Variational current-coupling gyrokinetic-MHD

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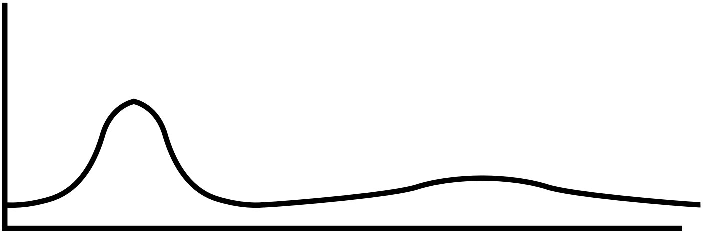
PPPL theory seminar

Review of hybrid modeling

Physical picture

plasma consists of two populations



cold, fluid-like bulk

hot, kinetic minority

Review of hybrid modeling

Basic mathematical approach

MHD models cold bulk, kinetic theory models hot minority

$$\rho \frac{\partial \boldsymbol{U}}{\partial t} = \ldots \qquad \frac{\partial F}{\partial t} = \ldots$$

GK-MHD hybrid models can be derived as follows:

Step 1: Couple two-fluid model to gyrokinetic equation *via* Maxwell's equations.

$$m_\sigma n_\sigma(\partial_t \mathbf{u}_\sigma + \mathbf{u}_\sigma \cdot \nabla \mathbf{u}_\sigma) = -\nabla \mathrm{p}_\sigma + q_\sigma n_\sigma(\boldsymbol{E} + \mathbf{u}_\sigma \times \boldsymbol{B})$$

$$\partial_t n_\sigma + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = 0$$

$$\partial_t F + \nabla \cdot (F \mathbf{u}_{\mathrm{gy}}) + \partial_{v_\parallel}(F a_{\parallel \mathrm{gy}}) = 0$$

$$\nabla \times \boldsymbol{B} = \mu_o\Big(\sum_\sigma q_\sigma n_\sigma \mathbf{u}_\sigma + \boldsymbol{J}_h\Big) + \mu_o \epsilon_o \partial_t \boldsymbol{E}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\epsilon_o \nabla \cdot \boldsymbol{E} = \sum_\sigma q_\sigma n_\sigma + q_h n_h$$

$$\nabla \cdot \boldsymbol{B} = 0$$

GK-MHD hybrid models can be derived as follows:

Step 2: Set displacement current and total charge to zero.

$$m_\sigma n_\sigma(\partial_t \mathbf{u}_\sigma + \mathbf{u}_\sigma \cdot \nabla \mathbf{u}_\sigma) = -\nabla \mathrm{p}_\sigma + q_\sigma n_\sigma(\boldsymbol{E} + \mathbf{u}_\sigma \times \boldsymbol{B})$$

$$\partial_t n_\sigma + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = 0$$

$$\partial_t F + \nabla \cdot (F\mathbf{u}_{\mathrm{gy}}) + \partial_{v_\parallel}(F a_{\parallel \mathrm{gy}}) = 0$$

$$\nabla \times \boldsymbol{B} = \mu_o\Big(\sum_\sigma q_\sigma n_\sigma \mathbf{u}_\sigma + \boldsymbol{J}_h\Big) + \cancelto{0}{\mu_o \epsilon_o \partial_t \boldsymbol{E}}$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\cancelto{0}{\epsilon_o \nabla \cdot \boldsymbol{E}} = \sum_\sigma q_\sigma n_\sigma + q_h n_h$$

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$$\partial_t n_\sigma + \nabla \cdot (n_\sigma \mathbf{u}_\sigma) = 0$$

$$\partial_t F + \nabla \cdot (F \mathbf{u}_{\mathrm{gy}}) + \partial_{v_\parallel}(F a_{\parallel \mathrm{gy}}) = 0$$

$$\nabla \times \boldsymbol{B} = \mu_o\Big(\sum_\sigma q_\sigma n_\sigma \mathbf{u}_\sigma + \boldsymbol{J}_h\Big)$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$0 = \sum_\sigma q_\sigma n_\sigma + q_h n_h$$

$$\nabla \cdot \boldsymbol{B} = 0$$

GK-MHD hybrid models can be derived as follows:

Step 3: Sum fluid momentum equations, assume ideal Ohm's law.

$$\rho(\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}) = -\nabla \mathrm{p} - q_h n_h \boldsymbol{E} + (\mu_o^{-1} \nabla \times \boldsymbol{B} - \boldsymbol{J}_h) \times \boldsymbol{B}$$

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0$$

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{U}) = 0$$

$$\partial_t F + \nabla \cdot (F \mathbf{u}_{\mathrm{gy}}) + \partial_{v_\parallel} (F a_{\parallel \mathrm{gy}}) = 0$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

GK-MHD hybrid models can be derived as follows:

Step 4: Choose current coupling or pressure coupling closure.

- **Current coupling:** System is closed by expressing n_h and $\boldsymbol{J}_h$ in terms of moments of F
- **Pressure coupling:** Perpendicular component of hot momentum equation is added to cold momentum equation. Hot perpendicular momentum density is neglected. System is closed by expressing the hot pressure tensor P_h in terms of moments of F

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In this talk I will only discuss current coupling

Why current coupling?

I. Fewer approximations, not much additional complexity

II. Second-order moments noisier than first-order

x_i independent and same distribution as x

$$\langle x \rangle - \frac{1}{N}\sum_i x_i = \frac{1}{N}\sum_i \delta x_i$$

$$\langle x^2 \rangle - \frac{1}{N}\sum_i x_i^2 = \frac{1}{N}\sum_i (\delta x_i)^2 - \sigma_x^2 + 2\langle x \rangle \frac{1}{N}\sum_i \delta x_i$$

An important GK-MHD model is the BDC model

The current-coupling model of Belova, Denton, and Chan (J. Comput. Phys. 1997):

- **Current-coupling system:**

$$\rho(\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}) = -\nabla \mathrm{p} - q_h n_h \boldsymbol{E} + (\mu_o^{-1} \nabla \times \boldsymbol{B} - \boldsymbol{J}_h) \times \boldsymbol{B}$$

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0$$

$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{U}) = 0$$

$$\partial_t F + \nabla \cdot (F \mathbf{u}_{\mathrm{gy}}) + \partial_{v_\parallel} (F a_{\parallel \mathrm{gy}}) = 0$$

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B}$$

$$\nabla \cdot \boldsymbol{B} = 0$$

An important GK-MHD model is the BDC model

The current-coupling model of Belova, Denton, and Chan (J. Comput. Phys. 1997):

- **Hot charge and current densities:**

$$q_h n_h(\boldsymbol{x}) = q_h \iint_\mu \langle \delta(\boldsymbol{X} + \boldsymbol{\rho} - \boldsymbol{x}) \rangle F \, d^4\mathbf{z}$$

$$\boldsymbol{J}_h(\boldsymbol{x}) = q_h \iint_\mu \langle (\mathbf{u}_{\rm gy} + \boldsymbol{v}_\perp) \delta(\boldsymbol{X} + \boldsymbol{\rho} - \boldsymbol{x}) \rangle F \, d^4\mathbf{z},$$

An important GK-MHD model is the BDC model

The current-coupling model of Belova, Denton, and Chan (J. Comput. Phys. 1997):

- **Gyrocenter dynamics:**

$$a_{\| \mathrm{gy}} = \frac{q_h}{m_h} \frac{\boldsymbol{B}^{**}}{B_{\|}^{**}} \cdot \boldsymbol{E}^{**}$$

$$\mathbf{u}_{\mathrm{gy}} = \frac{1}{B_{\|}^{**}} \left[\boldsymbol{B}^{**} v_{\|} + \boldsymbol{E}^{**} \times \boldsymbol{b}_{\mathrm{eq}} \right]$$

$$\boldsymbol{E}^{**} = \langle \tilde{\boldsymbol{E}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle - q_h^{-1} \nabla([\mu + \delta\mu] B_{\mathrm{eq}})$$

$$\boldsymbol{B}^{**} = \boldsymbol{B}_{\mathrm{eq}} + \langle \tilde{\boldsymbol{B}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle$$

$$B_{\|}^{**} = \boldsymbol{B}^{**} \cdot \boldsymbol{b}_{\mathrm{eq}}$$

$$\delta\mu = -q_h B_{\mathrm{eq}}^{-1} \langle \boldsymbol{v}_{\perp} \cdot \tilde{\boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle = \frac{q_h^2}{2\pi m_h} \int_{D(\boldsymbol{X})} \tilde{\boldsymbol{B}} \cdot d\mathbf{S}$$

Why would anyone want a new GK-MHD model?

Previous models have broken conservation laws when $\boldsymbol{B}_{\text{eq}} \neq const.$

- **Momentum conservation in BDC model:**

$$\mathbf{N} = \iint_{\mu} m_h v_{\|} \boldsymbol{b}_{\text{eq}} \, F \, d^4\mathbf{z} + \int \rho \, \boldsymbol{U} \, d^3\boldsymbol{x}$$

satisfies

$$\frac{d\mathbf{N}}{dt} = \iint_{\mu} \Big(m_h v_{\|} \mathbf{u}_{\text{gy}} \cdot \nabla \boldsymbol{b}_{\text{eq}} - q_h \mathbf{u}_{\text{gy}} \times \langle \Delta \boldsymbol{B} \rangle - q_h \langle \boldsymbol{v}_{\perp} \times \Delta \boldsymbol{B} \rangle$$
$$- \nabla([\mu + \delta\mu] B_{\text{eq}}) - q_h \langle \boldsymbol{v}_{\perp} \times \tilde{\boldsymbol{B}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle \Big) F \, d^4\mathbf{z},$$

where $\Delta \boldsymbol{B} = \boldsymbol{B}_{\text{eq}}(\boldsymbol{X} + \boldsymbol{\rho}) - \boldsymbol{B}_{\text{eq}}(\boldsymbol{X})$.

Only zero when $\boldsymbol{B}_{\text{eq}} = const.$

Why would anyone want a new GK-MHD model?

Previous models have broken conservation laws when $\boldsymbol{B}_{\text{eq}} \neq const.$

- **Hot charge conservation in BDC model:**

$$\partial_t q_h n_h + \nabla \cdot \boldsymbol{J}_h =$$
$$\iint_\mu q_h \langle (\mathbf{u}_{\text{gy}} + \boldsymbol{v}_\perp) \cdot \nabla \boldsymbol{\rho} \cdot (\nabla \delta)(\boldsymbol{X} + \boldsymbol{\rho} - \boldsymbol{x}) \rangle F \, d^4\mathbf{z}$$

Only zero when $\boldsymbol{B}_{\text{eq}} = const.$

Why would anyone want a new GK-MHD model?

Previous models have broken conservation laws when $\boldsymbol{B}_{\text{eq}} \neq const.$

- **Phase space volume conservation in BDC model:**

$$\partial_t B_{\|}^{**} + \nabla \cdot (B_{\|}^{**} \mathbf{u}_{\text{gy}}) + \partial_{v_{\|}} (B_{\|}^{**} a_{\|\text{gy}}) =$$
$$\boldsymbol{b}_{\text{eq}} \cdot [\nabla \times \langle \tilde{\boldsymbol{E}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle - \langle (\nabla \times \tilde{\boldsymbol{E}})(\boldsymbol{X} + \boldsymbol{\rho}) \rangle]$$
$$+ v_{\|} \nabla \cdot \langle \tilde{\boldsymbol{B}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle - \boldsymbol{E}^{**} \cdot \nabla \times \boldsymbol{b}_{\text{eq}},$$

Only zero when $\boldsymbol{B}_{\text{eq}} = const.$

What is our new model?

Our model is a slight modification of the BDC model

- **We change the hot current:**

$$\boldsymbol{J}_h(\boldsymbol{x}) = \int_\mu \int q_h \langle (\mathbf{u}_{\mathrm{gy}} + \boldsymbol{v}_\perp)\delta(\boldsymbol{X} + \boldsymbol{\rho} - \boldsymbol{x})\rangle F \, d^4\mathbf{z}$$
$$+ \int_\mu \int q_h \langle\langle \mathbf{u}_{\mathrm{gy}} \cdot (\nabla_{\boldsymbol{X}} + \nabla_{\boldsymbol{x}})[\delta(\boldsymbol{X} + \lambda\boldsymbol{\rho} - \boldsymbol{x})\boldsymbol{\rho}]\rangle\rangle F \, d^4\mathbf{z}$$

"gyrodisk average":

$$\langle\langle Q \rangle\rangle = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} Q \, d\theta \, d\lambda$$

What is our new model?

Our model is a slight modification of the BDC model

- **And the gyrocenter dynamics:**

$$a_{\parallel \mathrm{gy}} = \frac{q_h}{m_h} \frac{\boldsymbol{B}^*}{B_\parallel^*} \cdot \boldsymbol{E}^*$$

$$\mathbf{u}_{\mathrm{gy}} = \frac{v_\parallel \boldsymbol{B}^*}{B_\parallel^*} + \frac{\boldsymbol{E}^* \times \boldsymbol{b}_{\mathrm{eq}}}{B_{\parallel *}}$$

$$\begin{aligned}
\boldsymbol{E}^* &= \tilde{\boldsymbol{E}}(\boldsymbol{X}) - q_h^{-1} \nabla([\mu + \delta\mu] B_{\mathrm{eq}}) + \langle\langle (\nabla \times \tilde{\boldsymbol{E}})(\boldsymbol{X} + \lambda \boldsymbol{\rho}) \times \boldsymbol{\rho} \rangle\rangle \\
&\quad + \nabla \langle\langle \tilde{\boldsymbol{E}}(\boldsymbol{X} + \lambda \boldsymbol{\rho}) \cdot \boldsymbol{\rho} \rangle\rangle \\
\boldsymbol{B}^* &= \boldsymbol{B}(\boldsymbol{X}) + m_h q_h^{-1} v_\parallel \nabla \times \boldsymbol{b}_{\mathrm{eq}} + \nabla \times \langle\langle \tilde{\boldsymbol{B}}(\boldsymbol{X} + \lambda \boldsymbol{\rho}) \times \boldsymbol{\rho} \rangle\rangle \\
B_\parallel^* &= \boldsymbol{B}^* \cdot \boldsymbol{b}_{\mathrm{eq}}
\end{aligned}$$

What are our model's conservation laws?

Energy is conserved:

$$E = \iint_{\mu} \left(\frac{1}{2} m_h v_{\|}^2 + [\mu + \delta\mu] B_{\mathrm{eq}} \right) F \, d^4\boldsymbol{z} + \int \left(\frac{1}{2}\rho |\boldsymbol{U}|^2 + \rho\, \mathcal{U}(\rho) + \frac{1}{2\mu_0} |\boldsymbol{B}|^2 \right) d^3\boldsymbol{x},$$

Note: This same energy is conserved in the BDC model.

What are our model's conservation laws?

Momentum is conserved assuming symmetry:

$$N_\phi = \iint_\mu m_h v_\| \boldsymbol{b}_{\mathrm{eq}} \cdot \mathbf{e}_z \times \boldsymbol{X}\, F\, d^4\mathbf{z} + \int \rho\, \boldsymbol{U} \cdot \mathbf{e}_z \times \boldsymbol{x}\, d^3\boldsymbol{x}$$
$$+ \int_\mu \int q_h \langle\langle [\boldsymbol{\rho} \times \boldsymbol{B}_{\mathrm{eq}}(\boldsymbol{X} + \lambda\boldsymbol{\rho})] \cdot [\mathbf{e}_z \times \boldsymbol{X}] \rangle\rangle F\, d^4\mathbf{z}$$
$$+ \iint_\mu q_h \langle\langle \lambda \mathbf{e}_z \cdot \boldsymbol{\rho}\boldsymbol{\rho} \cdot \boldsymbol{B}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) - \lambda|\boldsymbol{\rho}|^2 \mathbf{e}_z \cdot \boldsymbol{B}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) \rangle\rangle F\, d^4\mathbf{z}$$

Note: This the toroidal momentum conserved assuming an axisymmetric background. When the background is uniform, this model and the BDC model conserve the same linear momentum.

What are our model's conservation laws?

Hot charge is conserved:

$$\partial_t(q_h n_h) + \nabla \cdot \boldsymbol{J}_h = 0.$$

What are our model's conservation laws?

Phase space volume is conserved:

$$\partial_t B_{\parallel}^* + \nabla \cdot (B_{\parallel}^* \mathbf{u}_{\mathrm{gy}}) + \partial_{v_{\parallel}} (B_{\parallel}^* a_{\parallel \mathrm{gy}}) = 0$$

How is our model derived?

I. Construct system Lagrangian L by summing the net gyrocenter Lagrangian L_{p} and the MHD fluid Lagrangian L_{MHD},

$$L = L_{\mathrm{p}} + L_{\mathrm{MHD}}.$$

II. Express all quantities in L in terms of the Lagrangian configuration maps $\boldsymbol{q}(\boldsymbol{x}_o)$ and $\boldsymbol{z}(\mathbf{z}_o)$ associated with the MHD fluid and phase space fluid, respectively

III. Vary the action $S = \int L\, dt$ by varying $\boldsymbol{q}$ and $\boldsymbol{z}$ to find Euler-Lagrange equations.

I. The MHD fluid Lagrangian is the standard one.

The MHD Lagrangian:

$$L_{\mathrm{MHD}} = \frac{1}{2}\int \rho\,|\boldsymbol{U}|^2\,d^3\boldsymbol{x} - \int \rho\,\mathcal{U}(\rho)\,d^3\boldsymbol{x} - \frac{1}{2\mu_o}\int |\boldsymbol{B}_{\mathrm{eq}} + \tilde{\boldsymbol{B}}|^2\,d^3\boldsymbol{x}$$

I. The net gyrocenter Lagrangian is more subtle.

The relationship between L_p and the single-gyrocenter Lagrangian ℓ_gy is clear:

$$L_\mathsf{p} = \iint_\mu \ell_\mathrm{gy}\, F\, d^4\mathbf{z}$$

I. The net gyrocenter Lagrangian is more subtle.

But what is the ℓ_{gy} that brings us closest to the BDC model?

$$\ell_{\text{gy}} = ???$$

I. This ℓ_{gy} reproduces the BDC model's gyrocenter dynamics when $\boldsymbol{B}_{\mathrm{eq}} = const.$

If $\ell_{\mathrm{gy}} = \ell_{\mathrm{Br}}$, where

$$\ell_{\mathrm{Br}} = \left(q_h \boldsymbol{A}_{\mathrm{eq}} + m_h v_{\|} \boldsymbol{b}_{\mathrm{eq}}\right) \cdot \dot{\boldsymbol{X}} + q_h \langle \tilde{\boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle \cdot \dot{\boldsymbol{X}} - \left(\frac{1}{2} m_h v_{\|}^2 + \mu B_{\mathrm{eq}} + q_h \langle \tilde{\varphi}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle - q_h \langle \boldsymbol{v}_{\perp} \cdot \tilde{\boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho}) \rangle \right)$$

BDC gyrocenter dynamics are recovered when $\boldsymbol{B}_{\mathrm{eq}} = const.$.

Note: This Lagrangian was given originally by Brizard (J. Plasma Phys. 1989)

I. No ℓ_{gy} in literature gives BDC gyrocenter dynamics when $\boldsymbol{B}_{\mathrm{eq}} \neq const.$

The BDC gyrocenter equations of motion can be expressed entirely in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. Reminder:

$$a_{\parallel\mathrm{gy}} = \frac{q_h}{m_h}\frac{\boldsymbol{B}^{**}}{B_\parallel^{**}} \cdot \boldsymbol{E}^{**}$$

$$\mathbf{u}_{\mathrm{gy}} = \frac{1}{B_\parallel^{**}}\left[\boldsymbol{B}^{**} v_\parallel + \boldsymbol{E}^{**} \times \boldsymbol{b}_{\mathrm{eq}}\right]$$

$$\boldsymbol{E}^{**} = \langle \tilde{\boldsymbol{E}}(\boldsymbol{X} + \boldsymbol{\rho})\rangle - q_h^{-1}\nabla([\mu + \delta\mu]B_{\mathrm{eq}})$$

$$\boldsymbol{B}^{**} = \boldsymbol{B}_{\mathrm{eq}} + \langle \tilde{\boldsymbol{B}}(\boldsymbol{X} + \boldsymbol{\rho})\rangle$$

$$B_\parallel^{**} = \boldsymbol{B}^{**} \cdot \boldsymbol{b}_{\mathrm{eq}}$$

$$\delta\mu = -q_h B_{\mathrm{eq}}^{-1}\langle \boldsymbol{v}_\perp \cdot \tilde{\boldsymbol{A}}(\boldsymbol{X} + \boldsymbol{\rho})\rangle = \frac{q_h^2}{2\pi m_h}\int_{D(\boldsymbol{X})} \tilde{\boldsymbol{B}} \cdot d\mathbf{S}$$

I. No $\ell_{\rm gy}$ in literature gives BDC gyrocenter dynamics when $\boldsymbol{B}_{\rm eq} \neq const.$

In contrast, all $\ell_{\rm gy}$ in literature give gyrocenter dynamics that require evaluating the potentials $\tilde{\varphi}, \tilde{\boldsymbol{A}}$. In particular,

$$\tilde{\boldsymbol{A}} \to \tilde{\boldsymbol{A}} + \nabla\psi$$

$$\Rightarrow \ell_{\rm Br} \to \ell_{\rm Br} + \frac{q_h}{c}\langle(\nabla\psi)(\boldsymbol{X}+\boldsymbol{\rho})\rangle \cdot \dot{\boldsymbol{X}}$$

$\Rightarrow$ Gauge invariance is spoiled by $\ell_{\rm Br}$

I. Why not just use ℓ_{Br} anyway?

Choosing $\ell_{\mathrm{gy}} = \ell_{\mathrm{Br}}$ spoils gauge invariance of the whole theory. There are two negative consequences.

I. **Hot charge is not conserved.**
II. **Spurious momentum transfer terms appear in the fluid momentum equation.**

I. If ℓ_{gy} were gauge invariant, problems disappear!

Fact: Noether's theorem guarantees that gauge-invariant Lagrangian systems conserve charge.

Consequence: Because we are deriving our model from a Lagrangian, finding a gauge-invariant ℓ_{gy} would ensure charge conservation. Spurious momentum transfer terms would disappear too!

I. With a small modification, ℓ_{Br} can be made gauge invariant.

First, add a special total time derivative to ℓ_{Br}:

$$\ell_{\mathrm{Br}} \to \ell_{\mathrm{Br}} - \frac{d}{dt} q_h \langle\langle \tilde{\boldsymbol{A}}(\boldsymbol{X} + \lambda \boldsymbol{\rho}) \cdot \boldsymbol{\rho} \rangle\rangle.$$

I. With a small modification, ℓ_{Br} can be made gauge invariant.

Next, replace the total time derivative with the approximation:

$$\frac{d}{dt} q_h \langle\langle \tilde{\boldsymbol{A}}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) \cdot \boldsymbol{\rho} \rangle\rangle \approx$$

$$q_h \langle\langle \dot{\boldsymbol{X}} \cdot \nabla \tilde{\boldsymbol{A}}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) \cdot \boldsymbol{\rho} + \partial_t \tilde{\boldsymbol{A}}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) \cdot \boldsymbol{\rho} \rangle\rangle$$

Neglected terms are proportional to products of the fluctuating fields and gradients of $\boldsymbol{B}_{\mathrm{eq}}$.

I. With a small modification, ℓ_{Br} can be made gauge invariant.

The single-gyrocenter Lagrangian:

$$\ell_{\mathrm{gy}} \equiv \left(q_h \boldsymbol{A}_{\mathrm{eq}} + m_h v_\| \boldsymbol{b}_{\mathrm{eq}}\right) \cdot \dot{\boldsymbol{X}} - \left(\frac{1}{2} m_h v_\|^2 + [\mu + \delta\mu] B_{\mathrm{eq}}\right)$$
$$+ q_h \tilde{\boldsymbol{A}}(\boldsymbol{X}) \cdot \dot{\boldsymbol{X}} - q_h \tilde{\varphi}(\boldsymbol{X})$$
$$+ q_h \langle\langle [\tilde{\boldsymbol{E}}(\boldsymbol{X} + \lambda\boldsymbol{\rho}) + \dot{\boldsymbol{X}} \times \tilde{\boldsymbol{B}}(\boldsymbol{X} + \lambda\boldsymbol{\rho})] \cdot \boldsymbol{\rho} \rangle\rangle.$$

This Lagrangian is manifestly gauge-invariant!

I. We now have our system Lagrangian!

The system Lagrangian:

$$\begin{aligned} L &= L_{\mathsf{p}} + L_{\mathsf{MHD}} \\ &= \iint_{\mu} \ell_{\mathrm{gy}} \, F \, d^4\mathbf{z} + \frac{1}{2} \int \rho \, |\boldsymbol{U}|^2 \, d^3\boldsymbol{x} \\ &- \int \rho \, \mathcal{U}(\rho) \, d^3\boldsymbol{x} - \frac{1}{2} \int |\boldsymbol{B}_{\mathsf{eq}} + \tilde{\boldsymbol{B}}|^2 \, d^3\boldsymbol{x} \end{aligned}$$

Note: We must set $\dot{\boldsymbol{X}} = \mathbf{u}_{\mathrm{gy}}(\mathbf{z})$ in ℓ_{gy} because we are integrating over the Eulerian phase space coordinates $\mathbf{z}$ and not the Lagrangian labels $\mathbf{z}_o$.

II. Now we must express L in terms of $\boldsymbol{q}(x_o)$ and $\boldsymbol{z}(\mathbf{z}_o)$

Expressing the Eulerian fluid velocities in terms of Lagrangian configuration maps is standard.

Eulerian fluid velocities:

$$\boldsymbol{U}(\boldsymbol{q}(\boldsymbol{x}_o)) = \frac{d\boldsymbol{q}(\boldsymbol{x_o})}{dt}$$

$$\boldsymbol{\mathcal{X}}(\boldsymbol{z}(\mathbf{z}_o)) = \frac{d\boldsymbol{z}(\mathbf{z}_o)}{dt}$$

where $\boldsymbol{\mathcal{X}} = (\mathbf{u}_{\mathrm{gy}}, a_{\|\mathrm{gy}})$.

II. Now we must express L in terms of $\boldsymbol{q}(\mathbf{x}_o)$ and $\boldsymbol{z}(\mathbf{z}_o)$

Expressing ρ and F in terms of Lagrangian configuration maps is also standard.

Eulerian mass density and distribution function:

$$\rho(\boldsymbol{q}(\boldsymbol{x}_o))\, d^3\boldsymbol{q} = \rho_0(\boldsymbol{x}_o)\, d^3\boldsymbol{x}_o$$
$$F(\boldsymbol{z}(\mathbf{z}_o))\, d^4\boldsymbol{z} = F_0(\mathbf{z}_o)\, d^4\mathbf{z}_o$$

where ρ_0 and F_0 are the initial ρ and F.

II. Expressing $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$ in terms of $\boldsymbol{q}(\boldsymbol{x}_o)$ is tricky

In order to express $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$ in terms of $\boldsymbol{q}(\boldsymbol{x}_o)$, we must invoke Ohm's law

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0$$

II. Expressing $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$ in terms of $\boldsymbol{q}(\boldsymbol{x}_o)$ is tricky

As usual, the curl of Ohm's law, together with Faraday's law, implies that the total magnetic field is frozen into the bulk flow

$$\partial_t \boldsymbol{B} = \nabla \times (\boldsymbol{U} \times \boldsymbol{B})$$

II. Expressing $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$ in terms of $\boldsymbol{q}(\boldsymbol{x}_o)$ is tricky

The curl of Ohm's law will be satisfied automatically if we freeze the total vector potential into the bulk flow.

The vector potential:

$$\Big(\boldsymbol{A}_{\text{eq}}(\boldsymbol{q}(\boldsymbol{x}_o)) + \tilde{\boldsymbol{A}}(\boldsymbol{q}(\boldsymbol{x}_o))\Big) \cdot d\boldsymbol{q} = \Big(\boldsymbol{A}_{\text{eq}}(\boldsymbol{x}_o) + \tilde{\boldsymbol{A}}_o(\boldsymbol{x}_o)\Big) \cdot d\boldsymbol{x}_o$$

II. Expressing $\tilde{\varphi}$ and $\tilde{\boldsymbol{A}}$ in terms of $\boldsymbol{q}(\boldsymbol{x}_o)$ is tricky

In order to satisfy Ohm's law completely, the potential $\tilde{\varphi}$ must therefore be expressed in the so-called hydrodynamic gauge.

The scalar potential:

$$\varphi = (\boldsymbol{A}_{\text{eq}} + \tilde{\boldsymbol{A}}) \cdot \boldsymbol{U}$$

III. We can now vary the action!

Our GK-MHD hybrid follows from the variational principle

$$\delta \int L \, dt = 0$$

where the quantities being varied are $\boldsymbol{q}(\boldsymbol{x}_o)$ and $\boldsymbol{z}(\mathbf{z}_o)$.

III. The variations of all Eulerian quantities can be calculated first

Eulerian variations implied by Lagrangian variations

$$\delta F = -\nabla \cdot (F \boldsymbol{\Xi}_{\boldsymbol{X}}) - \partial_{v_\parallel}(F \Xi_{v_\parallel}) = -\nabla_{\mathbf{z}} \cdot (F \boldsymbol{\Xi})$$

$$\delta \rho = -\nabla \cdot (\rho \boldsymbol{\xi})$$

$$\delta \boldsymbol{\mathcal{X}}_{\mathrm{gy}} = \partial_t \boldsymbol{\Xi} + \boldsymbol{\mathcal{X}}_{\mathrm{gy}} \cdot \nabla_{\mathbf{z}} \boldsymbol{\Xi} - \boldsymbol{\Xi} \cdot \nabla_{\mathbf{z}} \boldsymbol{\mathcal{X}}_{\mathrm{gy}}$$

$$\delta \boldsymbol{U} = \partial_t \boldsymbol{\xi} + \boldsymbol{U} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{U}$$

$$\delta \tilde{\boldsymbol{A}} = \boldsymbol{\xi} \times \boldsymbol{B} - \nabla(\boldsymbol{\xi} \cdot \boldsymbol{A})$$

$$\delta \tilde{\varphi} = -\boldsymbol{\xi} \cdot \nabla \tilde{\varphi} + \partial_t \boldsymbol{\xi} \cdot \boldsymbol{A}$$

$$\delta \tilde{\boldsymbol{B}} = \nabla \times (\boldsymbol{\xi} \times \boldsymbol{B})$$

$$\delta \tilde{\boldsymbol{E}} = \boldsymbol{\xi} \times (\nabla \times \tilde{\boldsymbol{E}}) - \nabla(\boldsymbol{\xi} \cdot \tilde{\boldsymbol{E}}) - (\partial_t \boldsymbol{\xi}) \times \boldsymbol{B}$$

III. The Euler-Lagrange equations are then given by:

$$q_h \boldsymbol{E}^* - q_h \boldsymbol{B}^* \times \mathbf{u}_{\rm gy} - m_h a_{\|{\rm gy}} \boldsymbol{b}_{\rm eq} = 0$$

$$m_h \mathbf{u}_{\rm gy} \cdot \boldsymbol{b}_{\rm eq} - m_h v_{\|} = 0$$

$$\rho(\partial_t \boldsymbol{U} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}) = -\nabla {\rm p} - q_h n_h \boldsymbol{E} + (\mu_o^{-1} \nabla \times \boldsymbol{B} - \boldsymbol{J}_h) \times \boldsymbol{B}.$$

$$q_h n_h = q_h \iint_\mu F(\boldsymbol{x}, v_{\|}) \, dv_{\|} - \nabla \cdot \boldsymbol{P}_{\rm gy}$$

$$\boldsymbol{J}_h = q_h \iint_\mu \mathbf{u}_{\rm gy}(\boldsymbol{x}, v_{\|}) \, F(\boldsymbol{x}, v_{\|}) \, dv_{\|} + \nabla \times \boldsymbol{M}_{\rm gy} + \partial_t \boldsymbol{P}_{\rm gy}$$

Note: These equations must be supplemented by the evolution laws implicitly built into the variational principle.

$$\partial_t F + \nabla \cdot (F \mathbf{u}_{\rm gy}) + \partial_{v_{\|}} (F a_{\|{\rm gy}}) = 0$$

$$\partial_t \rho + \nabla \cdot (\rho \, \boldsymbol{U}) = 0$$

$$\boldsymbol{E} + \boldsymbol{U} \times \boldsymbol{B} = 0.$$

III. The gyrocenter polarization and magnetization densities are given by:

Polarization and Magnetization densities:

$$\boldsymbol{P}_{\rm gy} = \frac{\delta L_{\sf p}}{\delta \tilde{\boldsymbol{E}}}\,, \qquad \boldsymbol{M}_{\rm gy} = \frac{\delta L_{\sf p}}{\delta \tilde{\boldsymbol{B}}}.$$

III. The earlier expression of our model is straightforward to recover

The functional derivatives can be evaluated explicitly giving:

$$\boldsymbol{P}_{\rm gy}(\boldsymbol{x}) = q_h \iint_\mu \langle\langle \delta(\boldsymbol{X} + \lambda\boldsymbol{\rho} - \boldsymbol{x})\boldsymbol{\rho}\rangle\rangle F \, d^4\mathbf{z}$$

$$\boldsymbol{M}_{\rm gy}(\boldsymbol{x}) = q_h \iint_\mu \langle\langle \delta(\boldsymbol{X} + \lambda\boldsymbol{\rho} - \boldsymbol{x})\boldsymbol{\rho} \times [\mathbf{u}_{\rm gy} + \lambda \boldsymbol{v}_\perp]\rangle\rangle F \, d^4\mathbf{z}.$$

The hot charge and current densities calculated using these $\boldsymbol{P}_{\rm gy}$ and $\boldsymbol{M}_{\rm gy}$ agree with our earlier expressions.

Conclusion

We have identified a variational GK-MHD hybrid model in the current-coupling scheme with the following properties.

- It recovers the BDC model when the background magnetic field is uniform.
- It is the first GK-MHD model to simultaneously conserve energy, momentum, hot charge, and phase space volume in a general background magnetic field.
- By using a new gauge-invariant gyrocenter Lagrangian, it is expressed entirely in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$.

Using the same approach, we have also formulated a new drift-kinetic-MHD model with similar strengths.

- Cesare will present the DK-MHD model at PPPL in October!