Optical collapse, nonlinear laser beam combining and around

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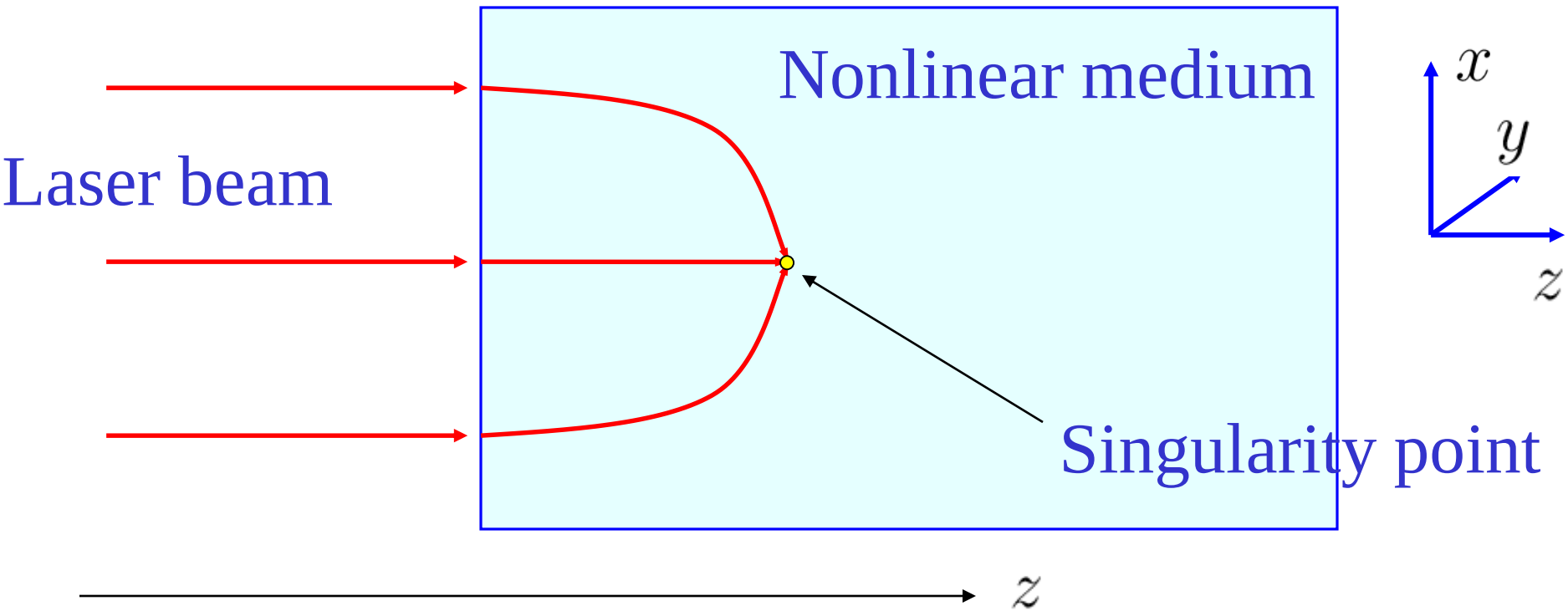
Support: NSF DMS-0719895, NSF DMS-0807131, NSF PHY-1004118, NSF DMS-1412140, DOE 6834403

- Explosive instability (blow-up)
  – formation of singularity in a finite time

- Collapse – blow-up with the contraction of the spatial extent of solution to zero

Self-focusing (collapse) of laser beam

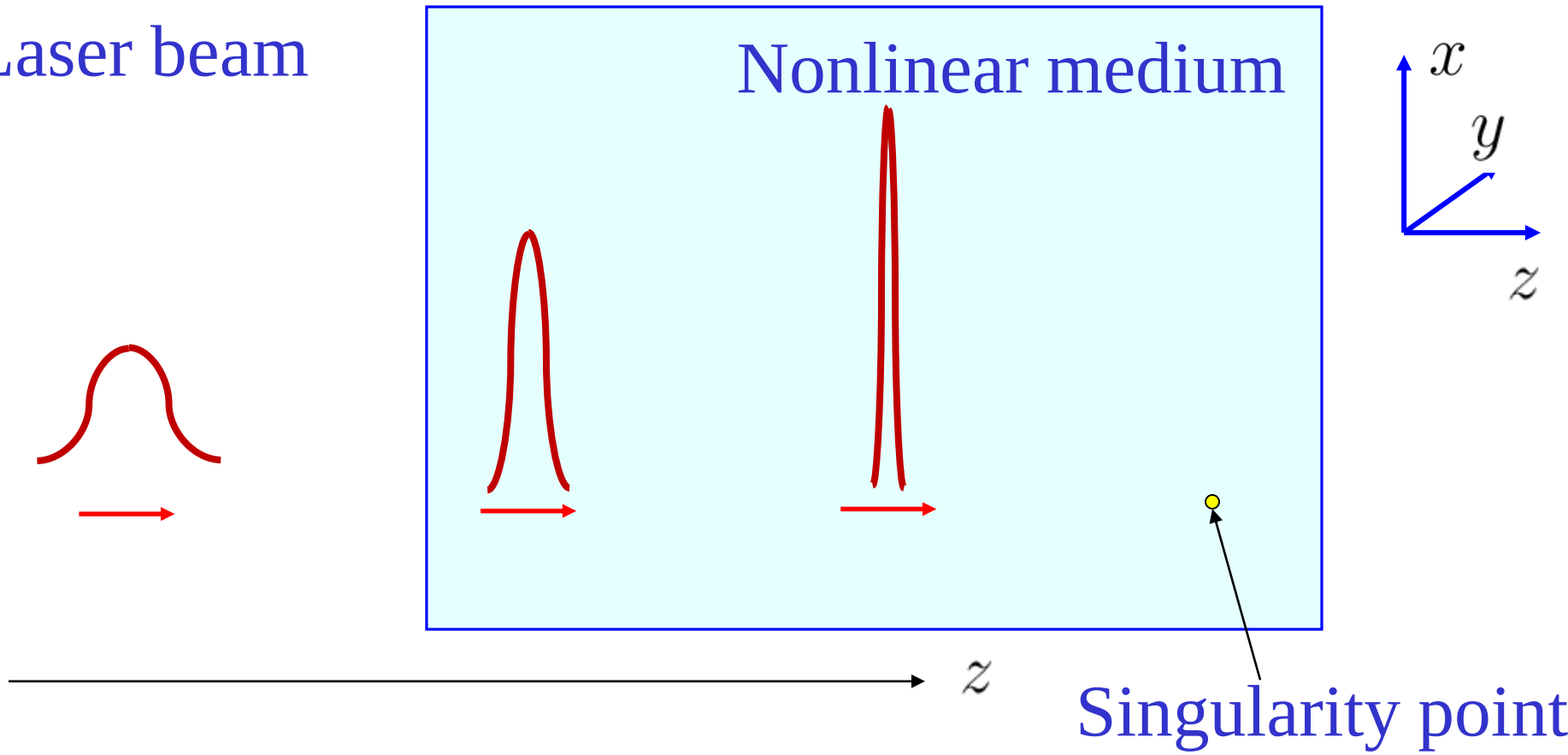


$$i\frac{\partial}{\partial z}\psi + \nabla^2\psi + |\psi|^2\psi = 0$$ - 2D Nonlinear Schrödinger Equation

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

ψ - amplitude of light

Self-focusing of optical bullet



$$i\frac{\partial\psi}{\partial z} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \beta\frac{\partial^2}{\partial t^2}\right)\psi + |\psi|^2\psi = 0$$

- 3D Nonlinear Schrödinger Equation (NLSE)

Nonlinear Schrödinger Equation (NLSE)

$$i\frac{\partial}{\partial t}\psi + \triangle\psi + |\psi|^2\psi = 0$$

$H = \int (|\nabla\psi|^2 - \frac{1}{2}|\psi|^4) d^D\mathbf{r}$ - Hamiltonian: $i\psi_t = \frac{\delta H}{\delta \psi^*}$

$N = \int |\psi|^2 d^D\mathbf{r}$ -optical power (in optics) or number of particles (in quantum mechanics) or wave action in oceanology

Conserved Integrals: $\frac{d}{dt}N = \frac{d}{dt}H = 0$

Mean square width: $A \equiv \int |\mathbf{r}|^2 |\psi|^2 d^D\mathbf{r}$

$\boxed{D = 2}$

Virial theorem[1]: $A_{tt} = 8H$

$$\Rightarrow \qquad A = 4Ht^2 + c_1 t + c_2$$

Singularity formation:

$$H < 0 \qquad \Rightarrow \qquad A\big|_{t\to t_0} \to 0 \qquad \Rightarrow \qquad \max_{t\to t_0} |\psi| \to \infty$$

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[1] S.N.Vlasov, V.A Petrishchev, and V.I. Talanov (1971).
V.E. Zakharov JETP, (1972)

Critical collapse of 2D Nonlinear Schrödinger Equation: Self-similar solution near singularity

$$i\frac{\partial}{\partial t}\psi + \triangle\psi + |\psi|^2\psi = 0$$

$$\Psi(\mathbf{x},t) = \Psi(r,t), \quad r = (x^2+y^2)^{1/2}$$

$$\Psi(r,t) \simeq \frac{1}{L}V(\rho)e^{i\tau + iLL_t\rho^2/4}, \quad L \to 0,$$

$$\rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')},$$

Soliton solution of NLSE:

$$\triangle V - V + |V|^2 V = 0$$

LogLog law[1]:

$$L = \left(2\pi\frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}$$

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[1]G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); V.M. Malkin (1993); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992) F. Merle and P. Raphael (2006).

Strong vs. weak collapse of NLSE

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0$$

D=2: Strong critical collapse (mass critical) as above
D=3: Weak supercritical collapse

$$|\psi_{c,weak}(\mathbf{r},t)| \simeq \frac{1}{L(t)}\eta\left(\frac{\mathbf{r}}{L(t)}\right), \; L(t) \to 0 \quad \text{for } t \to t_0$$

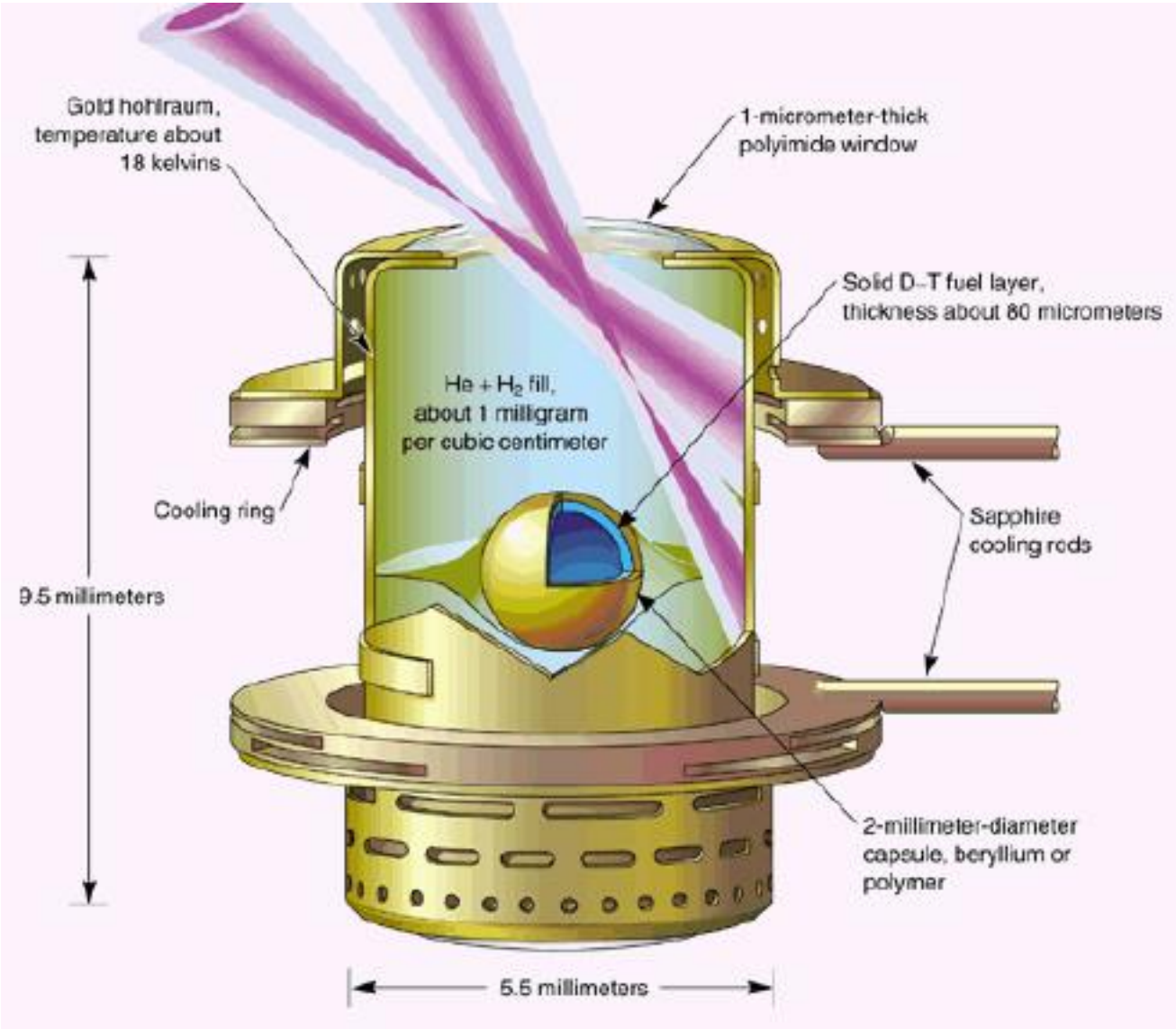
Self-similar variable $\xi \equiv \mathbf{r}/L(t)$

Number of particles in collapsing region

$$N_{collapse,weak} \simeq \int_{|\mathbf{r}|<\xi_c L(t)} |\psi_{c,weak}(\mathbf{r},t)|^2 d^3\mathbf{r}$$

$$= L(t)\int_{|\xi|<\xi_c} \eta^2(\xi) d^3\xi \sim L(t) \to 0 \text{ for } t \to t_0$$

Laser fusion of the National Ignition Facility



Goal: propagation of laser light in plasma with minimal distortion

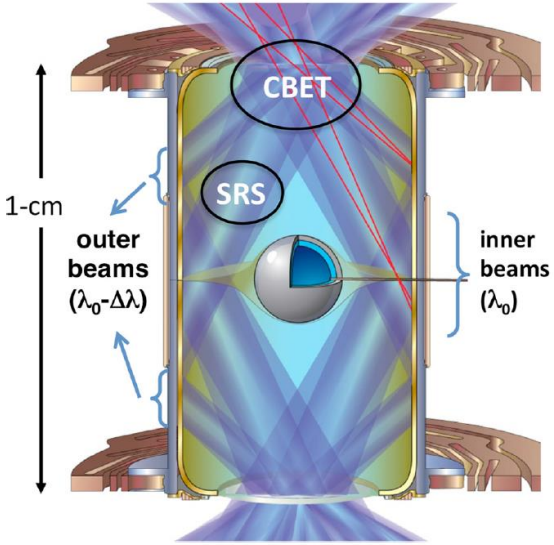
Difficulties: collapses from self-focusing of light and Langmuir wave collapses

Strong beam spray

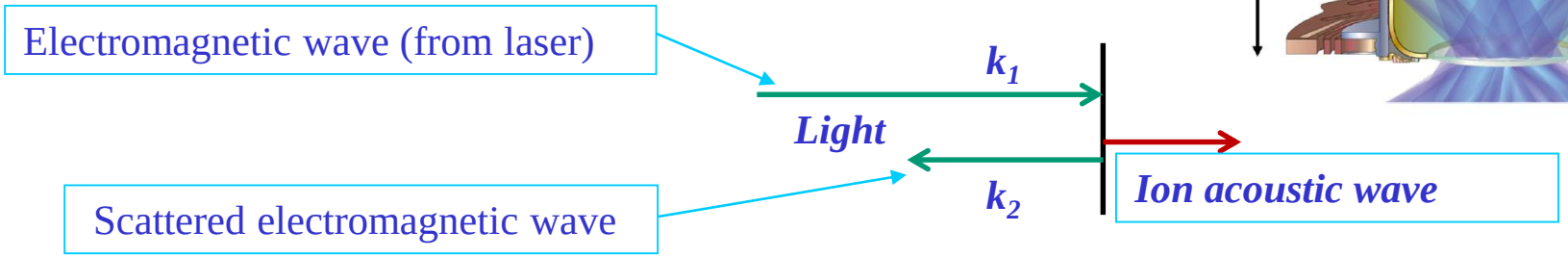
No spray

Laser propagation in plasma

Main laser-plasma interaction effects at The National Ignition Facility

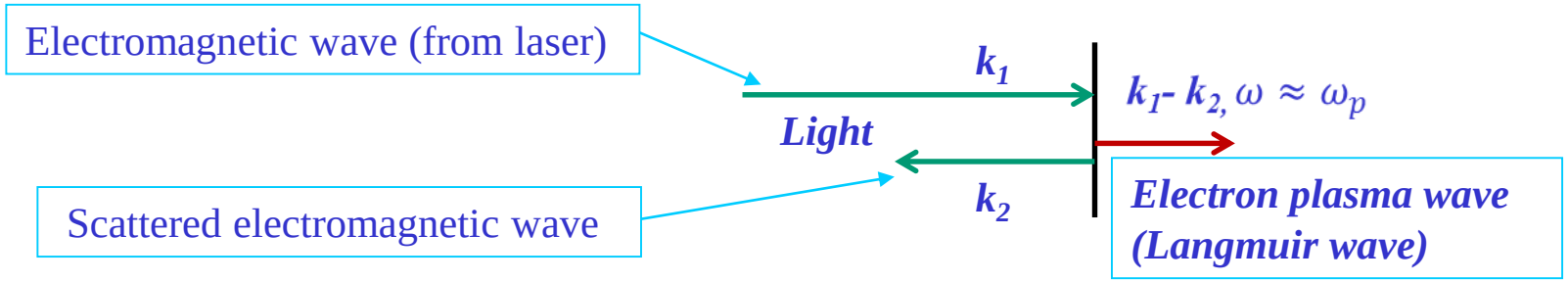


1. Stimulated Brillouin Scattering (SBS)



A particular version of SBS: cross-beam energy transfer (CBET) with both electromagnetic waves corresponding to two laser beams

2. Stimulated Raman Scattering (SRS)

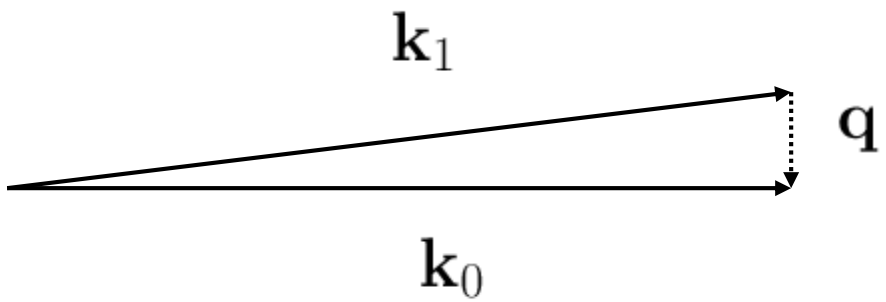


Forward Stimulated Brillouin Scattering

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{q}$$

$$c \gg c_s$$

$$c|k_0| = c|k_1| + c_s|q|$$



$\mathbf{k}_0,\ \mathbf{k}_1$ - light

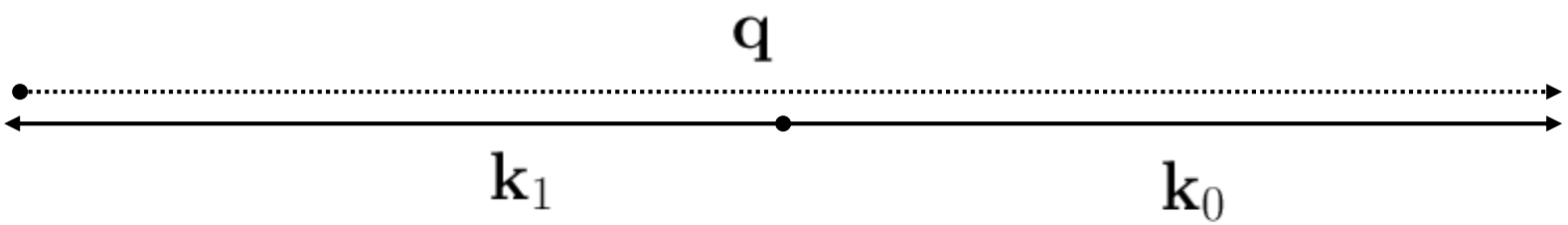
$\mathbf{q}$ - ion acoustic wave

Backward Stimulated Brillouin instability:

$$\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{q}$$

$$c|k_0| = c|k_1| + c_s|q|$$

$$c \gg c_s \quad \Rightarrow \quad \mathbf{k}_1 \simeq -\mathbf{k}_0 \text{ and } \mathbf{q} \simeq 2\mathbf{k}_0$$



$\mathbf{k}_0, \mathbf{k}_1$ - light

$\mathbf{q}$ - ion acoustic wave

Collapse from Forward Stimulated Brillouin Scattering

$$\mathcal{E} = E(\mathbf{r}, z, t)e^{ik_0 z - i\omega_0 t} + c.c.$$ - amplitude of light

$$\mathbf{r} = (x, y)$$

$$i\frac{\partial}{\partial z}E + \nabla^2 E = \rho E, \qquad \nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

$$\left(\frac{\partial^2}{\partial t^2} + 2\tilde{\nu}\frac{\partial}{\partial t} - c_s^2\nabla^2\right)\rho = c_s^2\nabla^2\left[|E|^2 + \frac{\delta T_e}{T_e}\right]$$

ρ - low frequency plasma density fluctuation

$\tilde{\nu} = \nu k c_s$ - Landau damping

c_s - speed of sound

$$\rho = -|E|^2 \quad \Rightarrow \quad \boxed{\mathrm{i}\frac{\partial E}{\partial z} + \nabla_\perp^2 E + |E|^2 E = 0}$$

$$i\frac{\partial}{\partial z}E + \nabla^2 E = \rho E$$

Large correlation time limit $T_c \to \infty$

$$\rho = -|E|^2$$

$$i\frac{\partial}{\partial z}E + \nabla^2 E + E|E|^2 = 0$$ - Nonlinear Schrödinger Eq.

Small correlation time limit $T_c \to 0$

$\rho \to 0$ - light intensity is constant

Laser power and critical power

Typical power of each NIF's 192 beams: **$P=8\times10^{12}$ Watts**

Critical power for self-focusing: **$P_{cr}=1.6\times10^{9}$ Watts**

$$P_{cr} = m_e \frac{c^3 T_e}{e^2} \frac{n_c}{n_e}$$

$P/P_{cr}=5000$ for each NIF beam
or $P/P_{cr}=10^5$ for all NIF beams

Thermal fluctuations

$$\frac{\partial}{\partial t}\frac{\delta T}{T} - \left(\frac{\kappa}{n_e}\right)\Delta\frac{\delta T}{T} = \frac{1}{2}\nu_{ei}\delta\left(\frac{\mathrm{v}_{\mathrm{osc}}^2}{\mathrm{v}_{\mathrm{e}}^2}\right) = 2\nu_{ei}\delta I$$

$$\kappa = \frac{\kappa_{SH}}{1+(7k\lambda_{ei})^{4/3}(Z^*)^{2/3}}$$ - thermal conductivity

$$\kappa_{SH} = \frac{128}{3\pi} n_e \mathrm{v}_{\mathrm{e}} \lambda_{ei},$$

$$\mathrm{v}_{\mathrm{osc}} = eE/m_e\omega_0$$ - electron oscillation speed

λ_{ei} - electron-ion mean free path

$\nu_{ei} = \mathrm{v}_{\mathrm{e}} / \lambda_{ei}$ -electron-ion collision rate

Boundary condition for SBS: Spatial and temporal incoherence of laser beam

$$\hat{E}(\mathbf{k}, z=0, t) = |\hat{E}(\mathbf{k})| \exp\Big[i\phi_{\mathbf{k}}(t)\Big],$$

$$\Big\langle \exp i\Big[\phi_{\mathbf{k}}(t) - \phi_{\mathbf{k}'}(t')\Big]\Big\rangle = \delta_{\mathbf{k}\mathbf{k}'} \exp\Big(-|t-t'|/T_c\Big).$$

"Top hat" model of NIF optics:

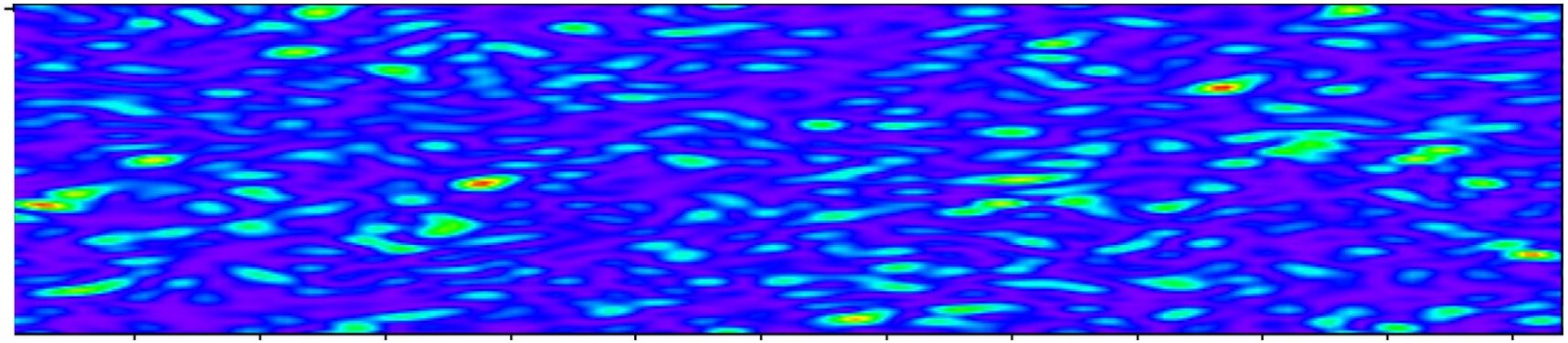
$$|\hat{E}(\mathbf{k})| = const,\ k < k_m;\ |\hat{E}(\mathbf{k})| = 0,\ k > k_m,$$

$$1/l_c \equiv k_m \simeq k_0/(2F),$$

F - optic $f/\#$

Idea of spatial and temporal incoherence of laser beam is to suppress self-focusing

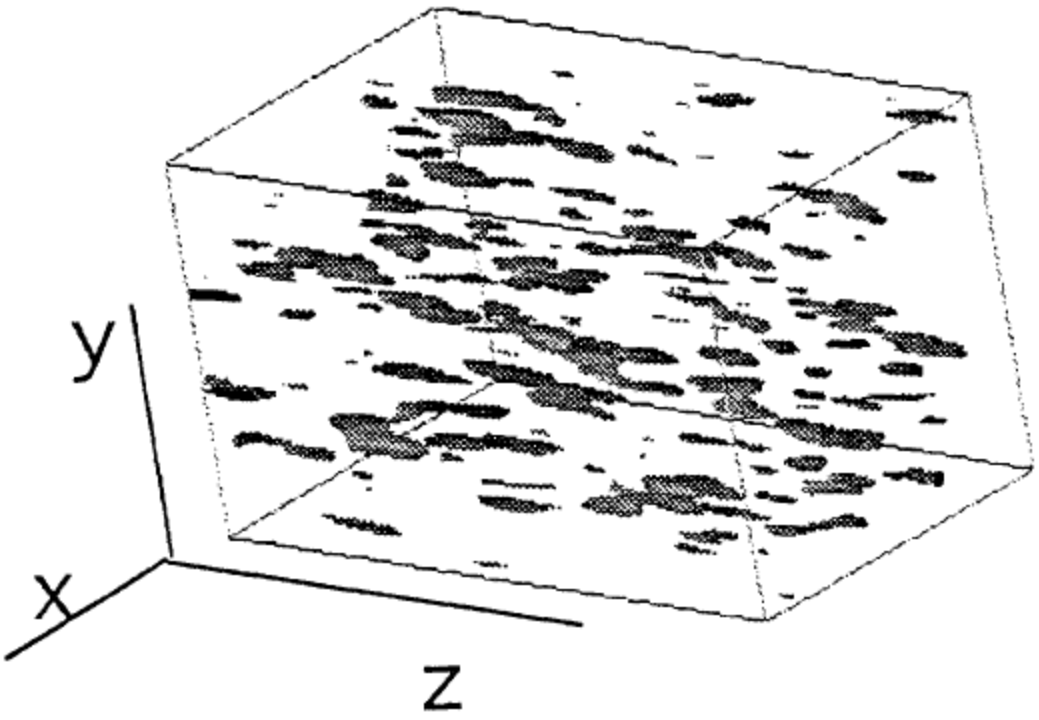
Intensity fluctuations fluctuate, in vacuum, on time scale $\boldsymbol{T_c}$



Laser propagation direction, z $\longrightarrow$

$I = |E|^2$ = intensity

3D picture of intensity fluctuations



Fraction of power in speckles with intensity above critical per unit length

$$M(I) = \frac{\pi^{3/2}\sqrt{5}V}{27F^4\lambda_0^3\pi}\left[\left(\frac{I}{\langle I\rangle}\right)^{3/2} - \frac{3}{10}\left(\frac{I}{\langle I\rangle}\right)^{1/2}\right]\exp\left(-\frac{I}{\langle I\rangle}\right),$$

$$P_{beam}^{-1}dP_{scattered}/dz = \frac{P_c M(I_c)}{\langle I\rangle V}.$$

$$I_{cr} = P_c\Big/\left(F\lambda\right)^2 \approx 2\times10^{16}\,\mathrm{W/cm^2}$$

For NIF:

$$\langle I\rangle = 2\mathrm{x}10^{15}\,\mathrm{W/cm^2} \Rightarrow P_{beam}^{-1}dP_{scattered}/dz = 0.8/\mathrm{cm}$$

$P_{scattered}$ **- amount of power lost for collapses per 1 cm of plasma**

Temporal incoherence of laser beam

$$\hat{E}(\mathbf{k}, z=0, t) = |\hat{E}(\mathbf{k})| \exp\Big[i\phi_{\mathbf{k}}(t)\Big],$$

$$\Big\langle \exp i\Big[\phi_{\mathbf{k}}(t) - \phi_{\mathbf{k}'}(t')\Big]\Big\rangle = \delta_{\mathbf{k}\mathbf{k}'} \exp\Big(-|t-t'|/T_c\Big).$$

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F - optic $f/\#$

Duration of collapse event $T_{collapse} \approx l_c \Big/ \left(c_s \sqrt{P/P_{cr}} \right)$

l_c / c_s **- acoustic transit time across speckle**

$l_c = F\lambda_0$ **- Transverse correlation length (speckle width)**

Condition for collapse to develop:

$$T_{collapse} \leq T_c \Longrightarrow P \geq P_c \left(\frac{l_c}{c_s T_c} \right)^2$$

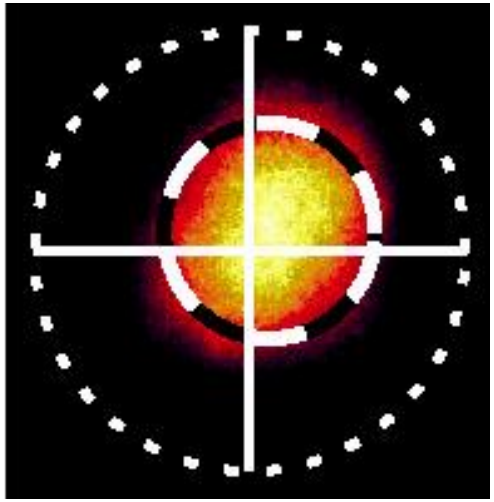
-probability of collapse decreases with T_c

But: Experiments (Niemann, *et al* , 2005) at the Omega laser facility

Conclusion: beam spray can not be explained based on collapses. Collective effects dominate[1].

No beam spray

Beam spray



Cross section of laser beam intensity after propagation through plasma
Dashed circles correspond to beam width for propagation in vacuum.

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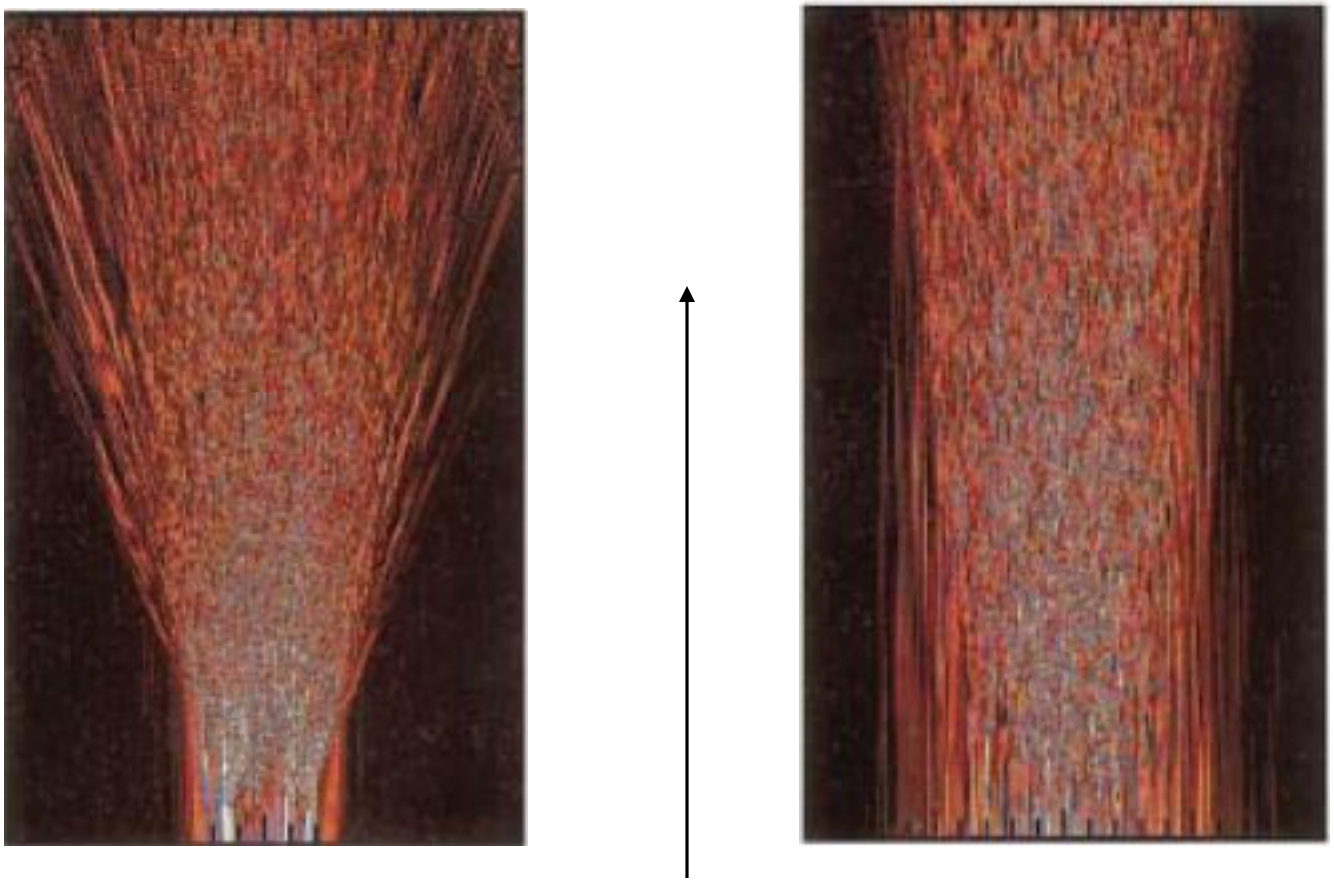
[1]P.M. Lushnikov and H.A. Rose. Plasma Physics and Controlled Fusion, **48**, 1501 (2006).

Ponderomotive beam spray

No RPP **With RPP**



Laser propagation in plasma

C. H. Still *et al.*, PoP, volume 7, page 2023 (2000).

Explanation of beam spray through collective forward stimulated Brillouin instability[1]

Even for very small correlation time, $T_c \to 0$,
there is forward stimulated Brillouin instability (FSBS)

Instability is controlled by the single parameter:

$$\tilde{I} = \frac{1}{\nu} F^2 \frac{n_e}{n_c} \left\langle \left(\frac{\mathrm{v}_{\mathrm{osc}}}{\mathrm{v}_{\mathrm{e}}} \right)^2 \right\rangle \approx \frac{1}{\nu} \frac{\langle P_{\mathrm{speckle}} \rangle}{P_{\mathrm{critical}}}$$

- dimensionless laser intensity

ν - La dau damping

F - optic f-number

$$\tilde{I}_0 = \frac{F^2}{\nu} \frac{n_e}{n_c} \frac{I\, e^2}{4 m_e \omega_0^2 T_e}$$

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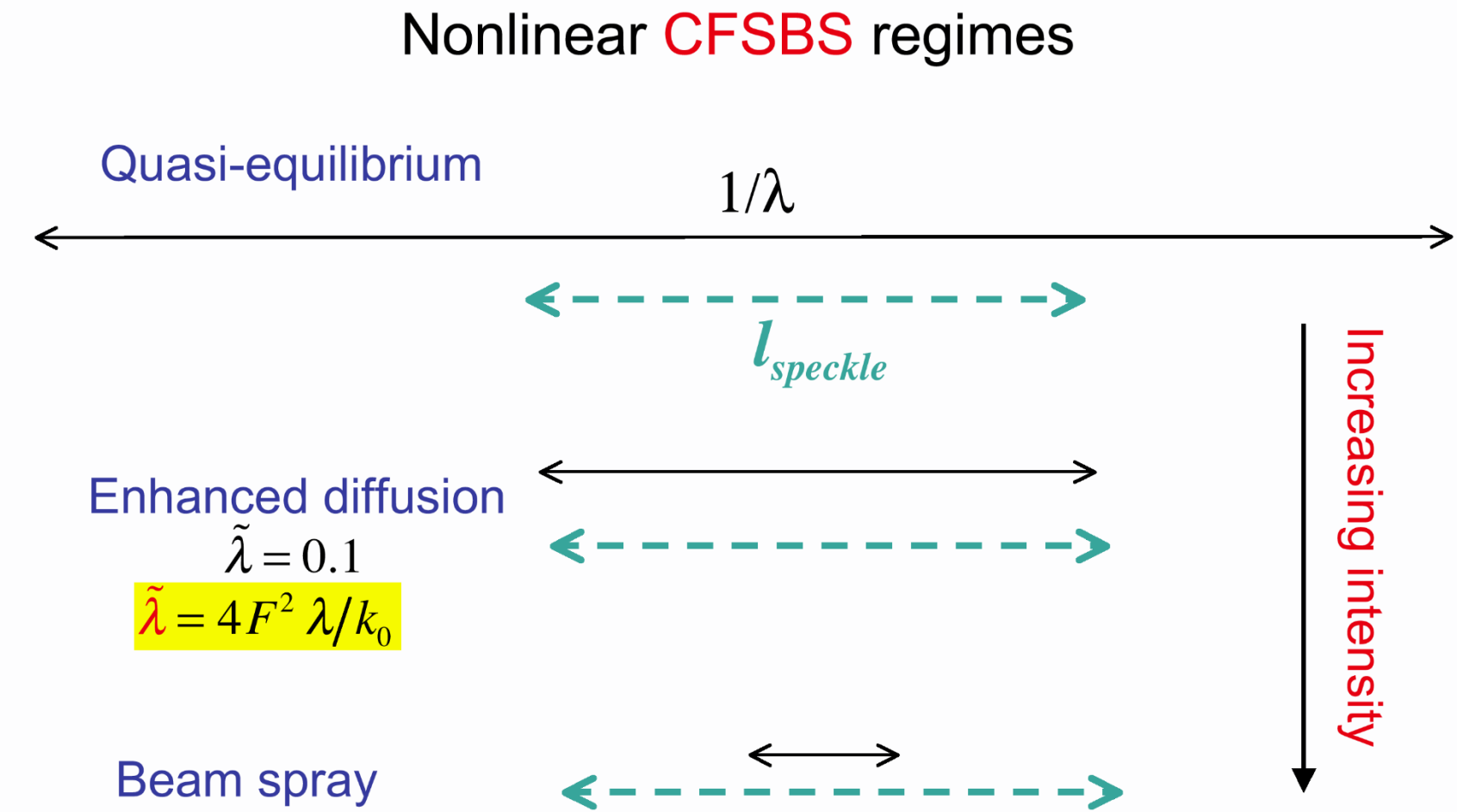
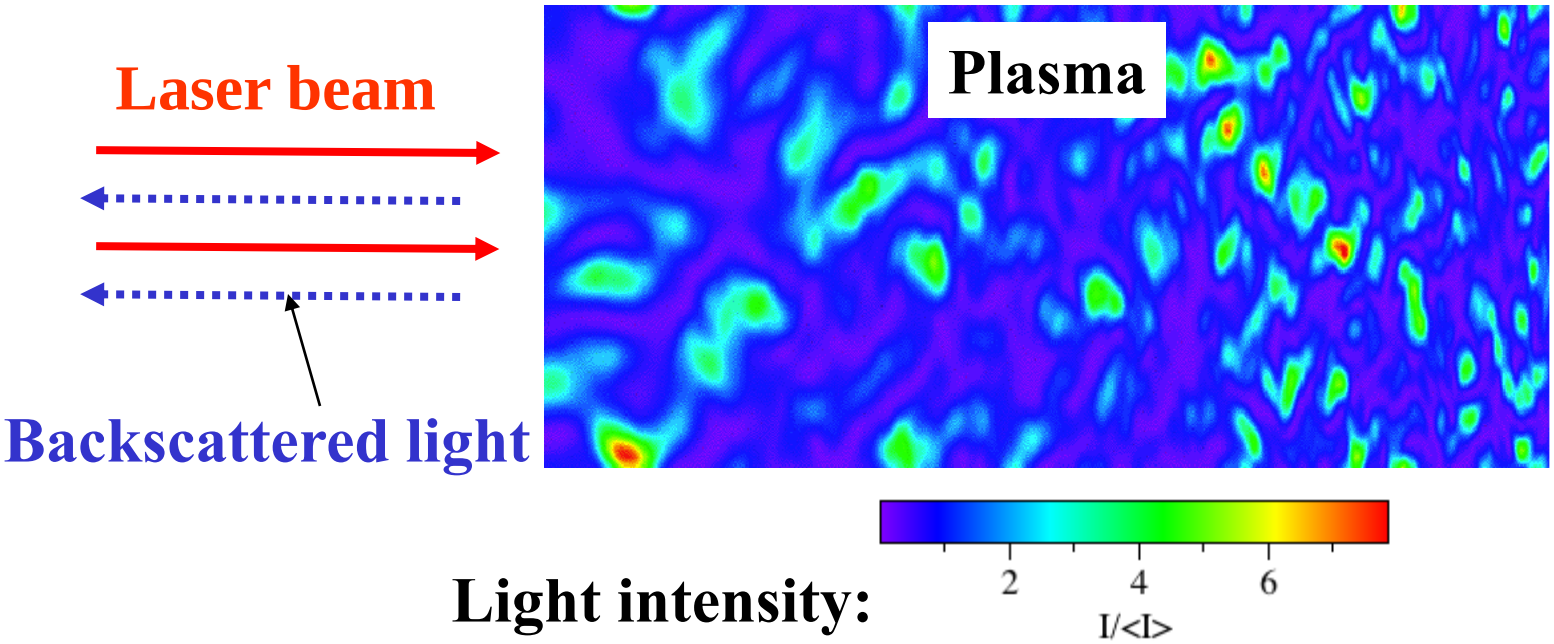
[1]P.M. Lushnikov and H.A. Rose. Physical Review Letters, **92**, 255003 (2004).

λ -convective growth rate

perturbations ~ $e^{\lambda z + i\mathbf{k}\cdot\mathbf{x} - i\omega t}$



Indirect interaction of FSBS with backscatter[1]: in intermediate regime of FBSB instability, the effective laser beam correlation length rapidly decreases with propagation distance



and ***backscatter is suppressed*** due to decrease of correlation length

[1]A.O. Korotkevich, P.M. Lushnikov, and H.A. Rose, Phys. of Plasmas, **22**, 012107 (2015).

Stimulated Brillouin Scattering (SBS) is strongly affected by Z number both for forward SBS[1,2] (at the nonlinear stage results in multiple collapses[3,4]) and backward SBS[5].

E.g. gold plasma is subject to strong SBS for ~10 times smaller laser intensities than low Z plasma

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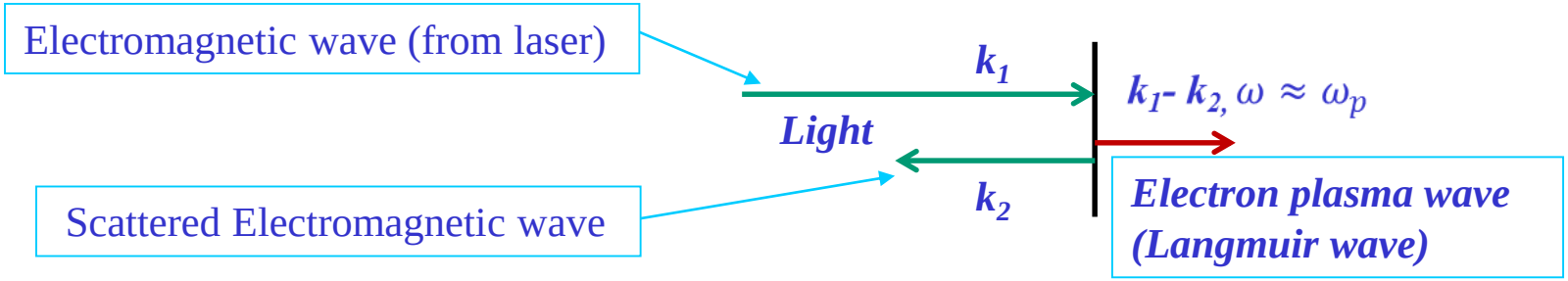
[1]P.M. Lushnikov and H.A. Rose. Physical Review Letters, **92**, 255003 (2004).
[2]P.M. Lushnikov and H.A. Rose. Plasma Physics and Controlled Fusion, **48**, 1501 (2006).
[3]P.M. Lushnikov and N. Vladimirova. Optics Letters, **35**, 1965-1967 (2010).
[4]Y. Chung and P.M. Lushnikov, Physical Review E, **84**, 036602 (2011).
[5]A.O. Korotkevich, P.M. Lushnikov, and H.A. Rose, Phys. of Plasmas, **22**, 012107 (2015).

Stimulated Raman Scattering (SRS)



In the kinetic regime:

$$0.25 < k\lambda_D < 0.45$$

λ_D – Debye length

Found that SRS can be reduced by the filamentation from the collapse of Langmuir waves

Collapse of Langmuir waves: Generalized Zakharov Eq.

$$i\left[\frac{\partial}{\partial t}+\mathrm{v}_{\mathrm{group}}\frac{\partial}{\partial x_{\mathrm{P}}}+\nu_{\mathrm{Landau}}\left(k,|\psi|\right)\right]\psi=\left[-D_{\perp}\Delta_{\perp}+\Delta\omega_{\mathrm{trapped}}+\frac{1}{2}\frac{\delta n}{n}\omega_{\mathrm{pe}}\right]\psi$$

$$\left(\frac{1}{c_{\mathrm{ia}}^{2}}\frac{\partial^{2}}{\partial t^{2}}-\Delta_{\perp}\right)\frac{\delta n}{n}=\frac{1}{4}\left(k\lambda_{D}\right)^{2}\Delta_{\perp}\left|\frac{e\psi}{T_{e}}\right|^{2}$$

ψ - amplitude of Langmuir wave

$\nu_{\mathrm{Landau}}\left(k,|\psi|\right)$ - nonlinear Landau damping

$\Delta\omega_{\mathrm{trapped}}$ - nonlinear frequency shift

c_{ia} - speed of ion-acoustic waves

δn - density of low frequency fluctuations

$$\delta n\propto-|\psi|^{2}\quad\Longrightarrow\quad \mathrm{i}\frac{\partial\psi}{\partial z}+\nabla_{\perp}^{2}\psi+|\psi|^{2}\psi=0$$

Kinetic effects in Generalized Zakharov Eq:

$\nu_{\text{Landau}}\left(k,|\psi|\right)$ - nonlinear Landau damping

$\Delta\omega_{\text{trapped}}$ - nonlinear frequency shift

$$i\left[\frac{\partial}{\partial t}+\mathrm{v}_{\text{group}}\frac{\partial}{\partial x_{\mathrm{P}}}+\nu_{\text{Landau}}\left(k,|\psi|\right)\right]\psi=\left[-D_{\perp}\Delta_{\perp}+\Delta\omega_{\text{trapped}}+\frac{1}{2}\frac{\delta n}{n}\omega_{\text{pe}}\right]\psi$$

$$\left(\frac{1}{c_{\text{ia}}^{2}}\frac{\partial^{2}}{\partial t^{2}}-\Delta_{\perp}\right)\frac{\delta n}{n}=\frac{1}{4}\left(k\lambda_{D}\right)^{2}\Delta_{\perp}\left|\frac{e\psi}{T_{e}}\right|^{2}$$

Kinetic effects require to solve 3+3 Vlasov equation (3 velocity dimensions and 3 spatial dimensions) for the phase space distribution function $f(\mathbf{r},\mathbf{v},t)$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}}\right)f=0$$

$$\mathbf{E}=-\nabla\phi \qquad \Delta\phi=-\rho=-\int f\,d\mathrm{v}_x d\mathrm{v}_y d\mathrm{v}_z$$

Scaled units: electron thermal units

[1]P.M. Lushnikov, H.A. Rose, D.A. Silantyev, and N. Vladimirova. Physics of Plasmas, **21**, 72103 (2014).

[2]D.A. Silantyev, P.M. Lushnikov and H.A. Rose. Submitted to Physics of Plasmas, ArXiv:1610.06137 (2016).

[3]D.A. Silantyev, P.M. Lushnikov and H.A. Rose. Submitted to Physics of Plasmas, ArXiv: 1610.10071 (2016).

Focus on Critical collapse of 2D Nonlinear Schrödinger Equation: Self-similar solution near singularity

$$i\frac{\partial}{\partial t}\psi + \triangle\psi + |\psi|^2\psi = 0$$

$$\Psi(\mathbf{x}, t) = \Psi(r, t), \quad r = (x^2 + y^2)^{1/2}$$

$$\Psi(r, t) \simeq \frac{1}{L}V(\rho)e^{i\tau + iLL_t\rho^2/4}, \quad L \to 0,$$

$$\rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')},$$

Soliton solution of NLSE:

$$\triangle V - V + |V|^2 V = 0$$

LogLog law[1]:

$$L = \left(2\pi\frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}$$

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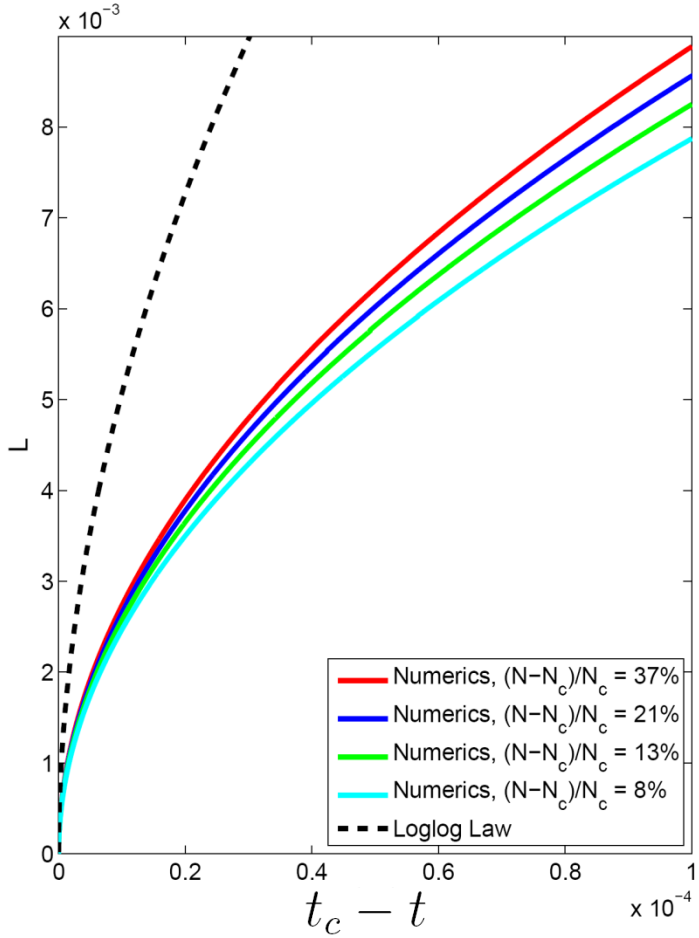
[1]G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992); V.M. Malkin (1993); F. Merle and P. Raphael (2006).

But simulations failed to confirm log-log law in a convincing way[1] although the exact proof of the existence of log-log scaling was given[2]

Example of NLS simulations:

***L(t)* depends on initial conditions**



[1]See e.g.: N.E. Kosmatov, V.F. Shvets and V.E. Zakharov (1991);
G. D. Akrivis, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney (2003).
[2]F. Merle and P. Raphael (2006).

A little of history of 2D NLS collapse

- **1962** G.A. Askaryan: Self-focusing of laser beam

- **1964** R.Y. Chiao, E. Garmire and C.H. Townes: self-trapping of laser beam, Townes soliton as instability threshold

- **1970** V.I. Talanov: lens transform

- **1971** S.N.Vlasov, V.A Petrishchev, and V.I. Talanov: Virial theorem and exact proof of collapse formation

- **1983** M.I. Weinstein. Sharp estimates on collapse.

- **1985** E.A. Kuznetsov and S.K. Turitsyn: conformal symmetry and Noether theorem

- **1985** G. Fraiman : “almost” log-log scaling of collapse

- **1987** M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem: log-log scaling of collapse

- **1992** A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov: collapse turbulence

- **1993** V.M. Malkin: collapse in terms of the excess of number of particles above critical; limitations of log-log scaling

- **2006** F. Merle and P. Raphael: exact proof of existence of log-log scaling

2D NLSE collapse in blow-up variables

Blow-up variables (use *t* instead of *z*)

$$\rho = \frac{r}{L}, \quad \tau = \int^t \frac{dt'}{L^2(t')}$$

and lens transform

$$\psi(r,t) = \frac{1}{L} V(\rho,\tau) e^{i\tau + iLL_t\rho^2/4}$$

$$\Rightarrow \quad iV_\tau + \nabla^2 V - V + |V|^2 V + \frac{\beta}{4}\rho^2 V = 0,$$

where $\boxed{\beta = -L^3 L_{tt}}$ **- adiabatically slow small parameter**

$$\beta \ll 1$$

Looking for solution in the form

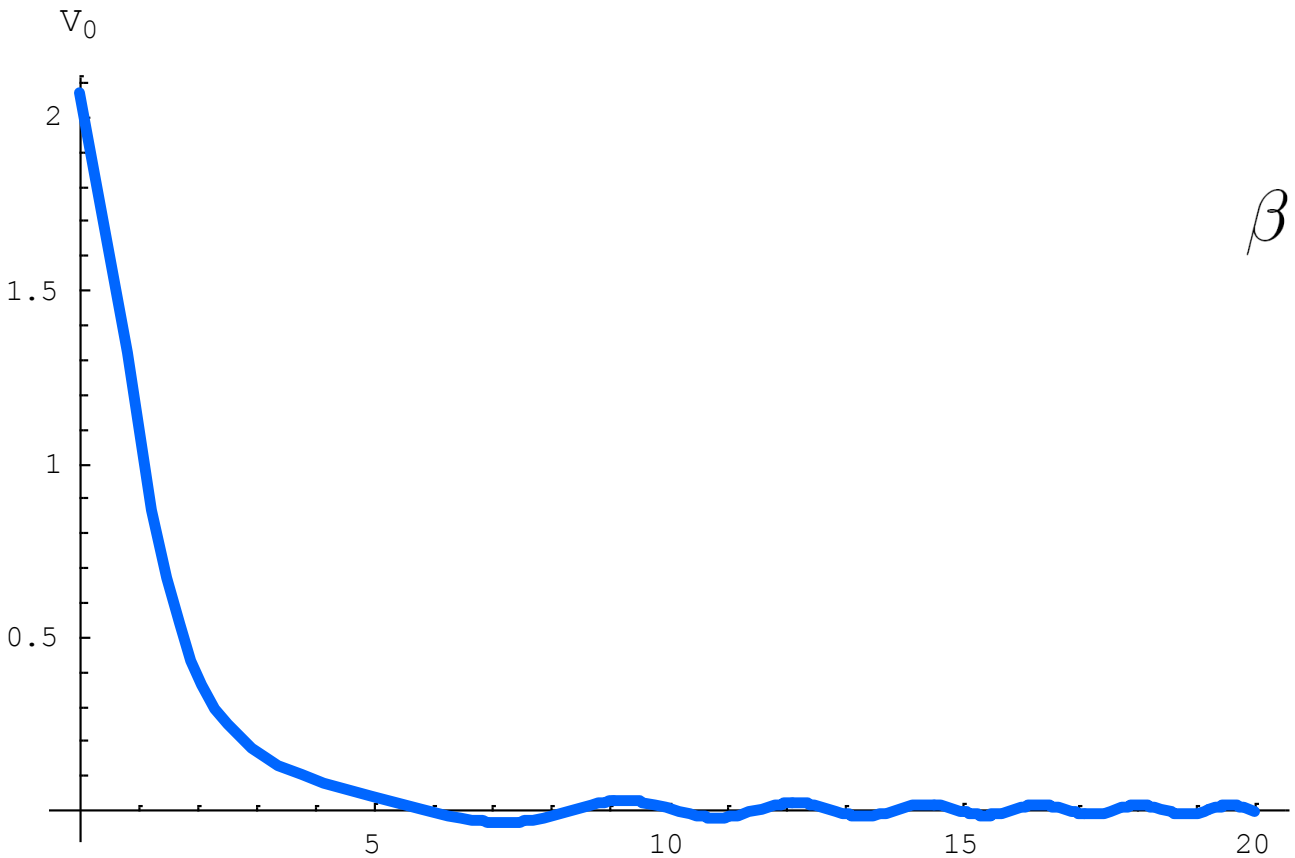
$$V = V_0 + V_1 + \ldots$$

Tail minimization principle: during collapse dynamics system dynamically select collapsing solution with minimal tail amplitude

Then we look for $\boldsymbol{V_0}$ with the minimal tail

NLSE: In adiabatic approximation of slow β minimizing tails by shooting method:

$$\nabla^2 V_0 - V_0 + |V_0|^2 V_0 + \frac{\beta}{4}\rho^2 V_0 = 0$$

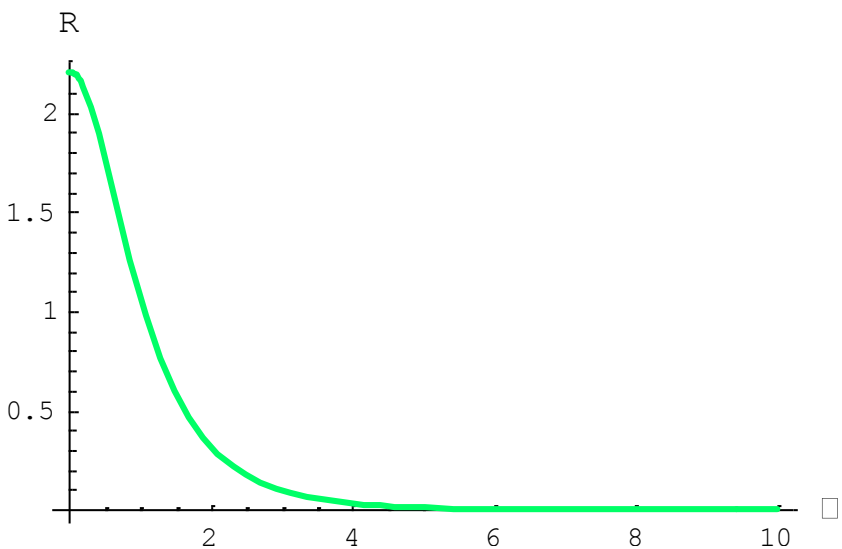


$$\beta = 0.2$$

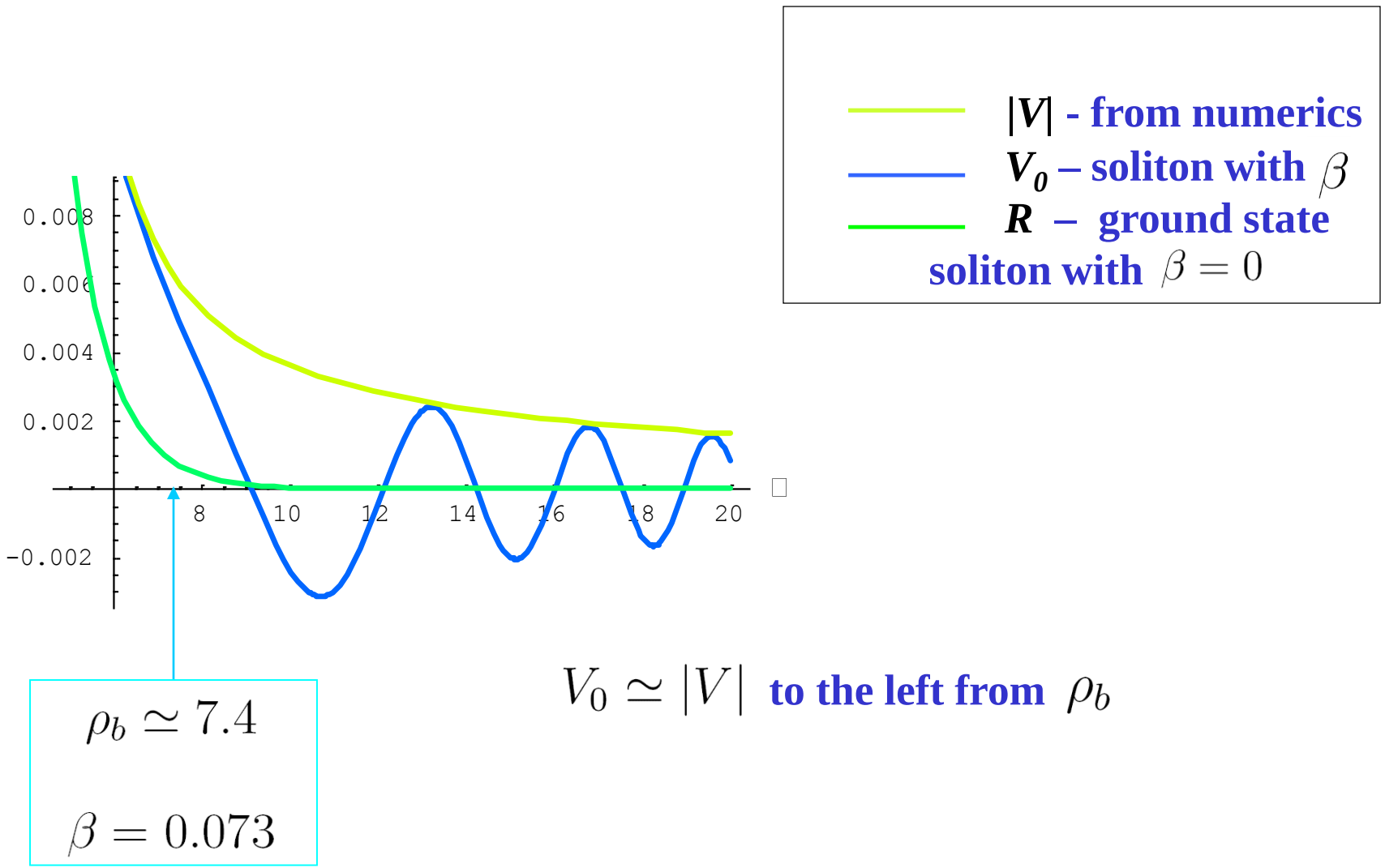
Approximation through ground state soliton $R(\rho)$

$$V_0 = R(\rho) + \beta \frac{\partial V_0}{\partial \beta}\Big|_{\beta=0} + O(\beta^2)$$

$$-R + \nabla^2 R + R^3 = 0$$



NLSE: Full solution V match the envelope of V_0 of in the tail:

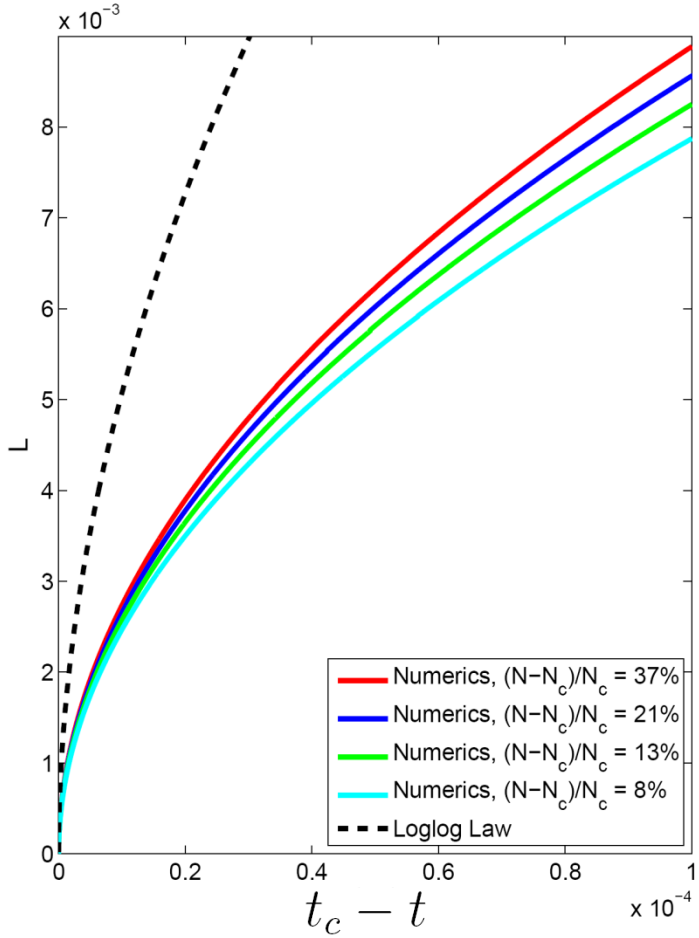


$V_0 \simeq |V|$ **to the left from** ρ_b

But simulations failed to confirm log-log law in a convincing way[1] although the exact proof of the existence of log-log scaling was given[2]

Example of NLS simulations:

***L(t)* depends on initial conditions**

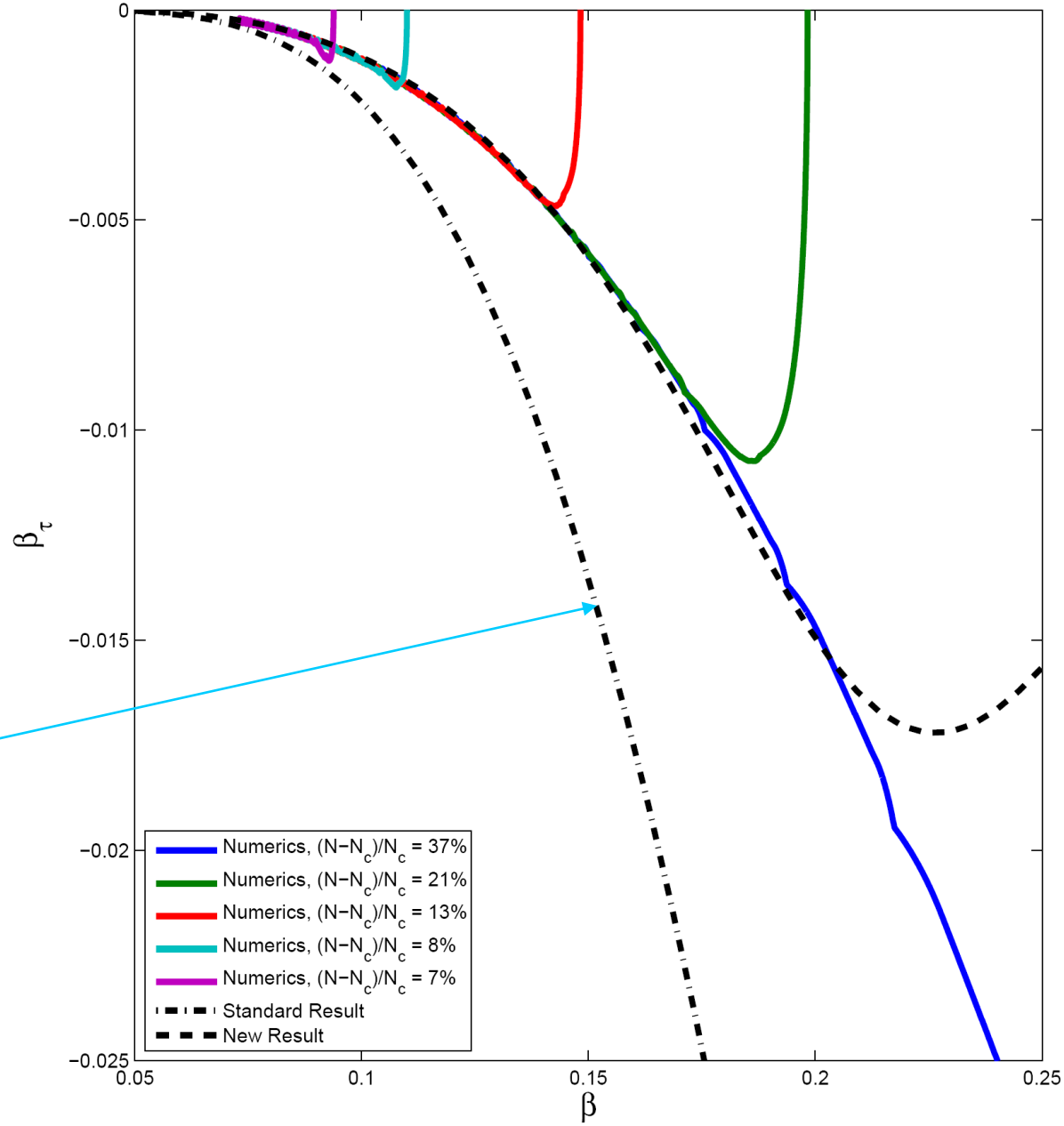


[1]See e.g.: N.E. Kosmatov, V.F. Shvets and V.E. Zakharov (1991);
G. D. Akrivis, V.A. Dougalis, O.A. Karakashian, and W.R. McKinney (2003).
[2]F. Merle and P. Raphael (2006).

NLSE: Modifying the standard theory

***L(t)* is not universal but**
$\beta_\tau(\beta)$ is universal:

$$\beta_\tau = -\tilde{M}\exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$



Recall that we are looking for solution in the form

$$V = V_0 + V_1 + \dots$$

V_1 **- has the imaginary part because of slow dependence of** β **on** τ **:**

$$i\frac{\partial V_0}{\partial \tau} = i\beta_\tau \frac{\partial V_0}{\partial \beta}$$

Also we need to make sure that V **has only outgoing waves for** $\rho \to \infty$

In analogy with Gamov α-decay theory introduce nonself-adjoint problem

$$\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4}\rho^2 \tilde{V}_0 - i\nu(\beta)\tilde{V}_0 = 0,$$

where ν **can be determined from the balance of the norm of *V* as**

$$\nu \sim e^{-\pi/\sqrt{\beta}}$$

In other words, we look at

$$\nabla^2 \tilde{V}_0 - \tilde{V}_0 + |\tilde{V}_0|^2 \tilde{V}_0 + \frac{\beta}{4}\rho^2 \tilde{V}_0 - i\nu(\beta)\tilde{V}_0 = 0,$$

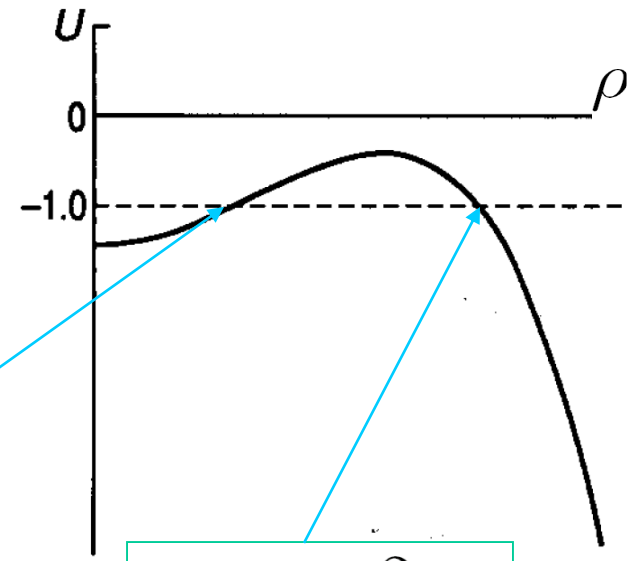
as the Schrodinger equation with the effective potential *U*:

$$U(\rho) = -|\tilde{V}_0|^2 + \frac{\beta}{4}\rho^2$$

and complex eigenvalue *E*:

$$E = -1 - i\nu(\beta)$$

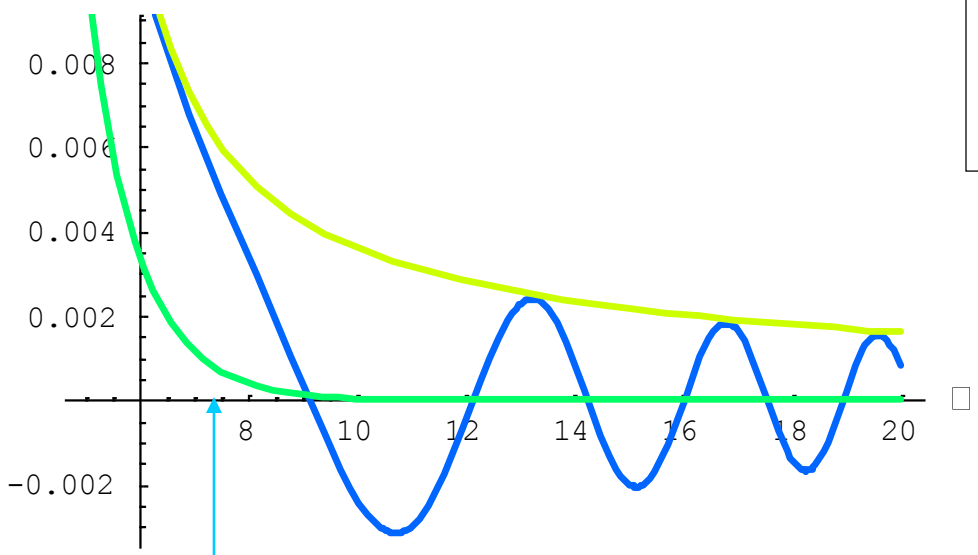
$\Rightarrow$ **2 turning points** ρ_a **and** ρ_b **of WKB:**



$$\rho_a \sim 1$$

$$\rho_b \simeq \frac{2}{\beta^{1/2}}$$

Solution near ρ_b



$|V|$ - from numerics

V_0 – soliton with β

R – ground state soliton with $\beta = 0$

$\rho_b \simeq 7.4$

$\beta = 0.073$

$V_0 \simeq |V|$ to the left from ρ_b

Oscillating tail is given by the linear combination of confluent hypergometric functions of the first and second kinds:

$$c_1 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}\ {}_1F_1(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2)+c_2 e^{-\frac{i}{4}\sqrt{\beta}\rho^2}U(\frac{1}{2}+i\frac{1}{2\sqrt{\beta}};1;i\sqrt{\beta}\rho^2).$$

Matching asymptotics and using WKB give

$$V_0(\beta,\rho)=\frac{2^{1/2}A_R}{\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\cos\left[\frac{\beta^{1/2}}{4}\rho^2-\beta^{-1/2}\ln\rho+\phi_0\right],\quad \rho\gg\rho_b$$

Here $A_R\equiv 3.52$ **is determined by the asymptotic of ground state soliton** $R_0(\rho)=\frac{A_R}{\rho^{1/2}}e^{-\rho},\quad \rho\gg 1$

$\Rightarrow$ **Asymptotics of complex solution**

$$V(\beta,\rho)=\frac{2^{1/2}A_R}{-\beta^{1/4}}e^{-\frac{\pi}{2\beta^{1/2}}}\frac{1}{\rho}\exp\left[i\frac{\beta^{1/2}}{4}\rho^2-i\beta^{-1/2}\ln\rho-i\phi_0\right],\quad \rho\gg\rho_b.$$

Introducing the number of particles to the left of the second turning point

$$N_b = \int\limits_{r<\rho_b L} |\psi|^2 d\mathbf{r} = 2\pi \int\limits_{\rho<\rho_b} |V|^2 \rho d\rho.$$

and balancing the flux of particles through that point

$$\frac{dN_b}{d\tau} = \rho \left[iV^* V_\rho + c.c. \right] |_{\rho=\rho_b}, \qquad \frac{dN_b}{d\tau} = \beta_\tau \frac{dN_b}{d\beta}$$

$\Rightarrow$ **New basic ODE system**

$$\boxed{\begin{cases} \beta_\tau = -\tilde{M} \left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6) \right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}} \right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}}$$

Here $$\frac{dN_b}{d\beta} = M \left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6) \right]$$

$$c_1 = 4.793,\ c_2 = 52.37,\ c_3 = 296.99,\ c_4 = -4660.87,\ c_5 = 10540.4$$

Compare with old basic ODE system of the standard theory

$$\begin{cases} \beta_\tau = -\tilde{M}\exp\left[-\frac{\pi}{\beta^{1/2}}\right], \tilde{M} = 45.056\ldots, \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

Asymptotic solution near collapse time t_c :

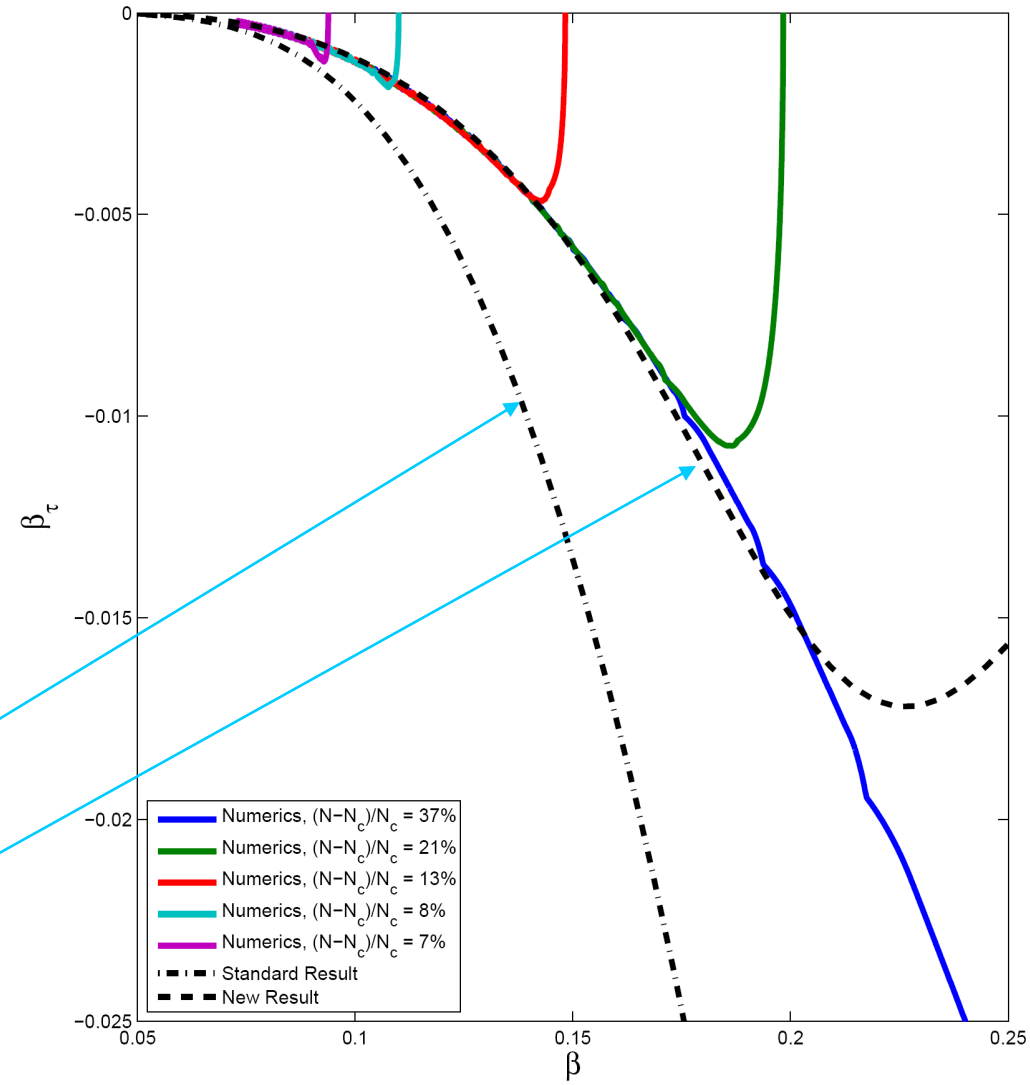
$$\boxed{L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}}$$

---

[1]G. Fraiman (1985); M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987); A. Dyachenko, A. Newell, A. Pushkarev and V.E. Zakharov (1992); V. F. Malkin (1993).

Returning to previous Figure

***L(t)* is not universal but**
***$\beta_\tau(\beta)$* is universal:**



$$\beta_\tau = -\tilde{M} \exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

$$\beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right]$$

Finding asymptotic of a new basic ODE system

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

$$-\ln\frac{L}{L_0} = \frac{2\pi^3 e^x}{\tilde{M}}\left[\frac{1}{x^4} + \frac{4}{x^5} + \frac{20+\pi^2 c_1}{x^6} + \frac{120+6\pi^2 c_1}{x^7} + \frac{840+42\pi^2 c_1+\pi^4 c_2}{x^8} + \frac{6720+336\pi^2 c_1+8\pi^4 c_2}{x^9}\right.$$

$$+\frac{60480+3024\pi^2 c_1+72\pi^4 c_2+\pi^6 c_3}{x^{10}} + \frac{604800+30240\pi^2 c_1+720\pi^4 c_2+10\pi^6 c_3}{x^{11}}$$

$$+\frac{6652800+332640\pi^2 c_1+7920\pi^4 c_2+110\pi^6 c_3+\pi^8 c_4}{x^{12}} + \frac{79833600+3991680\pi^2 c_1+95040\pi^4 c_2+1320\pi^6 c_3+12\pi^8 c_4}{x^{13}}$$

$$\left.+\frac{1037836800+51891840\pi^2 c_1+1235520\pi^4 c_2+17160\pi^6 c_3+156\pi^8 c_4+\pi^{10} c_5}{x^{14}} + O\left(\frac{1}{x^{15}}\right)\right]$$

$$x = \frac{\pi}{\beta^{1/2}}$$

$$\tau = \int_0^t \frac{dt'}{L^2(t')} \quad \Rightarrow$$

$$t_c - t = \int_t^{t_c} dt = \int_\tau^\infty L^2 d\tau = \int_\beta^0 L^2 \frac{d\tau}{d\beta} d\beta$$

$$= -\int_\beta^0 L^2 \frac{1}{\tilde{M}} \left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right] \exp\left[\frac{\pi}{\beta^{1/2}}\right] d\beta$$

Using $\beta(L)$ **from the inversion of previous expression and inverting that equation**

$$\Rightarrow$$

Asymptotic of new basic ODE system

$$\begin{cases} \beta_\tau = -\tilde{M}\left[1 + c_1\beta + c_2\beta^2 + c_3\beta^3 + c_4\beta^4 + c_5\beta^5 + O(\beta^6)\right]^{-1} \exp\left[-\frac{\pi}{\beta^{1/2}}\right], \\ L^3 L_{tt} = -\beta, \\ \tau = \int_0^t \frac{dt'}{L^2(t')} \end{cases}$$

$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A}\right)^{1/2} \left[1 + \frac{2(1 + 4\ln 3 - 4\ln\ln A)}{(\ln A)^2}\right.$$

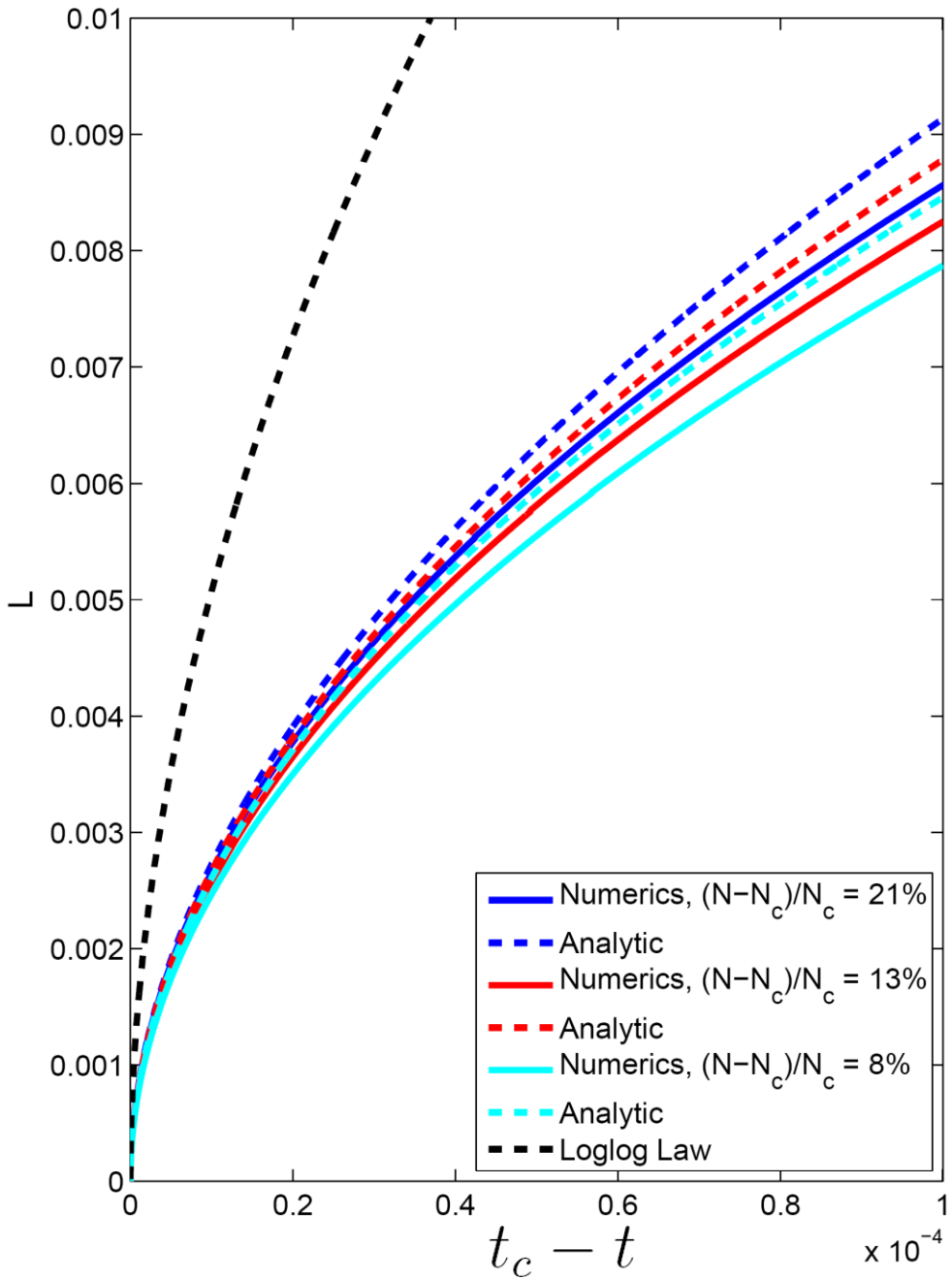
$$\left. + \frac{14 - 48\ln\ln A + 48(\ln\ln A)^2 + 48\ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} + O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\right]$$

$$A = -3^4 \frac{\tilde{M}}{2\pi^3} \ln\left[[2\pi(t_c - t)]^{1/2} \frac{e^{-a_0}}{L(z_0)}\right], \qquad \tilde{M} = 44.773\ldots, \quad \beta_0 = \beta(t_0),\ c_1 = 4.793\ldots, c_2 = 52.37\ldots$$

$$a_0 = \frac{e^{\frac{\pi}{\sqrt{\beta_0}}}}{\tilde{M}} \left(\frac{2\beta_0^2}{\pi} + \frac{8\beta_0^{5/2}}{\pi^2} + \frac{2\beta_0^3\,(20 + \pi^2 c_1)}{\pi^3} + \frac{12\beta_0^{7/2}\,(20\pi^3 + \pi^5 c_1)}{\pi^7} + \frac{2\beta_0^4\,(840\pi^3 + 42\pi^5 c_1 + \pi^7 c_2)}{\pi^8}\right)$$

Simulations vs. analytic

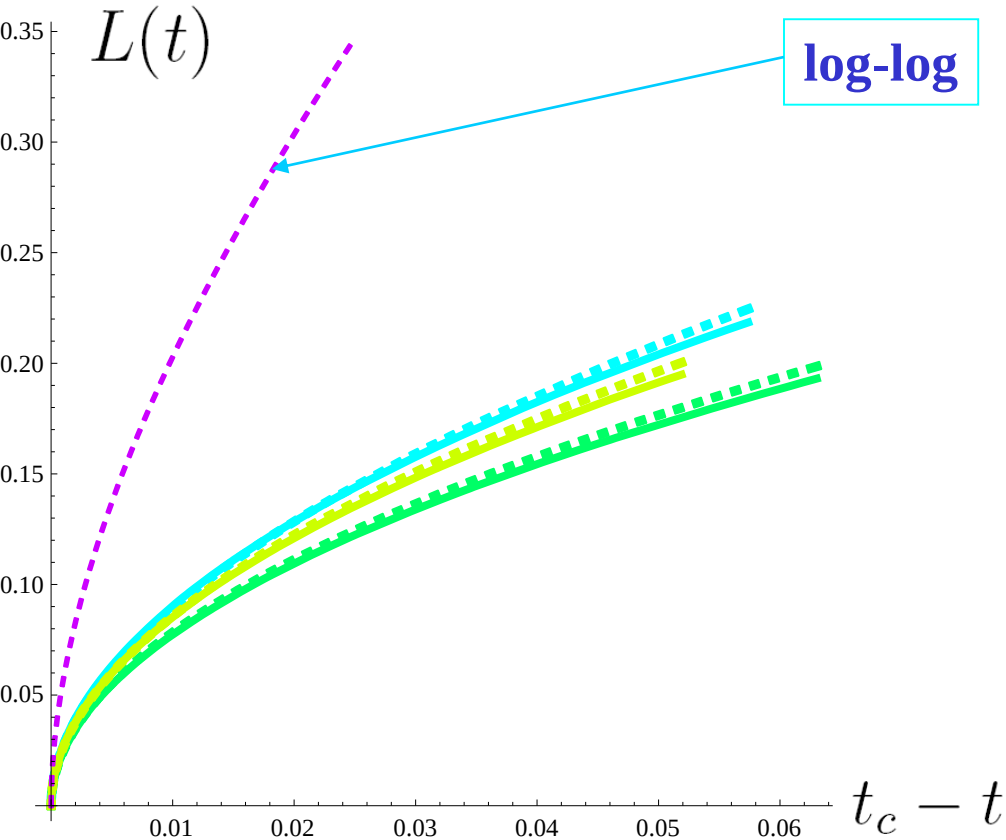
$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A} \right)^{1/2}$$



Simulations vs next order analytic

$$L = \left(\frac{2\pi(t_c - t)}{\ln A - 4\ln 3 + 4\ln\ln A}\right)^{1/2}\left[1 + \frac{2(1 + 4\ln 3 - 4\ln\ln A)}{(\ln A)^2}\right.$$

$$\left. + \frac{14 - 48\ln\ln A + 48(\ln\ln A)^2 + 48\ln 3 - 96(\ln A)(\ln 3) + 48(\ln 3)^2 + \frac{1}{2}\pi^2 c_1}{(\ln A)^3} + O\left(\frac{(\ln\ln A)^3}{(\ln A)^4}\right)\right]$$



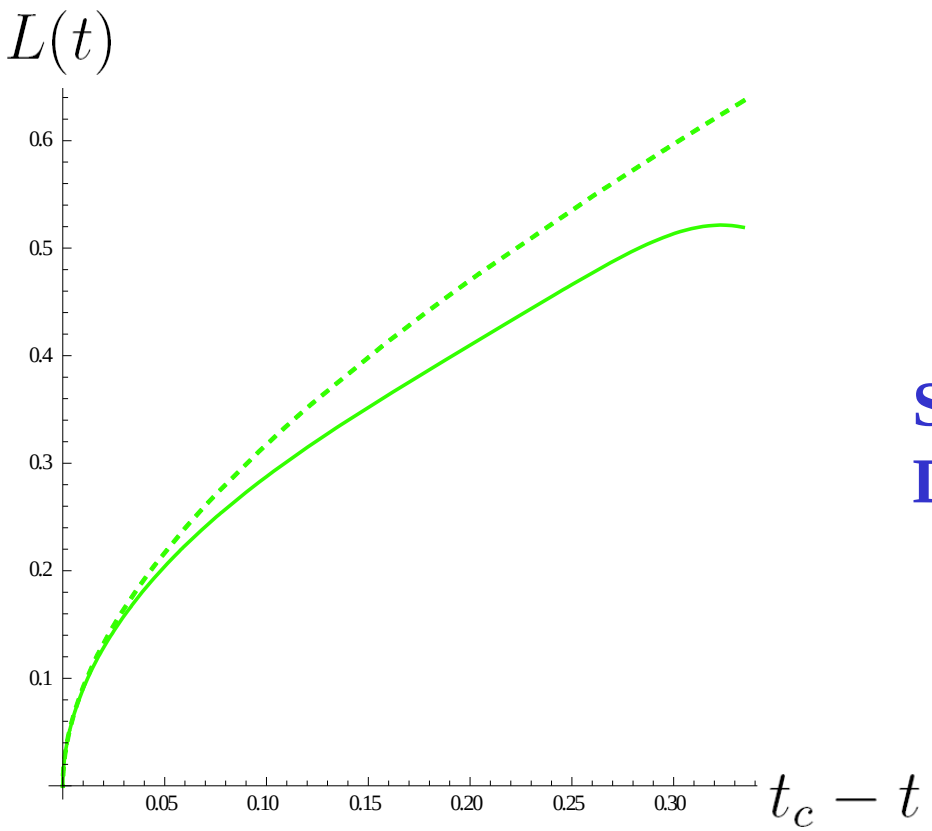
log-log

$$A = -3^4 \frac{\tilde{M}}{2\pi^3} \ln\left[[2\pi(t_c - t)]^{1/2} \frac{e^{-a_0}}{L(z_0)}\right]$$

Solid – numerics

Dashed - analytics

Simulations vs. analytic – larger interval starting from the initial Gaussian



Solid – numerics
Dashed - analytics

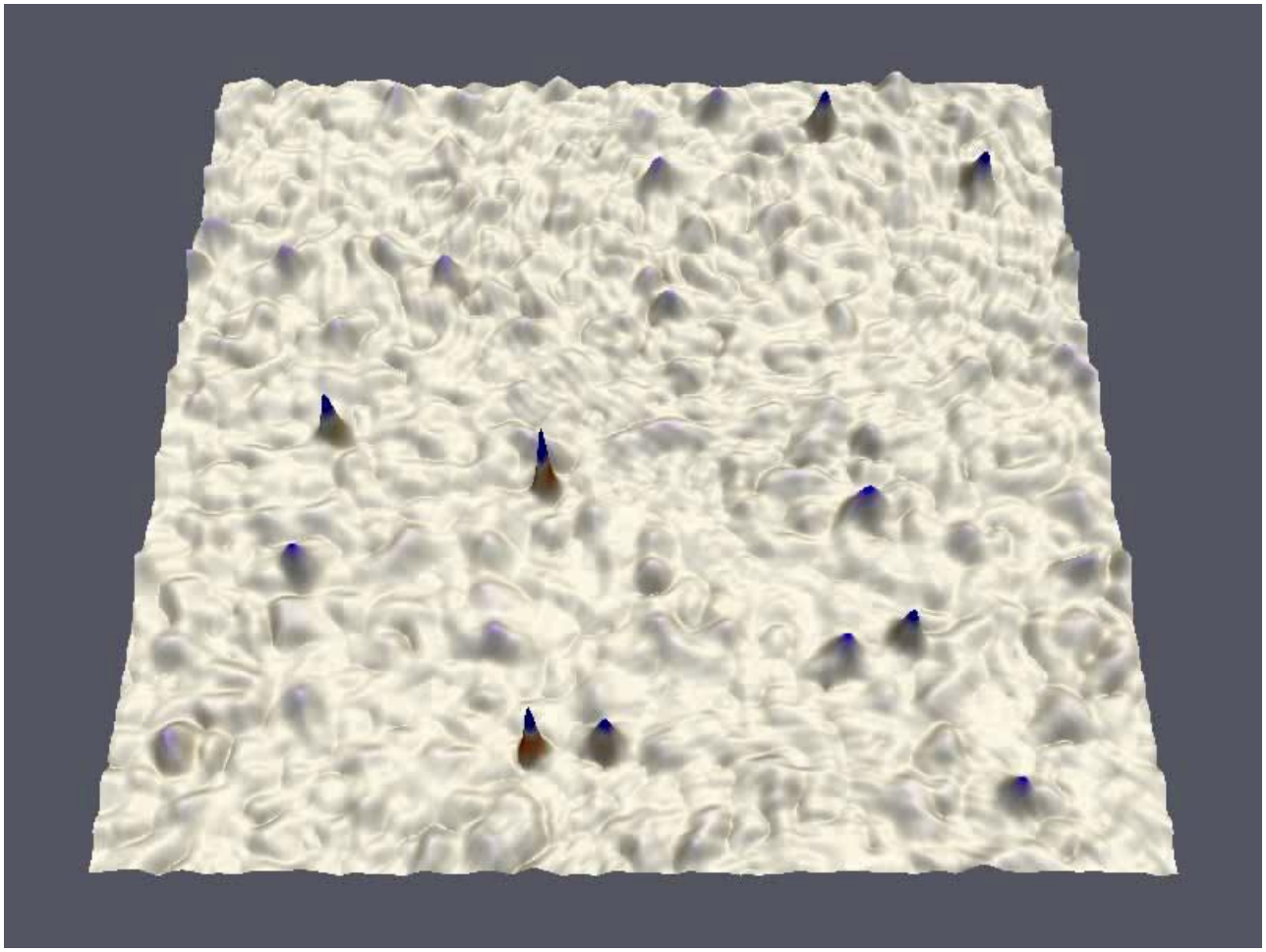
In comparison, the standard log-log scaling dominates only for amplitudes above[1]

$$L = \left(2\pi \frac{t_c - t}{\ln|\ln(t_c - t)|}\right)^{1/2}$$

$$10^{10^{100}} = 10^{\text{Googol}} = \text{Googolplex}$$

[1]P.M. Lushnikov, S.A. Dyachenko and N. Vladimirova. Physical Review A, v. 88, 013845 (2013).

Collapse turbulence (multiple filamentation or Rogue waves) in 2D Nonlinear Schrödinger Equation



$$i\psi_t + \nabla^2\psi + (1 + i\epsilon)|\psi|^2\psi = 0$$ - NLSE with two-photon absorbtion

[1]P.M. Lushnikov and N. Vladimirova, Optics Letters **35**, 1695 (2010).
[2]Y. Chung and P.M. Lushnikov, Phys. Rev E, **84**, 036602 (2011).

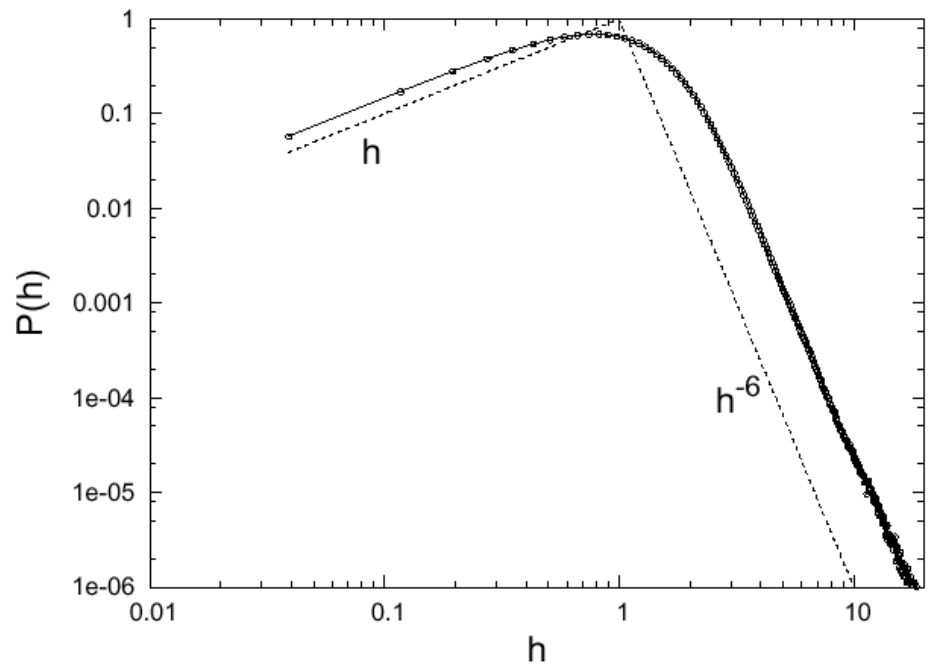
Collapse turbulence of Nonlinear Schrödinger Equation in 2D

$$i\psi_t + \nabla^2\psi + (1+i\epsilon)|\psi|^2\psi = 0$$

Intermittency of turbulence: Tail of PDF for amplitudes is dominated by NLSE collapses

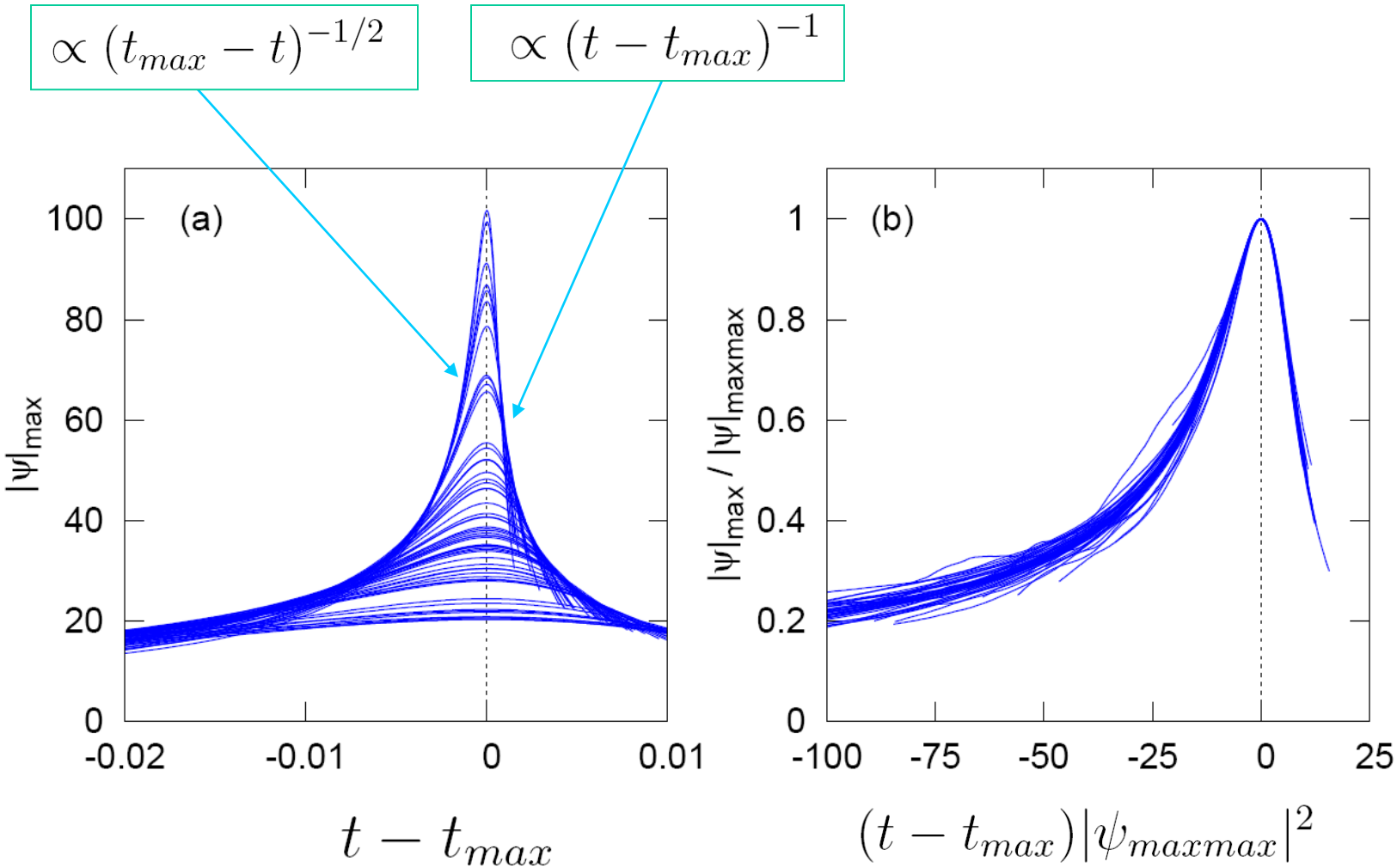
Probability density

$$\mathcal{P}(h) = \frac{\int \delta(|\psi(r,t)|-h)drdt}{\int drdt}$$

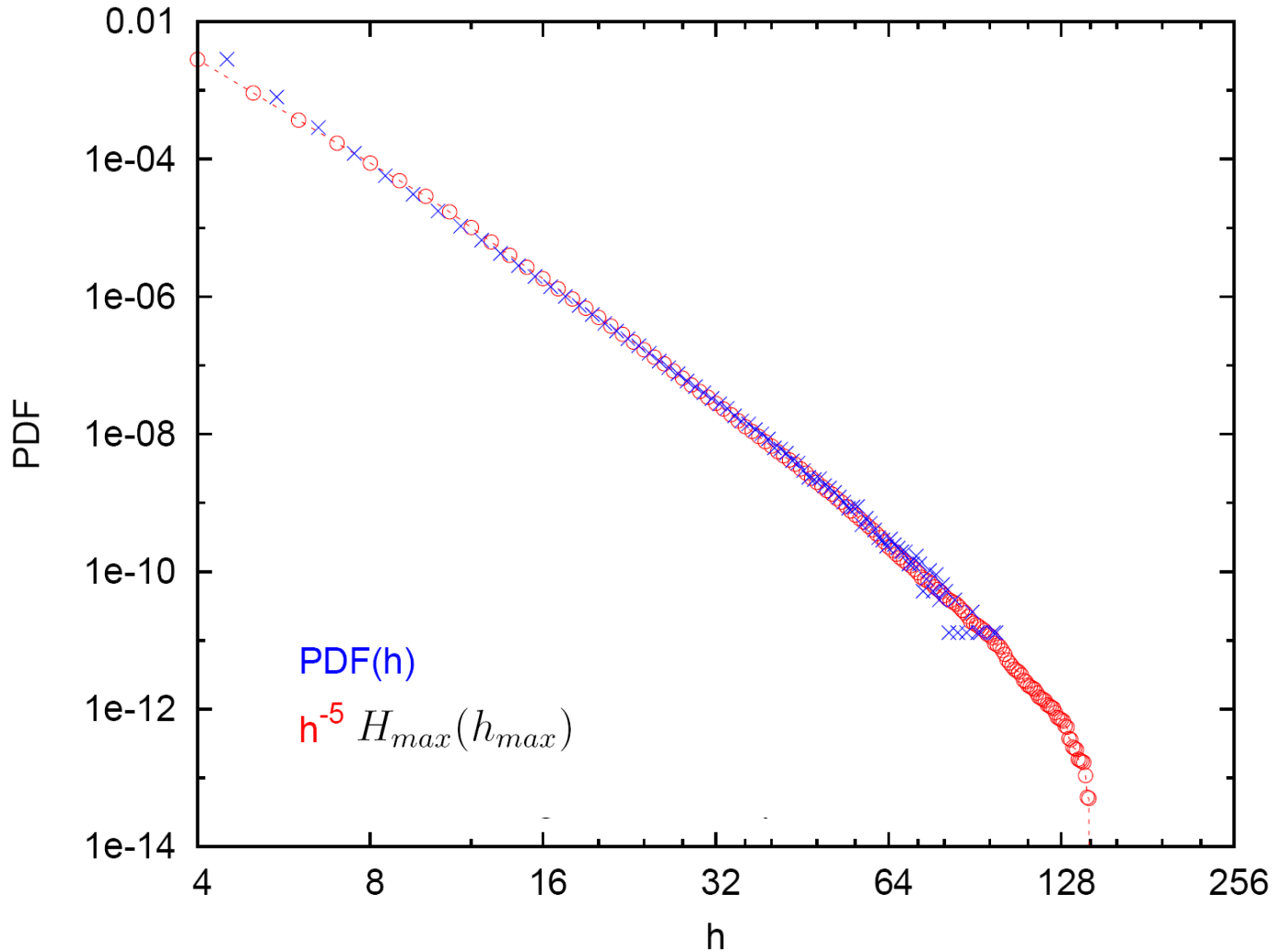


Universality of multiple collapses in rescaled variables vs. nonrescaled variables

$$i\partial_t\psi + (1 - ia\epsilon)\nabla^2\psi + (1 + ic\epsilon)|\psi|^2\psi = i\epsilon b\psi$$

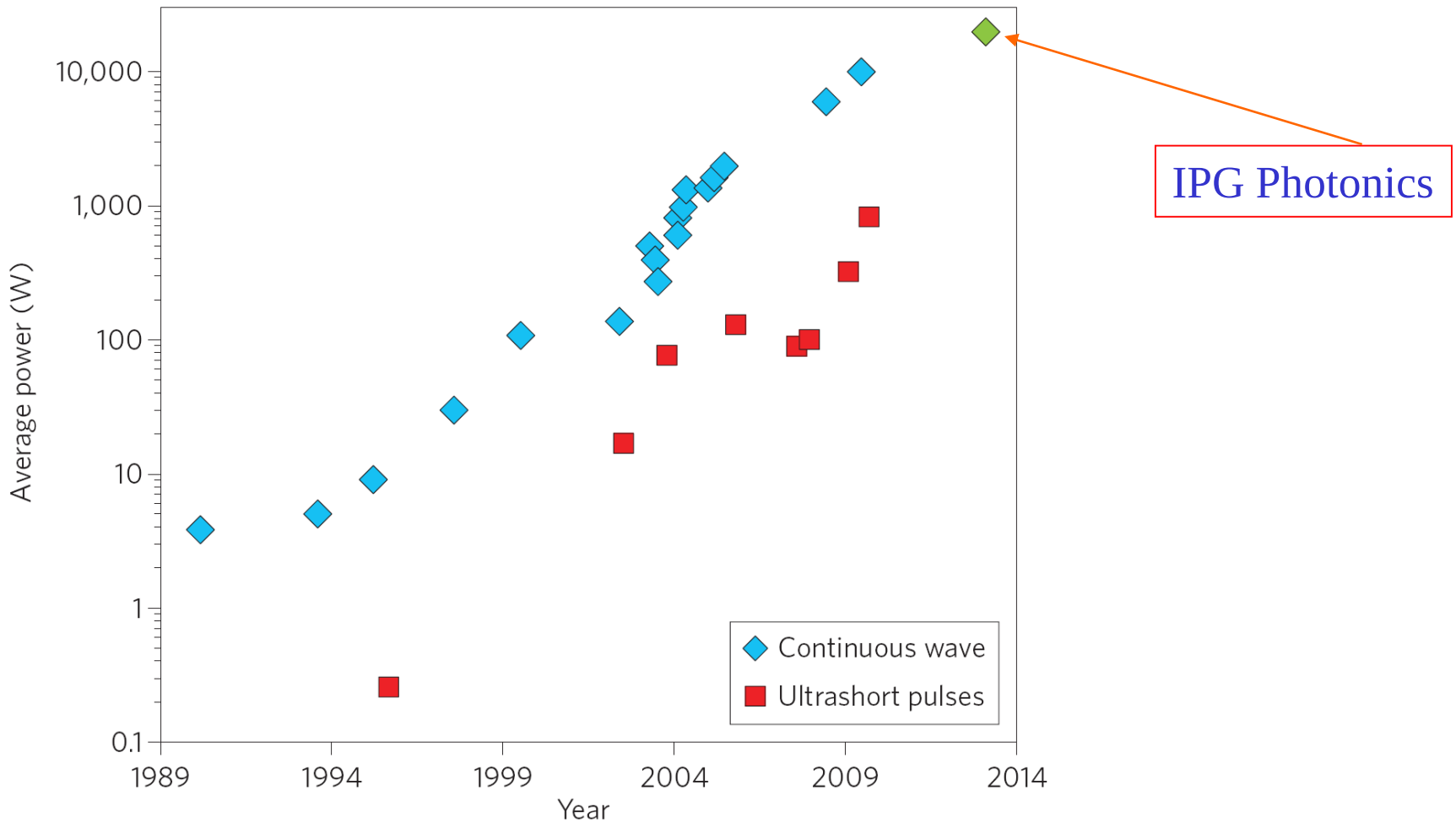


2D NLSE turbulence: Tail of probability density function (PDF) for the fluctuations of $|\psi|$ from simulations (blue crosses) compared with the analytical result (red circles) from collapse contributions [1]



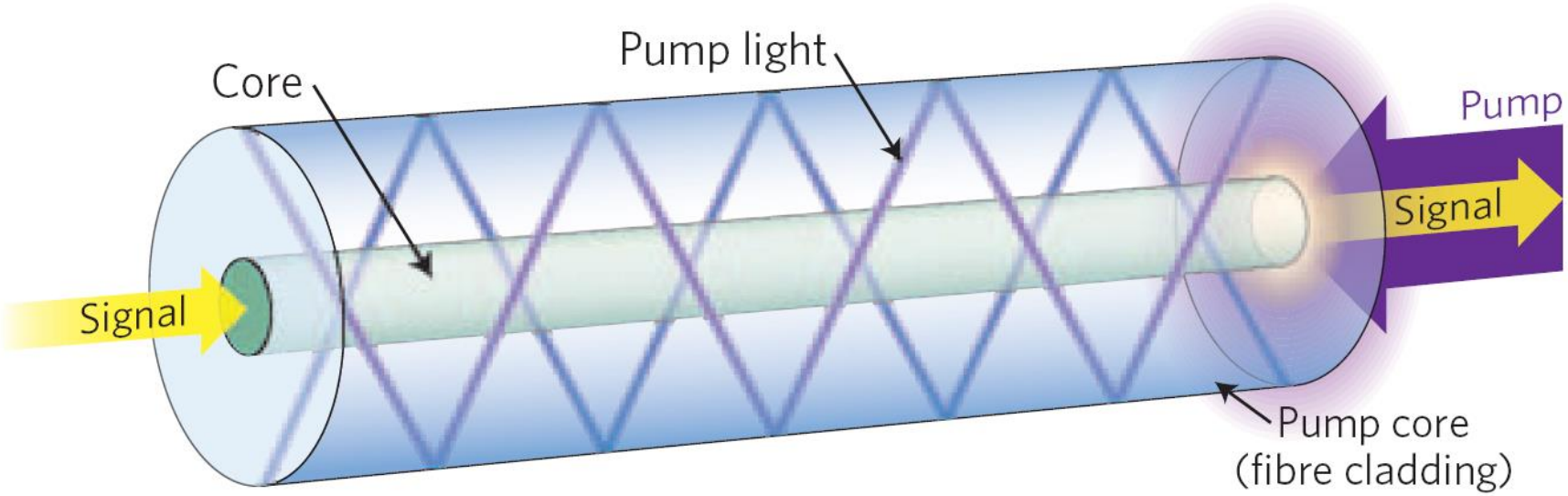
[1]P.M. Lushnikov and N. Vladimirova, Optics Letters, **35**, 1967 (2010);
[2]Y. Chung and P.M. Lushnikov, Phys. Rev E, **84**, 036602 (2011).

Evolution of the average output power of nearly diffraction limited fiber lasers (emitting either a continuous wave or ultrashort pulses) [1]



[1] C. Jauregui, J. Limpert, and A. Tunnermann, Nature Photonics **7**, 861 (2013)

Schematic of a high-power double-clad fiber amplifier



The signal is coupled in the fiber core, which contains the active material, whereas the pump is coupled in the fiber cladding. This structure allows the pump to be progressively absorbed by the active material in the core as the pump propagates along the fiber. This absorbed pumped energy is used to amplify the signal.

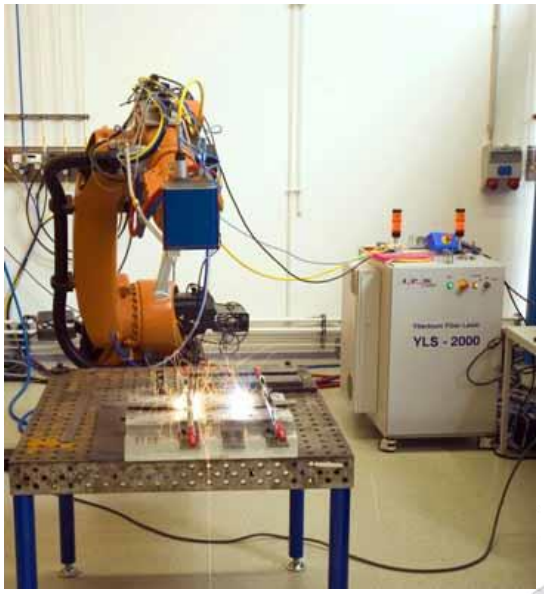
Advantages of fiber lasers

- Alignment-free laser systems
- High efficiency (50-80%)
- Compact design
- Maintenance-free operation

Disadvantage of fiber lasers

- Mode instabilities limiting average power

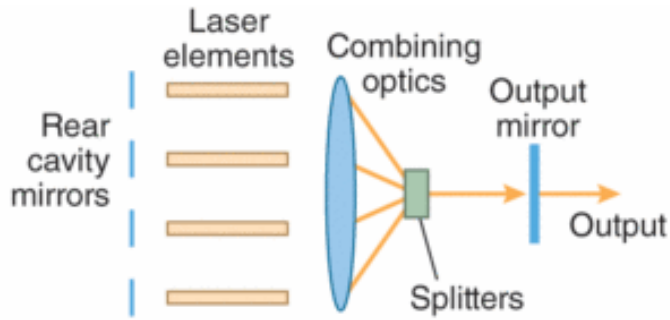
Commercially available IPG Photonics Fiber lasers up to 50kW[1]

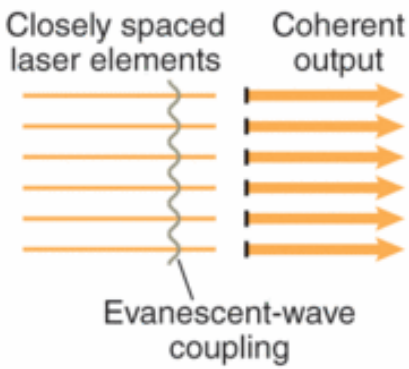


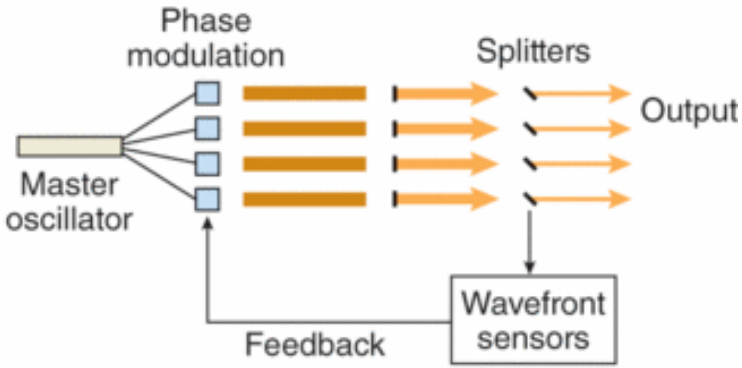
[1]www.ipgphotonics.com

Overcoming power limitations: Laser beam combining

Standard schemes[1]







[1]http://www.laserfocusworld.com

Coherent beam combining:

Combine several laser beams such that the phase of each laser beam is controlled to ideally produce the combined beam with the coherent phase.

Example[1]: five 500W laser beams into 1.9kW Gaussian beam with a good beam quality $M^2 = 1.1$.

Difficulties in coherent beam combining:

-Complicated adaptive optics scheme

-Bad scaling with power due to nonlinearity

---

[1] S. M. Redmond, D. J. Ripin, C. X. Yu, S. J. Augst, T. Y. Fan, P. A. Thielen, J. E. Rothenberg, and G. D. Goodno, Opt. Lett. 37, 2832 (2012).

New proposal[1,2]:

Use nonlinearity to our advantage to achieve combining of multiple laser beams into a diffraction-limited beam by the strong self-focusing in a waveguide with the Kerr nonlinearity.

---

[1] P.M. Lushnikov and N. Vladimirova, Optics Letters **39**, 3429-3432 (2014).
[2] P.M. Lushnikov and N. Vladimirova, Optics Express **23**, 31120-31125 (2015).

Nonlinear laser beam combining

(1) Simpler limit: combining of a few laser beams into a single diffraction-limited beam

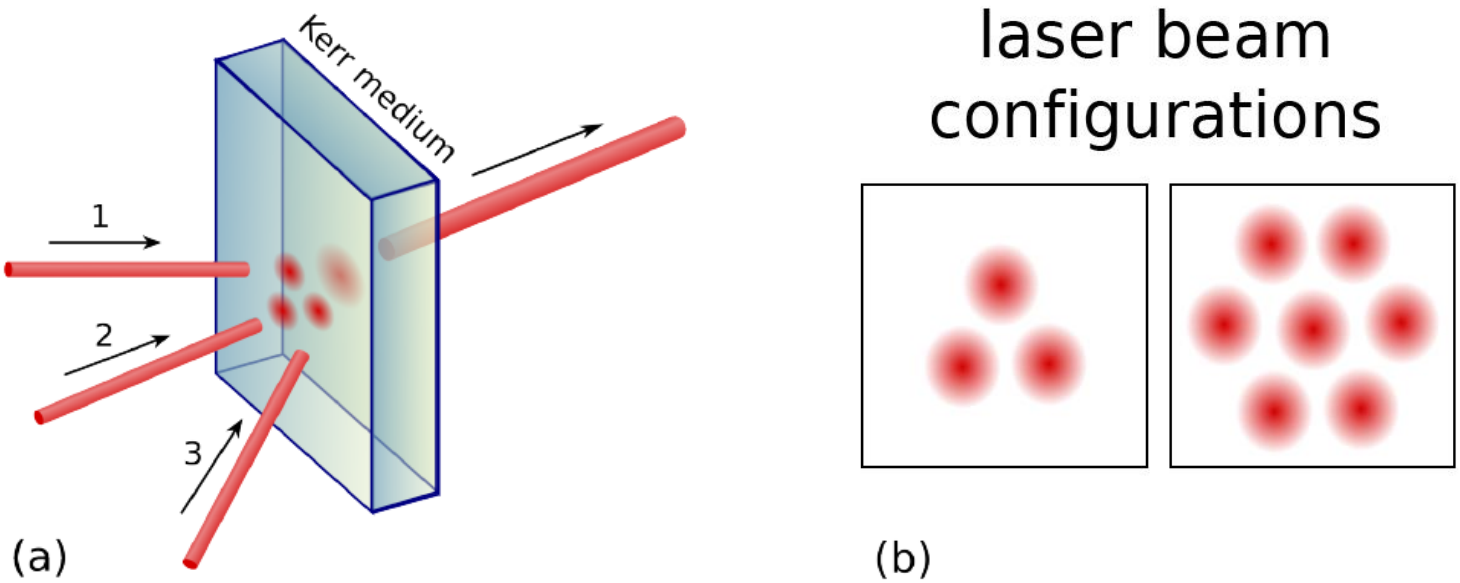
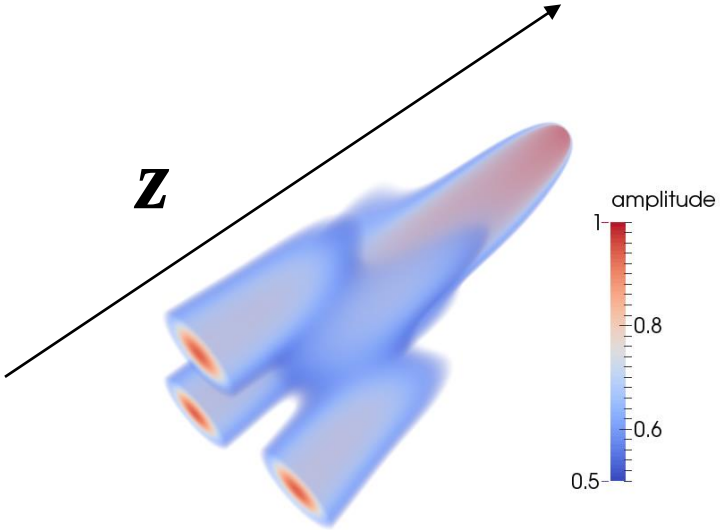
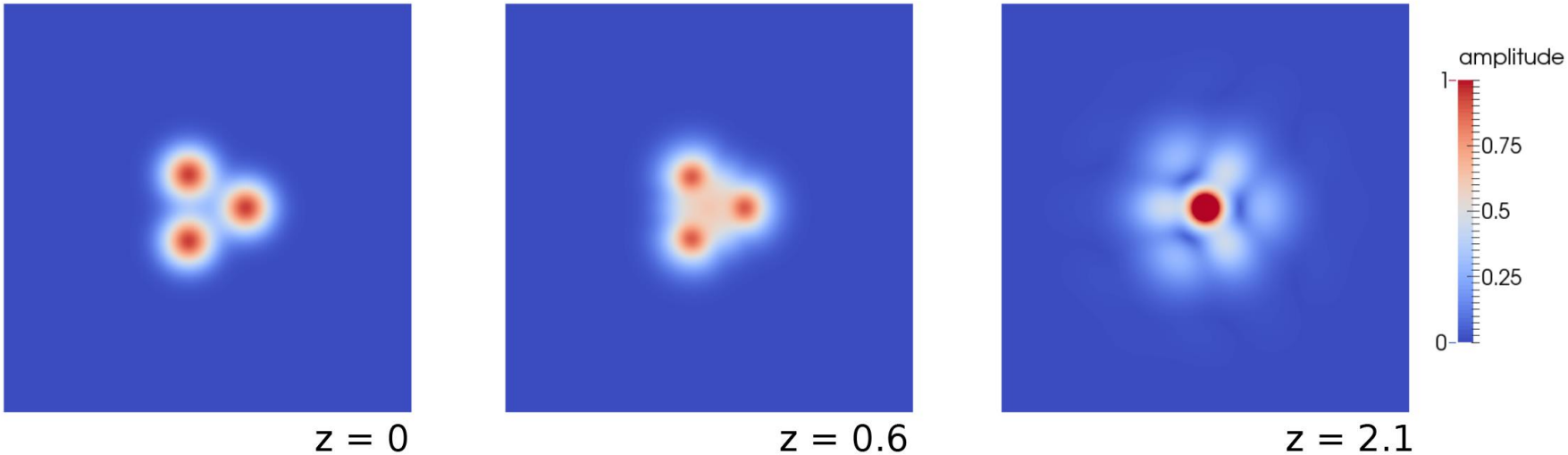


FIG. 1: Schematics of beam combining setup.

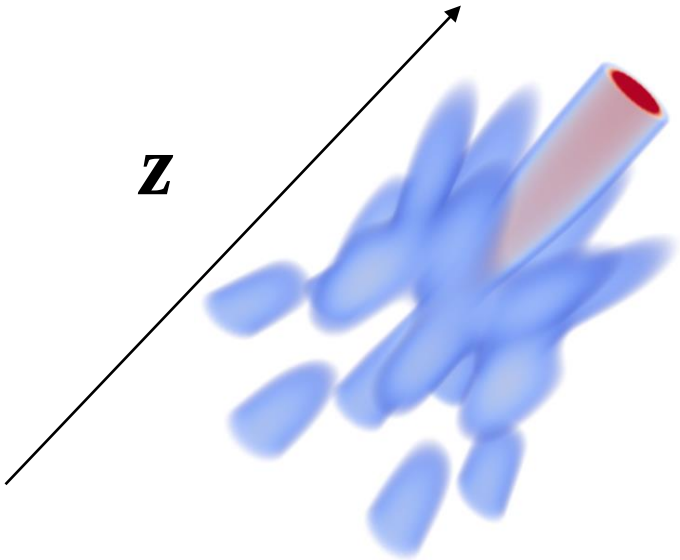
Propagation and combining of 3 beams along *z*



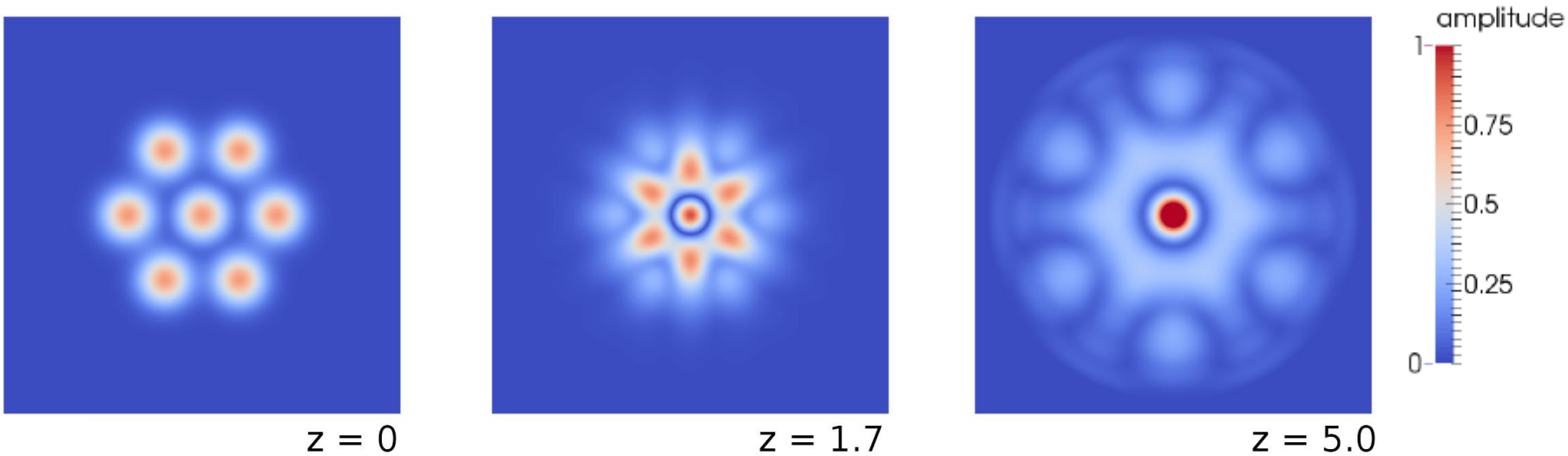
Cross sections at different *z*



Propagation and combining of 7 beams along *z*

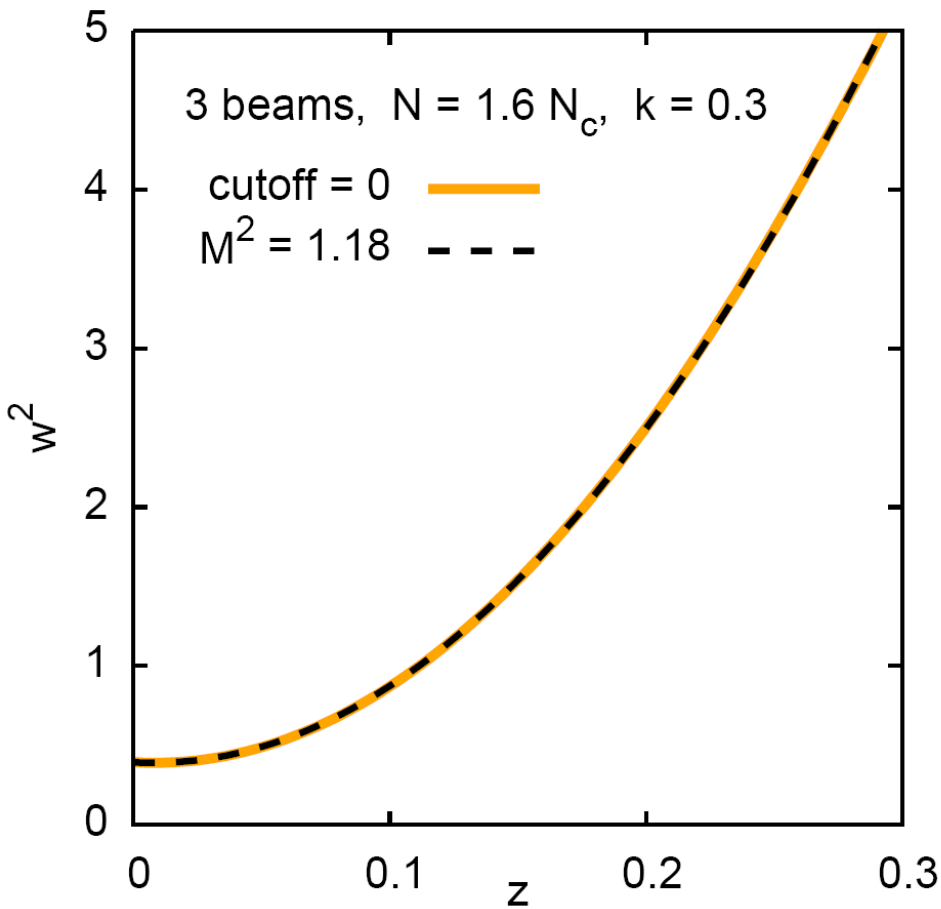


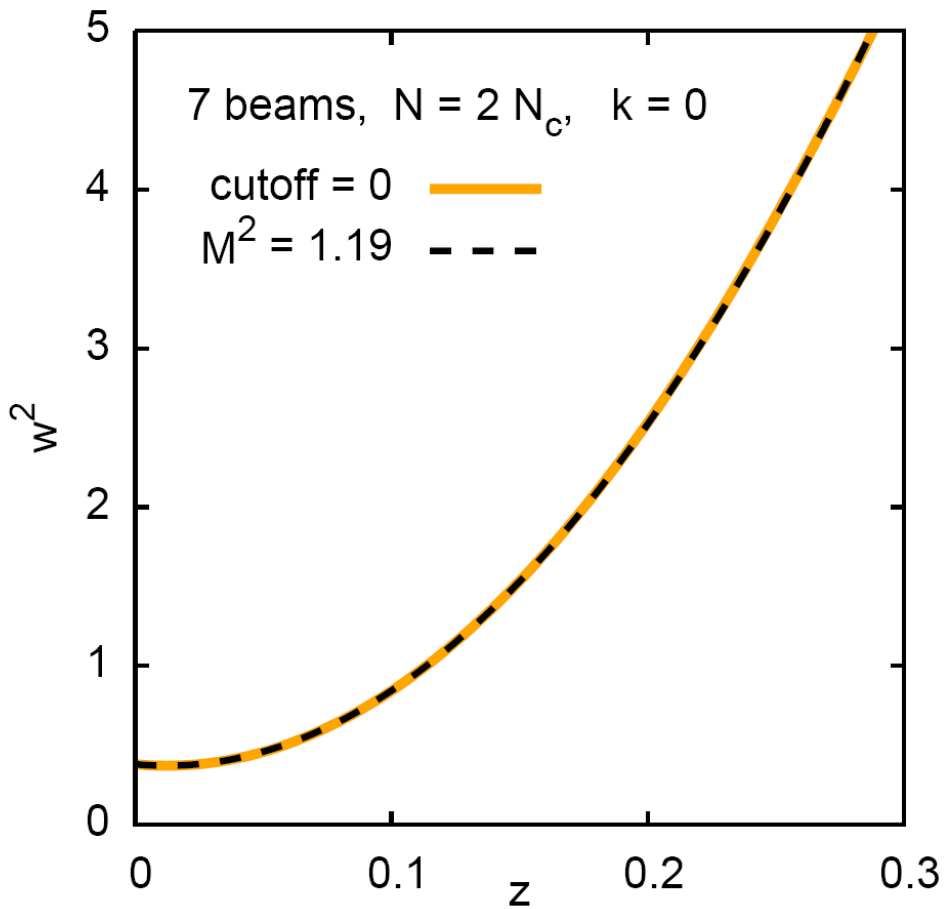
Cross sections at different *z*



Laser beam quality M^2 after exit of the collapsed beam from Kerr media based on least square fit of beam waste w on the propagation distance z [1]

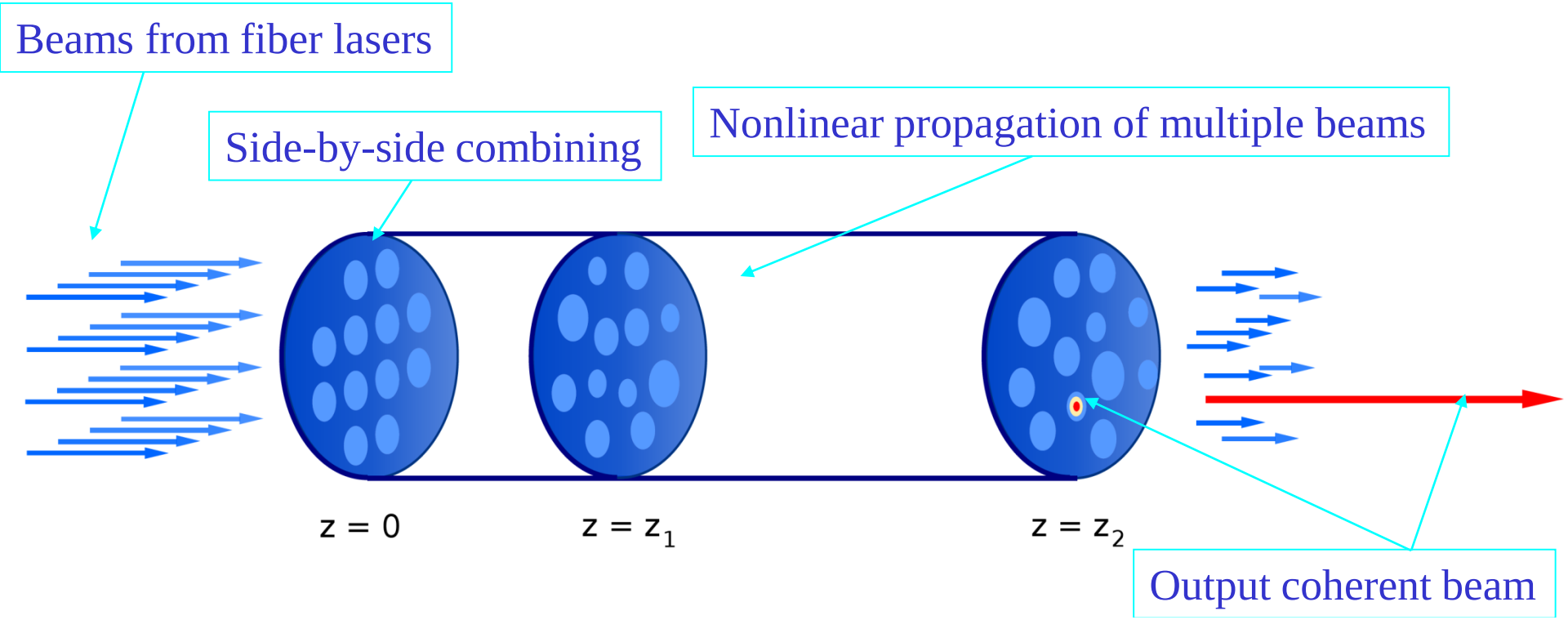
$$w^2(z) = w_0^2 + \left(\frac{2M^2}{k_0 w_0}\right)^2 (z - z_0)^2$$



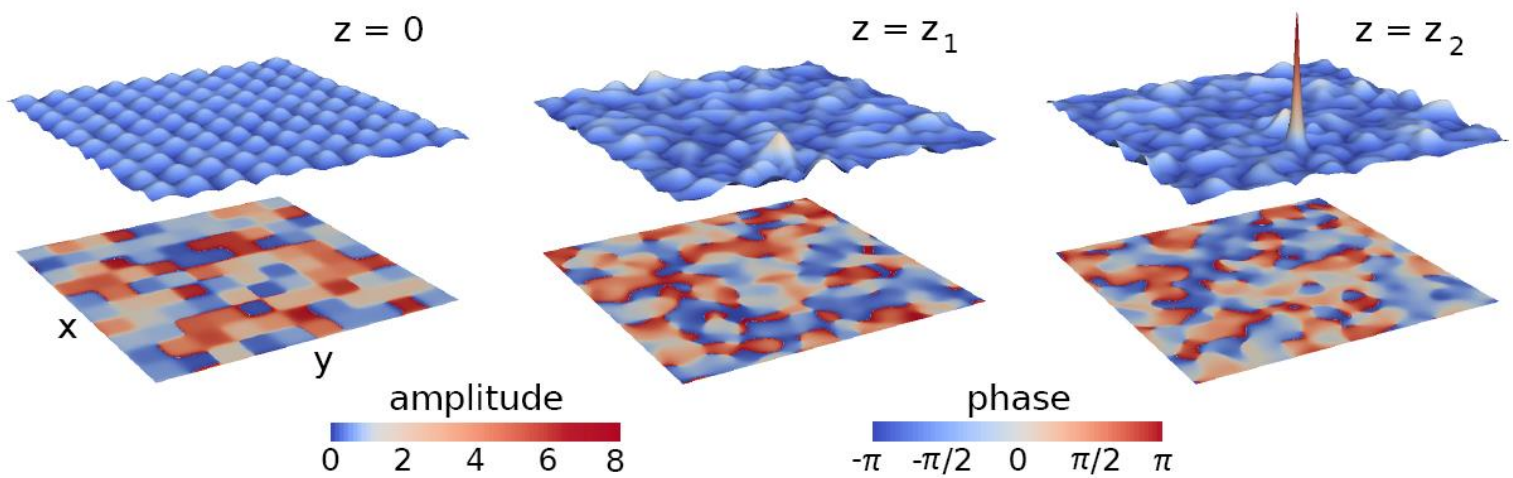


[1] T. Sean Ross, Laser Beam Quality Metrics, SPIE Press, 204 pages (2013)

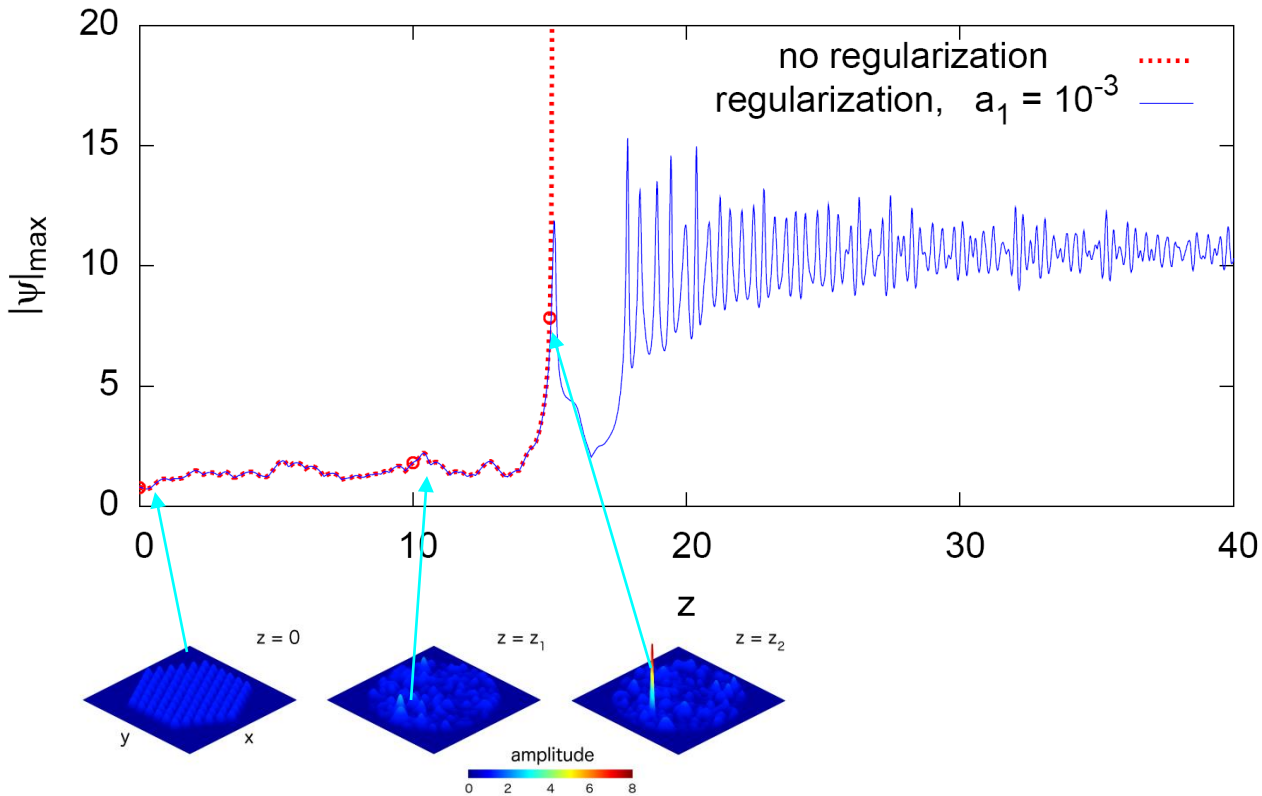
(2) Nonlinear laser beam combining of multiple beams



Laser amplitudes and phases in fiber cross sections:



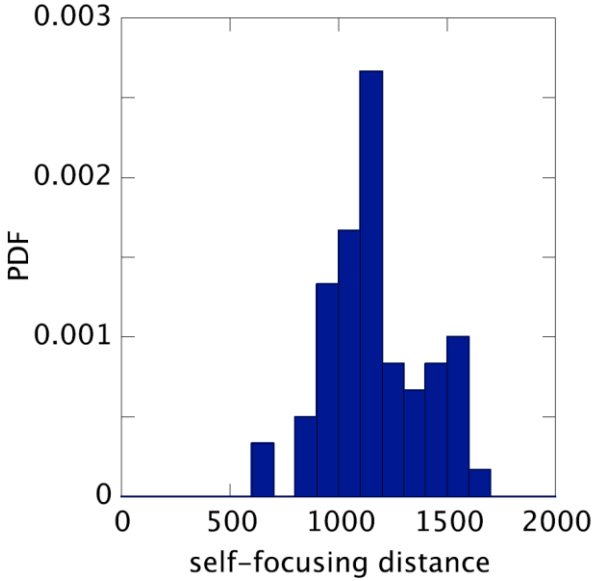
z-dependence of the maximum light amplitude at the cross-section vs. z:



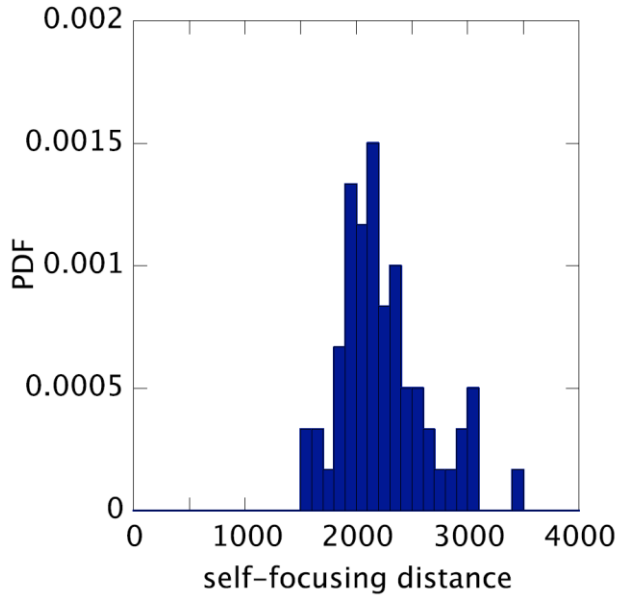
Regularization of collapse

$$i\partial_z\psi + \nabla^2\psi + |\psi|^2\psi - a_1|\psi|^4\psi = 0, \quad 0 < a_1 \ll 1$$

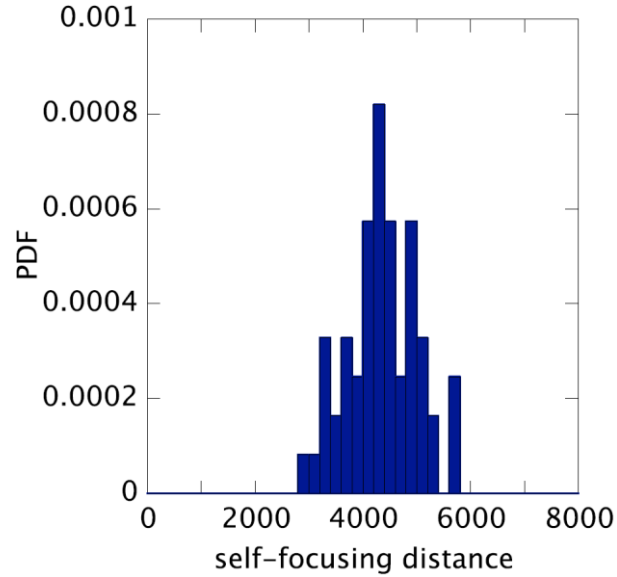
Probability density function (PDF) of the collapse distance



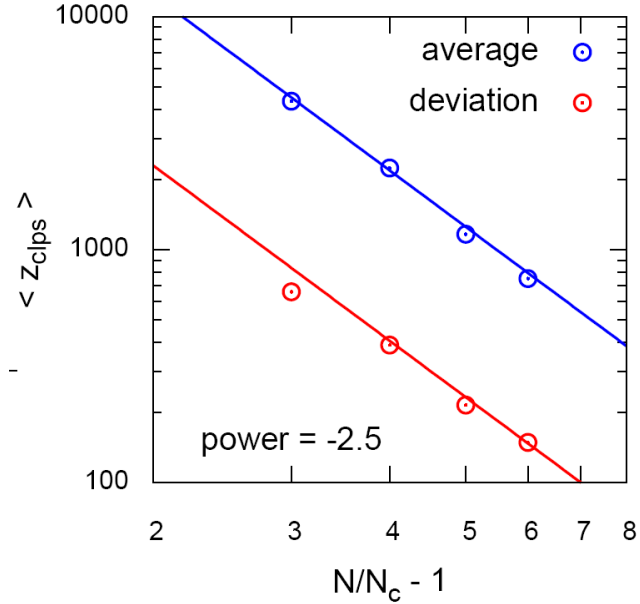
$N=6N_c$



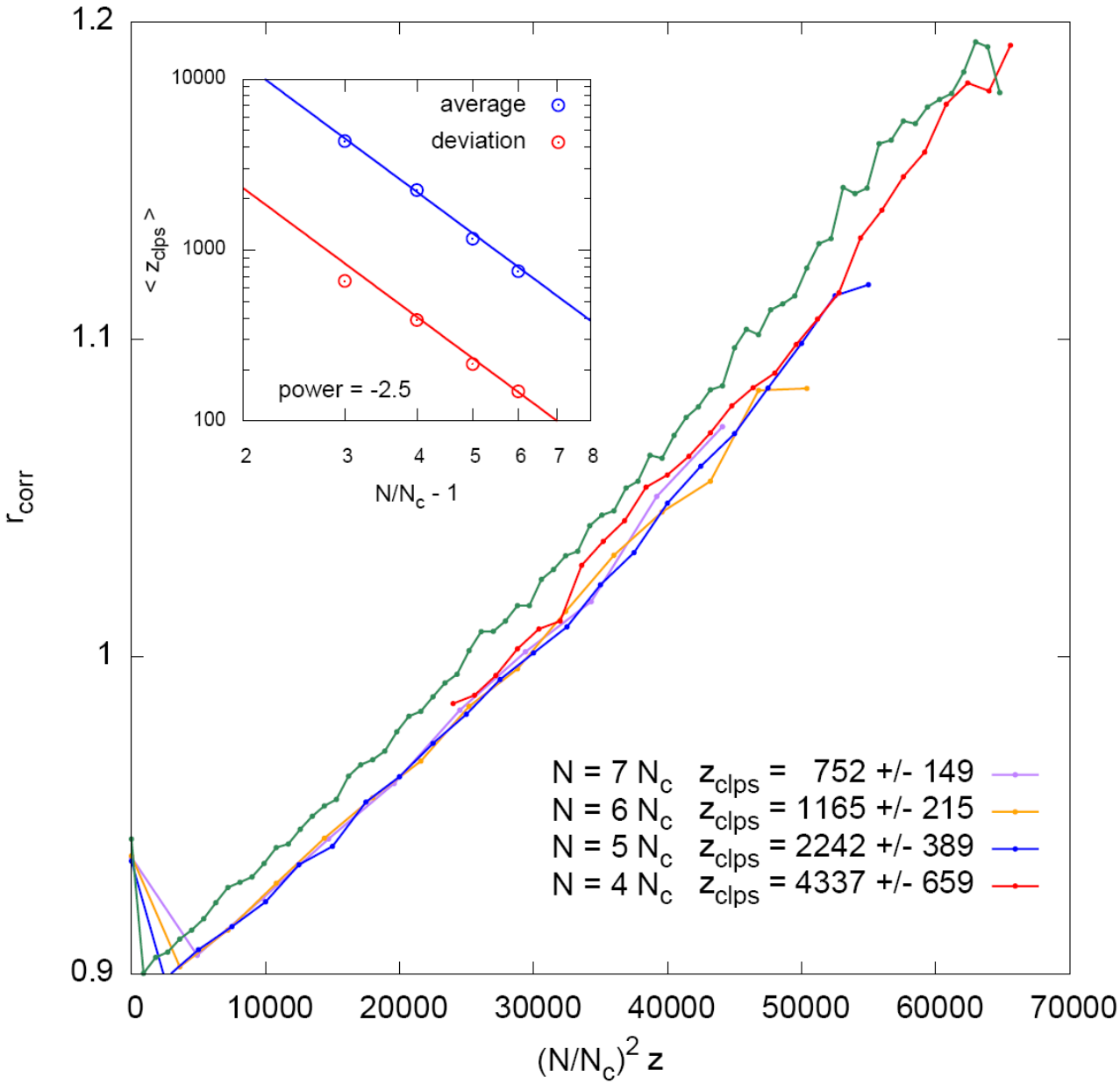
$N=5N_c$



$N=4N_c$



Inverse cascade before collapse



Physical units: $$i\partial_z\psi + \frac{1}{2k}\nabla^2\psi + \frac{kn_2}{n_0}|\psi|^2\psi = 0,$$

$k = 2\pi n_0/\lambda_0$ λ_0 **-wavelength in vacuum**

n_0 **- linear index of refraction**

n_2 **- nonlinear Kerr index with** $n = n_0 + n_2 I$

$I = |\psi|^2$ **- laser intensity**

$$n_0 = 1.4496,\ n_2 = 2.46 \cdot 10^{-16}\mathrm{cm}^2/\mathrm{W} \text{ for } \lambda_0 = 1070\mathrm{nm}$$

Critical power: $$\boxed{P_c = \frac{N_c\lambda_0^2}{8\pi^2 n_2 n_0} \simeq \frac{11.70\,\lambda_0^2}{8\pi^2 n_2 n_0}} \simeq 4.7\mathrm{MW}$$

$N_c \equiv 2\pi \int R^2 r dr = 11.7008965\ldots$

$\psi = e^{iz}R(r)$ $-R + \nabla^2 R + R^3 = 0$ **- ground state soliton**

Optical fiber and laser intensity parameters for the nonlinear beam combining in fused silica:

$I_0 = 10^9 \mathrm{W/cm^2}$ **- Laser intensity for the continuous wave operations**

$\Rightarrow$ **Optical fiber length** $\sim 4\mathrm{m}$

Optical fiber diameter $\sim 2\mathrm{mm}$

Combined beam power $P_c = 4.7\mathrm{MW}$

$\Rightarrow$ **Requires to combine several hundreds of the commercially available fiber lasers. The proposed scheme does not high quality beams for combining.**

Short pulse operations for the nonlinear beam combining in fused silica:

Fused silica optical damage threshold:

$$I_{\text{thresh}} \sim 5 \cdot 10^{11} \text{W/cm}^2 \text{ for } 8\,\text{ns pulses}$$

$$I_{\text{thresh}} \sim 1.5 \cdot 10^{12} \text{W/cm}^2 \text{ for } 14\text{ps pulses}$$

$\Rightarrow$ **Fiber length and cross section can be scaled down by a factor** $\sim 10^{-3}$ **compare with the continuous wave operations**

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[1] P.M. Lushnikov and N. Vladimirova, Optics Letters **39**, 3429-3432 (2014).
[2] P.M. Lushnikov and N. Vladimirova, Optics Express **23**, 31120-31125 (2015).

Conclusion

-Propose to achieve a nonlinear beam combining by propagating multiple laser beams in the waveguide with the Kerr nonlinearity.

- Large fluctuations during propagation seed the collapse event resulting in the formation of near diffraction-limited beam.

- Optical fiber length $\sim 4\mathrm{m}$ with optical fiber diameter $\sim 2\mathrm{mm}$ is sufficient to achieve the combined beam power

$$P_c = 4.7\mathrm{MW}$$