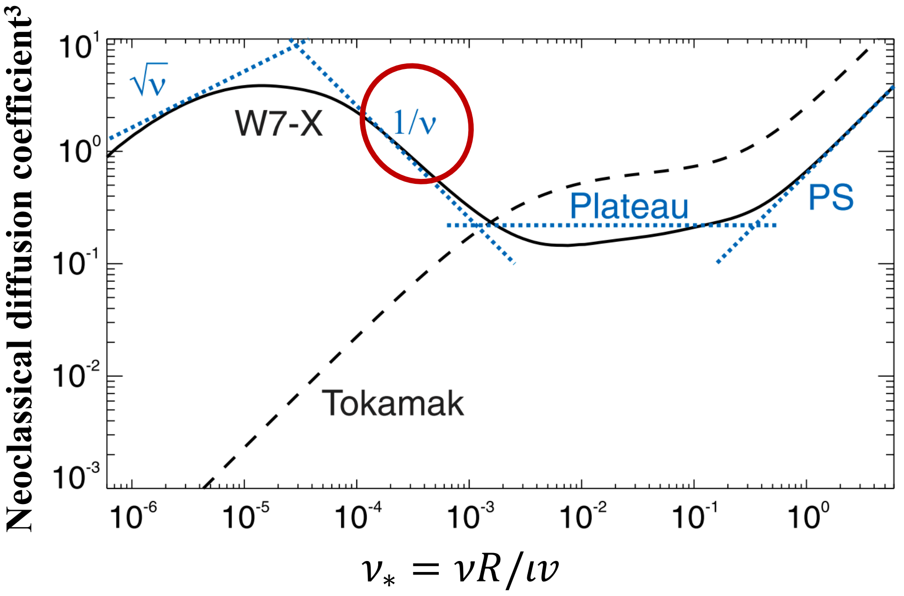
Many other applications of adjoint approach possible[1]

Effective Ripple[2] ($\epsilon_{\text{eff}}^{3/2}$)

- Proxy for low-collisionality neoclassical confinement
- Adjoint approach requires bulk force $\boldsymbol{\mathcal{F}} = \nabla \cdot \boldsymbol{P}(\psi, \alpha)$

$$f_{QS} = \int d^3x \, \epsilon_{\text{eff}}^{3/2}(\psi) w(\psi)$$



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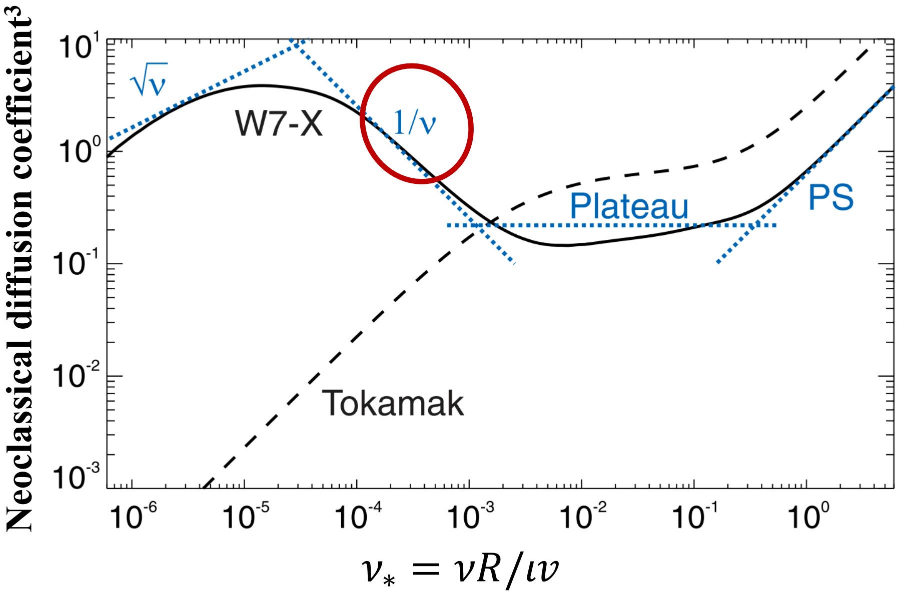
Cannot be implemented with ANIMEC



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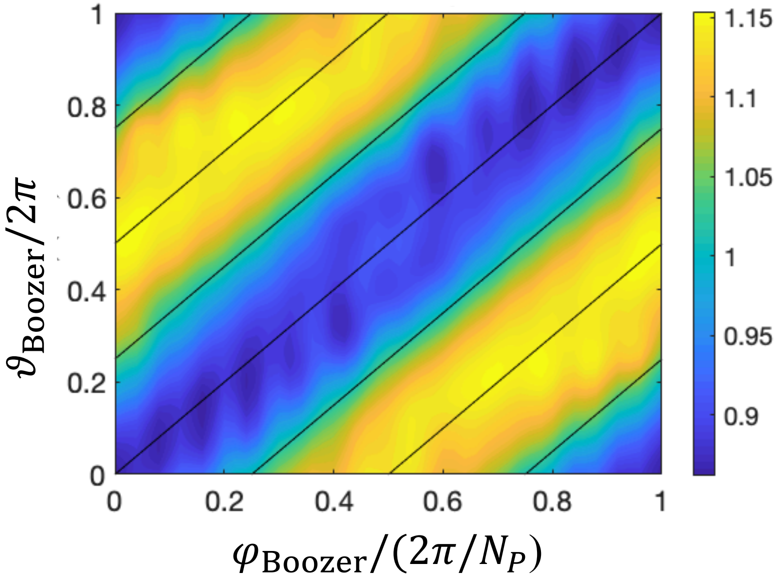
Departure from quasi-symmetry

- Quasi-symmetry → guiding center confinement, reduced neoclassical transport

$$f_{QS} = \int d^3x\, w(\psi)(\boldsymbol{B}\times\nabla B \cdot \nabla\psi - F(\psi)\boldsymbol{B}\cdot\nabla B)^2$$

Does not require Boozer coordinate transformation

HSX field strength[2]



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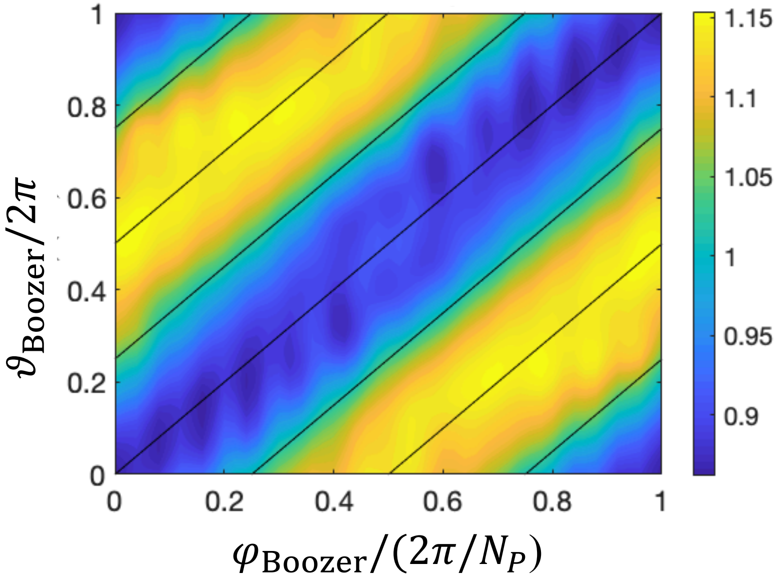
- Adjoint approach requires bulk force, $\boldsymbol{\mathcal{F}}(\boldsymbol{r})$

HSX field strength[2]



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Adjoint approach for coil shape gradient

Generalization of self-adjointness of MHD force operator[1]

$$\int_{V_{\text{plasma}}} d^3x \left(-\boldsymbol{F}(\boldsymbol{\xi}_1)\cdot\boldsymbol{\xi}_2 + \boldsymbol{F}(\boldsymbol{\xi}_2)\cdot\boldsymbol{\xi}_1\right) + \frac{1}{c}\sum_k \left(I_{C_k}\int_{C_k} dl\,\left(\delta\boldsymbol{r}_{C_{1,k}}\cdot\boldsymbol{t}\times\delta\boldsymbol{B}_2 - \delta\boldsymbol{r}_{2,k}\cdot\boldsymbol{t}\times\delta\boldsymbol{B}_1\right)\right)$$
$$-\frac{2\pi}{c}\int_{V_{\text{plasma}}} d\psi\,\left(\delta I_{T,2}(\psi)\delta\iota_1(\psi) - \delta I_{T,1}(\psi)\delta\iota_2(\psi)\right) = 0$$

[1]T. Antonsen Jr., E.J. Paul, M. Landreman, *J. Plasma Phys.*, 85 (2019).

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1. Compute shape derivative for figure of merit

$$\delta f(\boldsymbol{\xi}_1) = \int_{V_{\text{plasma}}} d^3x\ \boldsymbol{\xi}_1\cdot\boldsymbol{A}_1$$

2. Adjoint displacement $\boldsymbol{\xi}_2$ satisfies

$$\boldsymbol{F}(\boldsymbol{\xi}_2) = -\boldsymbol{A}_1$$

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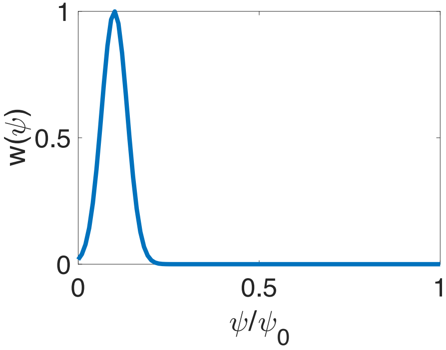
$$\sum_k \int_{C_k} dl\ \delta\boldsymbol{r}_{C_{1,k}} \cdot \underbrace{\boldsymbol{t}\times\delta\boldsymbol{B}_2\,\frac{I_{C_k}}{\mathrm{c}}}_{}$$

$\boldsymbol{S}_k$= coil shape gradient

[1]T. Antonsen Jr., E.J. Paul, M. Landreman, *J. Plasma Phys.*, 85 (2019).

Rotational transform coil shape gradient with VMEC

$$f_\iota = \int_{V_{\text{plasma}}} d\psi \, w(\psi) \iota(\psi)$$



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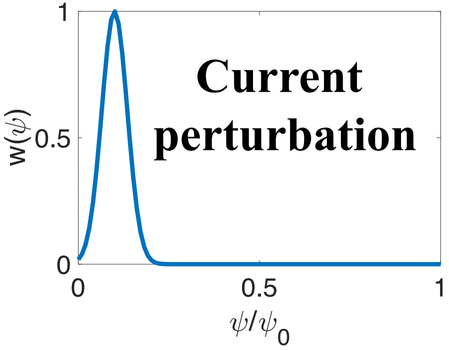

Adjoint problem

$$\boldsymbol{F}(\boldsymbol{\xi}_2) = 0$$

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$$\boldsymbol{S}_k = -I_{C_k} \boldsymbol{t} \times \frac{\delta \boldsymbol{B}_2}{2\pi}$$

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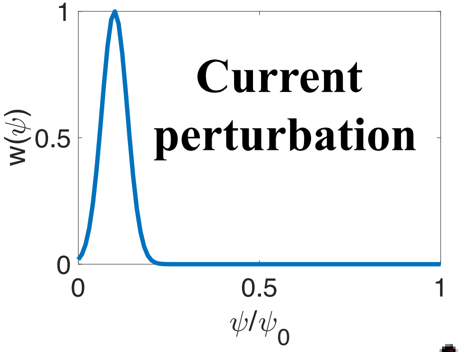

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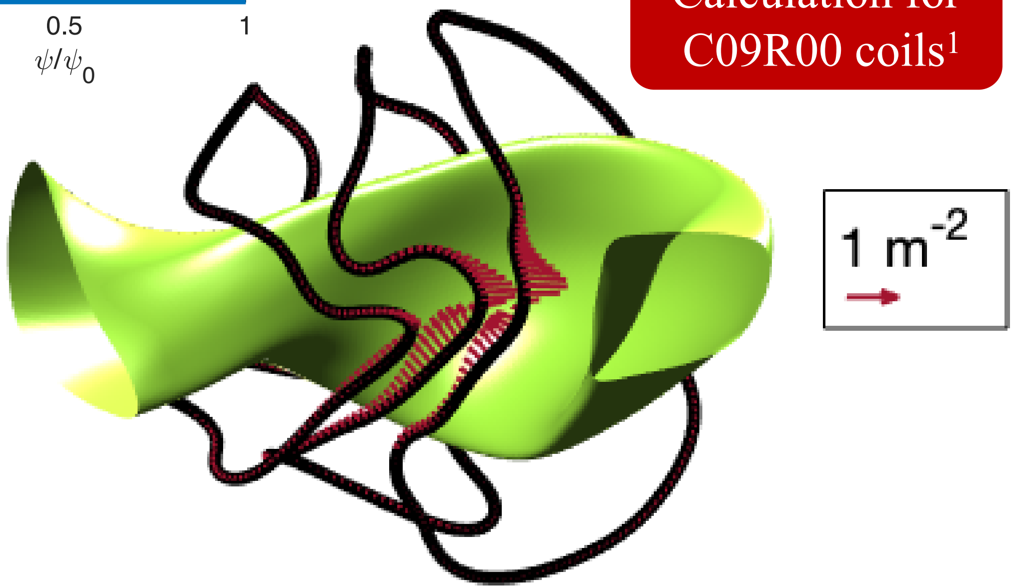
Computed with DIAGNO[2]

Calculation for C09R00 coils[1]



[1]D. Williamson et al, *Fusion Engineering*, (2005).
[2]S. Lazerson, *Plasma Phys. Control. Fusion*, 55 (2013).

Outline

- Introduction
- Shape gradients for MHD equilibria
- **Perturbed equilibrium approach**
  - Variational principle for linear MHD
  - Euler-Lagrange solutions
- Conclusions

Variational principle for perturbed MHD equilibria

Perturbed equilibrium with bulk force

$$\boldsymbol{F}(\boldsymbol{\xi}) + \delta \boldsymbol{F} = \frac{(\nabla\times\boldsymbol{B})\times\delta\boldsymbol{B} + (\nabla\times(\delta\boldsymbol{B}))\times\boldsymbol{B}}{4\pi} - \boldsymbol{\nabla}\delta p(\boldsymbol{\xi}) + \delta\boldsymbol{F} = 0$$

$$\delta\boldsymbol{B}(\boldsymbol{\xi}) = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})$$

$$\delta p = -\,\boldsymbol{\xi}\cdot\nabla p$$

$$\boldsymbol{\xi}\cdot\nabla\psi\,|_{\psi=0} = \boldsymbol{\xi}\cdot\nabla\psi\,|_{\psi=\psi_0} = 0$$

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$$\delta\boldsymbol{B}(\xi) = \nabla\times(\xi\times\boldsymbol{B})$$

$$\delta p = -\xi\cdot\nabla p$$

$$\xi\cdot\nabla\psi\,|_{\psi=0} = \xi\cdot\nabla\psi\,|_{\psi=\psi_0} = 0$$

$$\Updownarrow$$

Stationary points of $W[\xi]$

$$W[\xi] = \int_{V_P} d^3x\left(-\frac{\delta\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi} + \frac{\xi\cdot\boldsymbol{J}\times\delta\boldsymbol{B}}{c} - \xi\cdot\nabla p(\boldsymbol{\nabla}\cdot\xi) - 2\xi\cdot\delta\boldsymbol{F}\right)$$

$$\delta\xi\cdot\nabla\psi\,|_{\psi=0} = \delta\xi\cdot\nabla\psi\,|_{\psi=\psi_0} = 0$$

Spectral solution of Euler-Lagrange equation (I)

$\xi \cdot B = 0 \rightarrow$
2 independent components

$$W[\xi^\psi, \xi^\alpha] = \int_{V_P} d^3x \left(-\frac{\delta \boldsymbol{B} \cdot \delta \boldsymbol{B}}{4\pi} + \frac{\boldsymbol{\xi} \cdot \boldsymbol{J} \times \delta \boldsymbol{B}}{c} - \boldsymbol{\xi} \cdot \nabla p (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - 2 \boldsymbol{\xi} \cdot \delta \boldsymbol{F} \right)$$

$$\xi^\psi = \boldsymbol{\xi} \cdot \nabla \psi$$

$$\xi^\alpha = \boldsymbol{\xi} \cdot \nabla \theta - \iota(\psi)\, \boldsymbol{\xi} \cdot \nabla \phi$$

$$\xi^\psi|_{\psi=0} = \xi^\psi|_{\psi=\psi_0} = 0$$

Spectral solution of Euler-Lagrange equation (I)

$\xi \cdot B = 0 \rightarrow$ 2 independent components

$$W[\xi^\psi, \xi^\alpha] = \int_{V_P} d^3x \left(-\frac{\delta \boldsymbol{B} \cdot \delta \boldsymbol{B}}{4\pi} + \frac{\boldsymbol{\xi} \cdot \boldsymbol{J} \times \delta \boldsymbol{B}}{c} - \boldsymbol{\xi} \cdot \nabla p (\boldsymbol{\nabla} \cdot \boldsymbol{\xi}) - 2 \boldsymbol{\xi} \cdot \delta \boldsymbol{F} \right)$$

$$\xi^\psi = \boldsymbol{\xi} \cdot \nabla \psi$$

$$\xi^\alpha = \boldsymbol{\xi} \cdot \nabla \theta - \iota(\psi)\, \boldsymbol{\xi} \cdot \nabla \phi$$

$$\xi^\psi|_{\psi=0} = \xi^\psi|_{\psi=\psi_0} = 0$$

Expand in Fourier series

$$\xi^{\psi,\alpha}(\psi,\theta,\phi) = \sum_{m,n} \xi^{\psi,\alpha}_{mnc} \cos(m\theta - n\phi) + \xi^{\psi,\alpha}_{mns} \sin(m\theta - n\phi) = \boldsymbol{\Xi}^{\psi,\alpha}(\psi) \cdot \boldsymbol{\mathcal{F}}$$

$$W[\xi^\psi, \xi^\alpha] = \int_{V_P} d^3x \Big(\boldsymbol{\Xi}^\psi \cdot \overleftrightarrow{\boldsymbol{A}^{\psi\psi}} \boldsymbol{\Xi}^\psi + \boldsymbol{\Xi}^\psi \cdot \overleftrightarrow{\boldsymbol{A}^{\psi\psi'}} \boldsymbol{\Xi}^{\psi\prime}(\psi) + \boldsymbol{\Xi}^{\psi\prime}(\psi) \cdot \overleftrightarrow{\boldsymbol{A}^{\psi'\psi'}} \boldsymbol{\Xi}^{\psi\prime}(\psi) +$$
$$\boldsymbol{\Xi}^\alpha \cdot \overleftrightarrow{\boldsymbol{A}^{\alpha\psi}} \boldsymbol{\Xi}^\psi + \boldsymbol{\Xi}^\alpha \cdot \overleftrightarrow{\boldsymbol{A}^{\alpha\psi'}} \boldsymbol{\Xi}^{\psi\prime}(\psi) + \boldsymbol{\Xi}^\alpha \cdot \overleftrightarrow{\boldsymbol{A}^{\alpha\alpha}} \boldsymbol{\Xi}^\alpha + \boldsymbol{C}^\psi \cdot \boldsymbol{\Xi}^\psi + \boldsymbol{C}^\alpha \cdot \boldsymbol{\Xi}^\alpha \Big)$$

Spectral solution of Euler-Lagrange equation (II)

Variation w.r.t. Ξ^α

$$2\overleftrightarrow{A^{\alpha\alpha}}\Xi^{\alpha}(\psi) = \left(\overleftrightarrow{A^{\alpha\psi}}\Xi^{\psi}(\psi) + \overleftrightarrow{A^{\alpha\psi'}}\Xi^{\psi}{}'(\psi) + C^{\alpha}\right)$$

Spectral solution of Euler-Lagrange equation (II)

Variation w.r.t. Ξ^{α}

$$2\overleftrightarrow{A^{\alpha\alpha}}\Xi^{\alpha}(\psi) = \left(\overleftrightarrow{A^{\alpha\psi}}\Xi^{\psi}(\psi) + \overleftrightarrow{A^{\alpha\psi'}}\Xi^{\psi\prime}(\psi) + C^{\alpha}\right)$$

Variation w.r.t. Ξ^{ψ} (Ξ^{α} eliminated)

$$\overleftrightarrow{C^{\psi}}\,\Xi^{\psi}(\psi) + \overleftrightarrow{C^{\psi'}}\,\Xi^{\psi'}(\psi) + \overleftrightarrow{C^{\psi''}}\Xi^{\psi''}(\psi) + \overrightarrow{D} = \mathbf{0}$$

$$\Xi^{\psi}|_{\psi=0} = \Xi^{\psi}|_{\psi=\psi_0} = 0$$

Spectral solution of Euler-Lagrange equation (II)

Variation w.r.t. Ξ^α

$$2\overleftrightarrow{A^{\alpha\alpha}}\Xi^\alpha(\psi) = \left(\overleftrightarrow{A^{\alpha\psi}}\Xi^\psi(\psi) + \overleftrightarrow{A^{\alpha\psi'}}\Xi^{\psi\prime}(\psi) + C^{\alpha}\right)$$

Variation w.r.t. Ξ^ψ (Ξ^α eliminated)

$$\overleftrightarrow{C^\psi}\,\Xi^\psi(\psi) + \overleftrightarrow{C^{\psi'}}\,\Xi^{\psi\prime}(\psi) + \overleftrightarrow{C^{\psi''}}\Xi^{\psi\prime\prime}(\psi) + \vec{D} = \mathbf{0}$$

$$\Xi^\psi|_{\psi=0} = \Xi^\psi|_{\psi=\psi_0} = 0$$

- 2nd order coupled 2-point BVP solved with finite difference
- Similar to Euler-Lagrange eqn. solved by DCON[1]

[1]A. Glasser, *Phys. Plasmas,* 23 (2016).

Spectral solution of Euler-Lagrange equation (II)

Variation w.r.t. Ξ^{α}

$$2\overleftrightarrow{A^{\alpha\alpha}}\Xi^{\alpha}(\psi) = \left(\overleftrightarrow{A^{\alpha\psi}}\Xi^{\psi}(\psi) + \overleftrightarrow{A^{\alpha\psi'}}\Xi^{\psi\prime}(\psi) + C^{\alpha}\right)$$

Singular at rational surfaces in 3D

Variation w.r.t. Ξ^{ψ} (Ξ^{α} eliminated)

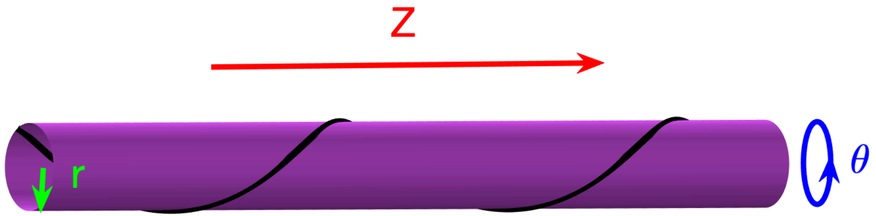
$$\overleftrightarrow{C^{\psi}}\,\Xi^{\psi}(\psi) + \overleftrightarrow{C^{\psi'}}\,\Xi^{\psi'}(\psi) + \overleftrightarrow{C^{\psi''}}\Xi^{\psi''}(\psi) + \vec{D} = 0$$

$$\Xi^{\psi}|_{\psi=0} = \Xi^{\psi}|_{\psi=\psi_0} = 0$$

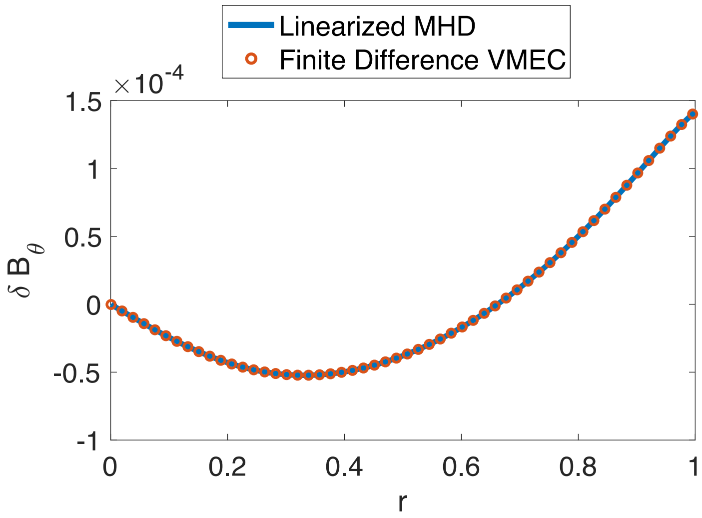
- 2nd order coupled 2-point BVP solved with finite difference
- Similar to Euler-Lagrange eqn. solved by DCON[1]

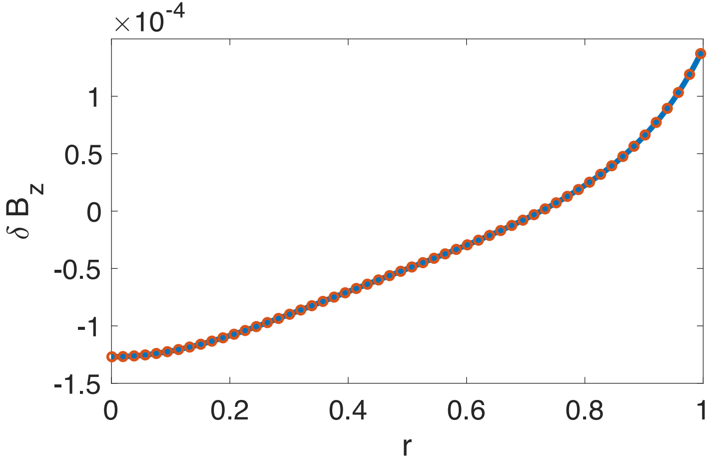
[1] A. Glasser, *Phys. Plasmas,* 23 (2016).

Preliminary perturbed equilibrium results



- Equilibrium with θ and z symmetry
- Bulk force = pressure perturbation
$$\delta \boldsymbol{F} = -\nabla \delta P(\psi)$$
- Benchmark with FD VMEC solution





Outline

- Introduction
- Shape gradients for MHD equilibria
- Perturbed equilibrium approach
- **Conclusions**

Conclusions (I)

Open questions

- How to extend linearized MHD approach to 3D (e.g. Frobenius analysis as in DCON[1])?
- Can we prevent flux surface overlap in linearized approach?
- Can adjoint approach be generalized to avoid assumption of surfaces?

[1]A. Glasser, *Phys. Plasmas,* 23 (2016).

Conclusions (I)

Open questions

- How to extend linearized MHD approach to 3D (e.g. Frobenius analysis as in DCON)?
- Can we prevent flux surface overlap in linearized approach?
- Can adjoint approach be generalized to avoid assumption of surfaces?

Future work

- Application of adjoint approach for fixed and free-boundary optimization (e.g. incorporation in STELLOPT).
- Demonstration for other important figures of merit (e.g. energetic particle confinement)

Conclusions (II)

- Adjoint methods allow efficient computation of geometric derivatives
  - Gradient-based optimization
  - Sensitivity and tolerance analysis
- Adjoint approach for MHD equilibria used to compute shape gradient for plasma boundary and coil shapes

✓ Magnetic well
✓ Magnetic ripple
✓ Rotational transform
❑ Effective ripple ($\epsilon_{\text{eff}}^{3/2}$)
❑ Quasisymmetry
? Energetic particles

Conclusions (II)

- Adjoint methods allow efficient computation of geometric derivatives
  - Gradient-based optimization
  - Sensitivity and tolerance analysis
- Adjoint approach for MHD equilibria used to compute shape gradient for plasma boundary and coil shapes

References for this work

- T.M. Antonsen, E.J. Paul, M. Landreman, *J. Plasma Phys.*, 85 (2019).
- E.J. Paul et al, *submitted to J. Plasma Phys.*, (arXiv:1910.14144).

Other adjoint methods for stellarators

- Optimization of coil shapes
  E.J. Paul et al, *Nuclear Fusion,* 58 (2018).
- Optimization of neoclassical quantities
  E.J. Paul et al, *J. Plasma Phys.*, 85 (2019).

Thank you for your attention