Exact collisional plasma fluid theories

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Fluid equations are $v^{(n)}$-moments of the kinetic equation:

- Kinetic equation combines free-streaming via the Vlasov operator d/dt and dissipation via the collision operator $C_{ss'}[f_s, f_{s'}]$, according to

$$\frac{df_s}{dt} = \sum_{s'} C_{ss'}[f_s, f_{s'}].$$

- Fundamental fluid quantities evolve according to the moment equations

$$\partial_t \rho_s + \partial_{\boldsymbol{x}} \cdot \boldsymbol{K}_s = \sum_{s'} m_s \int d\boldsymbol{v}\, C_{ss'}[f_s, f_{s'}],$$

$$\partial_t \boldsymbol{K}_s + \partial_{\boldsymbol{x}} \cdot \boldsymbol{\Pi}_s = \frac{e_s}{m_s}(\rho_s \boldsymbol{E} + \boldsymbol{K}_s \times \boldsymbol{B}) + \sum_{s'} m_s \int d\boldsymbol{v}\, \boldsymbol{v} C_{ss'}[f_s, f_{s'}]$$

$$\partial_t \mathcal{E}_s + \partial_{\boldsymbol{x}} \cdot \boldsymbol{Q}_s = \frac{e_s}{m_s} \boldsymbol{K}_s \cdot \boldsymbol{E} + \sum_{s'} \frac{m_s}{2} \int d\boldsymbol{v}\, v^2 C_{ss'}[f_s, f_{s'}]$$

- mass density, momentum density, energy density, stress tensor and energy-density flux are defined:

$$\{\rho_s, \boldsymbol{K}_s, \mathcal{E}_s, \boldsymbol{\Pi}_s, \boldsymbol{Q}_s\} \equiv m_s \int d\boldsymbol{v} \{1, \boldsymbol{v}, v^2/2, \boldsymbol{v}\boldsymbol{v}, \boldsymbol{v} v^2/2\} f_s$$

Every fluid theory faces two fundamental issues:

1. Evolution equation for $\boldsymbol{v}^{(n)}$-moment of the distribution function always depends on the $\boldsymbol{v}^{(n+1)}$-moment.
   In order to close the system, the highest moment must be prescribed in terms of lower moments :

$$\boldsymbol{\Pi} = ?(\rho, \boldsymbol{K}, \mathcal{E}), \quad \boldsymbol{Q} = ?(\rho, \boldsymbol{K}, \mathcal{E})$$

2. Moments of the collision operator must be expressed in terms of the fluid variables

$$m_s \int d\boldsymbol{v}\, \boldsymbol{v}^{(n)} C_{ss'}[f_s, f_{s'}] = ??(\rho, \boldsymbol{K}, \mathcal{E})$$

   Note: only in single-species systems with elastic collisions, the null-space of the collision operator allows this requirement to be relaxed

Chapman-Enskog (1916,1917) leads to Braginskii equations (1957)

Basic idea

- Expressions for stress tensor $\mathbf{\Pi}$ and energy flux $\boldsymbol{Q}$ obtained via multi-scale perturbation theory (space and time), $f = f_0 + f_1 + \ldots$.
- Assumption: dynamics is dominated by collisions and gyro-motion

$$C[f,f], \quad \frac{q}{m}\boldsymbol{v} \times \boldsymbol{B} \cdot \partial_{\boldsymbol{v}} f \quad \gg \quad \partial_t f, \quad \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} f, \quad \frac{q}{m}\boldsymbol{E} \cdot \partial_{\boldsymbol{v}} f$$

- Kinetic equation transformed into a sequence of linear integral equations

$$f_0: \qquad \frac{q_s}{m_s}(\boldsymbol{v} \times \boldsymbol{B}) \cdot \partial_{\boldsymbol{v}} f_{s0} = \sum_{s'} C[f_{s0}, f_{s'0}]$$

$$f_1: \quad D_s f_{s0} + \frac{q_s}{m_s}(\boldsymbol{v} \times \boldsymbol{B}) \cdot \partial_{\boldsymbol{v}} f_{s1} = \sum_{s'} (C[f_{s0}, f_{s'1}] + C[f_{s1}, f_{s'0}])$$

where $D_s = \partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} + \frac{q_s}{m_s}\boldsymbol{E} \cdot \partial_{\boldsymbol{v}}$

Consequences

- f_0 is a Maxwellian and f_1 depends on temperature and density gradients
- Stress and energy flux from f_1 introduce **viscosity** and **heat conductivity** into the momentum and energy equations
- Rarely carried beyond f_1 in plasmas (Burnett equations in neutral fluids)

Grad's approach (1949) is a Galerkin projection

Basic idea

- Fluid quantities are polynomial moments of the distribution function
- Expand f in terms of orthogonal polynomials and study the weak solution of the kinetic equation instead of multi-scale ordering
- Non-normal moments are treated on an equal footing to the fundamental fluid variables (extended set of dynamical variables)

Consequences

- Truncation of expansion provides a closed set of extended fluid equations
- Equations for mass, momentum, and energy **are** included
- Concepts of heat conductivity and viscosity **do not appear explicitly**

H. Grad, *"On the Kinetic Theory of Rarefied Gases"* (1949)

The indicated independence of initial values of the stresses and heat flow is only valid as an approximation for sufficiently slowly varying flows. This condition is violated, for example, in any but the weakest shock waves. Moreover, it is only in certain special cases of these quasi-equilibrium flows that the stresses and heat flow can be given explicitly in terms of gradients. The methods of Enskog and Chapman are aimed at obtaining these so-called "normal" solutions which depend only on the thermodynamic variables and their gradients. The validity of such results is limited to certain slowly varying flows.

Our objective is to apply Grad's method to plasmas

Compute Hermite polynomial moments of the Landau collision operator

- Apply Grad's expansion to multi-species plasma using Hermite polynomials
- Compute exact Hermite-moments of the nonlinear Landau collision operator
- Provide collisional fluid terms in a compact analytical form, ready for numerical implementation

Foreseen applications

- Role of nonlinear resistivity in magnetic reconnection
- Role of anisotropic resistivity in dynamo effect
- Edge and scrape-off layer physics as alternative to Braginskii

The Landau collision operator is viewed as a convolution

- Collision operator is the divergence of a velocity-space flux

$$C_{ss'}[f_s, f_{s'}] \equiv -\frac{c_{ss'}}{m_s}\partial_{\boldsymbol{v}} \cdot \boldsymbol{J}_{ss'}[f_s, f_{s'}], \quad c_{ss'} = \frac{e_s^2 e_{s'}^2 \ln \Lambda}{4\pi\varepsilon_0^2}$$

- The collisional velocity-space flux $\boldsymbol{J}_{ss'}$ is defined in terms of the so-called Rosenbluth-MacDonald-Judd-Trubnikov potential functions

$$\boldsymbol{J}_{ss'}[f_s, f_{s'}](\boldsymbol{v}) \equiv \mu_{ss'}(\partial_{\boldsymbol{v}}\phi_{s'})f_s - m_s^{-1}\partial_{\boldsymbol{v}} \cdot [(\partial_{\boldsymbol{v}}\partial_{\boldsymbol{v}}\psi_{s'})f_s], \quad \mu_{ss'} = \frac{m_s + m_{s'}}{m_s m_{s'}}$$

- The RMJT potentials are convolutions of the distribution function with 3D Laplacian Green's functions

$$\phi_s(\boldsymbol{v}) \equiv \int_{\mathbb{R}^3} d\boldsymbol{v}'\, f_s(\boldsymbol{v}')\, |\boldsymbol{v} - \boldsymbol{v}'|^{-1},$$
$$\psi_s(\boldsymbol{v}) \equiv \frac{1}{2}\int_{\mathbb{R}^3} d\boldsymbol{v}'\, f_s(\boldsymbol{v}')\, |\boldsymbol{v} - \boldsymbol{v}'|.$$

- Indeed $\partial_{\boldsymbol{v}} \cdot \partial_{\boldsymbol{v}}\psi_s = \phi_s$ and $\partial_{\boldsymbol{v}} \cdot \partial_{\boldsymbol{v}}\phi_s = -4\pi f_s$.
  These properties are useful to prove the energy and momentum conservation of the collisional moments.

Distribution functions are expanded in terms of Hermite-polynomials

- f_s is a Maxwellian with flow velocity $\boldsymbol{V}_s$ and variance $\sigma_s^2 = T_s/m_s$ multiplied by an orthogonal polynomial. The Gaussian envelope carries mass, momentum and energy (fundamental moments)

$$\frac{f_s(\boldsymbol{v})}{n_s} = \underbrace{\mathcal{N}_{\sigma_s^2}(\boldsymbol{v} - \boldsymbol{V}_s)}_{\text{Maxwellian}} \underbrace{\sum_{i=0}^{\infty} \frac{1}{i!} \boldsymbol{c}_{s(i)} \bar{\boldsymbol{G}}_{(i)}(\boldsymbol{v} - \boldsymbol{V}_s; \sigma_s^2)}_{\text{multi-index polynomial, } 1+\boldsymbol{v}+\boldsymbol{v}^2+\ldots} \tag{1}$$

- Expansion coefficients are the so-called *Hermite-moments* of the distribution function and their evolution carries the information for stress anisotropy and energy-density flux

$$\boldsymbol{c}_{s(j)} \equiv \int_{\mathbb{R}^3} d\boldsymbol{v} \, \frac{f_s(\boldsymbol{v})}{n_s} \bar{\boldsymbol{H}}_{(j)}(\boldsymbol{v} - \boldsymbol{V}_s; \sigma_s^2) \tag{2}$$

- The coefficients up to second rank are

$$c_{s(0)} = 1, \quad \boldsymbol{c}_{s(1)} = \boldsymbol{0}, \quad \boldsymbol{c}_{s(2)} = (\boldsymbol{P}_s - p_s \boldsymbol{I})/n_s m_s \tag{3}$$

Hermite polynomials and Maxwellians come in pairs

- The Gaussian distribution function

$$\mathcal{N}_{\sigma^2}(\boldsymbol{x}-\boldsymbol{\mu}) = \frac{1}{(2\pi\sigma^2)^{3/2}} \exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{\mu})^2}{2\sigma^2}\right] \tag{4}$$

- generates *covariant* Hermite polynomials via gradients wrt mean flow

$$\partial_{\boldsymbol{\mu}}^{(k)} \mathcal{N}_{\sigma^2}(\boldsymbol{x}-\boldsymbol{\mu}) = \mathcal{N}_{\sigma^2}(\boldsymbol{x}-\boldsymbol{\mu}) \bar{\boldsymbol{G}}_{(k)}(\boldsymbol{x}-\boldsymbol{\mu};\sigma^2), \tag{5}$$

- and *contravariant* Hermite polynomials

$$\bar{\boldsymbol{H}}_{(k)}(\boldsymbol{x}-\boldsymbol{\mu};\sigma^2) = \sigma^{2k} \bar{\boldsymbol{G}}_{(k)}(\boldsymbol{x}-\boldsymbol{\mu};\sigma^2), \tag{6}$$

- These polynomials are orthogonal under Gaussian measure

$$\int_{\mathbb{R}^3} d\boldsymbol{y} \bar{\boldsymbol{H}}_{(i)}(\boldsymbol{y};\sigma^2) \bar{\boldsymbol{G}}_{(j)}(\boldsymbol{y};\sigma^2) \mathcal{N}_{\sigma^2}(\boldsymbol{y}) = \partial_{\boldsymbol{x}}^{(j)} \boldsymbol{x}^{(i)} \Big|_{\boldsymbol{x}=\boldsymbol{0}} = \delta_{[(j)]}^{(i)}, \tag{7}$$

About notation: $\partial^{(k)} \equiv \underbrace{\partial \otimes \cdots \otimes \partial}_{k \text{ terms}}$, $\boldsymbol{x}^{(k)} \equiv \underbrace{\boldsymbol{x} \otimes \cdots \otimes \boldsymbol{x}}_{k \text{ terms}}$, $\bar{\boldsymbol{G}}_{(k)}(\boldsymbol{x}) = \bar{\boldsymbol{G}}_{(k)}^{k_1 \dots k_k}(\boldsymbol{x})$

Hermite-moments of the Landau operator are convenient

- Collisional moments are split into drag- and diffusion-related contributions

$$\begin{aligned}\boldsymbol{C}_{ss'(k+1)} &\equiv m_s \int_{\mathbb{R}^3} d\boldsymbol{v} \bar{\boldsymbol{H}}_{(k+1)}(\boldsymbol{v}-\boldsymbol{V}_s;\sigma_s^2) C_{ss'}[f_s, f_{s'}](\boldsymbol{v}) \\ &\equiv n_s n_{s'} c_{ss'}(k+1)\mathsf{Sym}\left[\mu_{ss'}\boldsymbol{R}_{ss'(k+1)} + \frac{k}{m_s}\boldsymbol{D}_{ss'(k+1)}\right],\end{aligned}$$

- The drag and diffusion terms are defined

$$\begin{aligned}\boldsymbol{R}_{ss'(k+1)} &= \int_{\mathbb{R}^3} d\boldsymbol{v} \left(\partial_{\boldsymbol{v}} \frac{\phi_{s'}}{n_{s'}}\right) \frac{f_s}{n_s} \bar{\boldsymbol{H}}_{(k)}(\boldsymbol{v}-\boldsymbol{V}_s;\sigma_s^2) \\ \boldsymbol{D}_{ss'(k+1)} &= \int_{\mathbb{R}^3} d\boldsymbol{v} \left(\partial_{\boldsymbol{v}}\partial_{\boldsymbol{v}} \frac{\psi_{s'}}{n_{s'}}\right) \frac{f_s}{n_s} \bar{\boldsymbol{H}}_{(k-1)}(\boldsymbol{v}-\boldsymbol{V}_s;\sigma_s^2)\end{aligned}$$

Steps to evaluate $R_{ss'(k+1)}$ and $D_{ss'(k+1)}$ analytically

- The products of Hermite polynomials are linearized

$$\frac{f_s}{n_s}\bar{\boldsymbol{H}}_{(k)}(\boldsymbol{v}-\boldsymbol{V}_s;\sigma_s^2)=\sum_{i=0}^{\infty}\sum_{l=0}^{i+k}\frac{\boldsymbol{c}_{s(i)}}{i!}\sigma_s^{k+l-i}\bar{\boldsymbol{a}}_{(i)(k)}^{(l)}\partial_{\boldsymbol{V}_s}^{(l)}\mathcal{N}_{\sigma_s^2}(\boldsymbol{v}-\boldsymbol{V}_s)$$

where the coefficient $\bar{\boldsymbol{a}}_{(i)(j)}^{(l)}=\frac{1}{l!}\partial_{\boldsymbol{x}}^{(i)}\partial_{\boldsymbol{y}}^{(j)}\partial_{\boldsymbol{z}}^{(l)}\left[e^{\boldsymbol{x}\cdot\boldsymbol{y}+\boldsymbol{y}\cdot\boldsymbol{z}+\boldsymbol{x}\cdot\boldsymbol{z}}\right]_{\boldsymbol{x}=\boldsymbol{0},\boldsymbol{y}=\boldsymbol{0},\boldsymbol{z}=\boldsymbol{0}}$

- Gradients of the potential functions are replaced by mean gradients

$$\partial_{\boldsymbol{v}}\frac{\phi_{s'}(\boldsymbol{v})}{n_{s'}}=-\partial_{\boldsymbol{V}_{s'}}\sum_{j=0}^{\infty}\frac{c_{s'(j)}}{j!}\partial_{\boldsymbol{V}_{s'}}^{(j)}\int_{\mathbb{R}^3}d\boldsymbol{v}'\frac{\mathcal{N}_{\sigma_{s'}^2}(\boldsymbol{v}'-\boldsymbol{V}_{s'})}{|\boldsymbol{v}-\boldsymbol{v}'|},$$

$$\partial_{\boldsymbol{v}}\partial_{\boldsymbol{v}}\frac{\psi_{s'}(\boldsymbol{v})}{n_{s'}}=\frac{1}{2}\partial_{\boldsymbol{V}_{s'}}\partial_{\boldsymbol{V}_{s'}}\sum_{j=0}^{\infty}\frac{\boldsymbol{c}_{s'(j)}}{j!}\partial_{\boldsymbol{V}_{s'}}^{(j)}\int_{\mathbb{R}^3}d\boldsymbol{v}'\mathcal{N}_{\sigma_{s'}^2}(\boldsymbol{v}'-\boldsymbol{V}_{s'})|\boldsymbol{v}-\boldsymbol{v}'|.$$

- Remaining double integral is a convolution of two Gaussians followed by a convolution with Green's functions

$$\iint_{\mathbb{R}^3}d\boldsymbol{v}d\boldsymbol{v}'\frac{\mathcal{N}_{\sigma_{s'}^2}(\boldsymbol{v}'-\boldsymbol{V}_{s'})\mathcal{N}_{\sigma_s^2}(\boldsymbol{v}-\boldsymbol{V}_s)}{|\boldsymbol{v}-\boldsymbol{v}'|}=\int_{\mathbb{R}^3}\frac{d\boldsymbol{x}}{|\boldsymbol{x}|}\mathcal{N}_{\sigma_s^2+\sigma_{s'}^2}[\boldsymbol{x}-(\boldsymbol{V}_s-\boldsymbol{V}_{s'})]$$

Exact expressions for $R_{ss'(k+1)}$ and $D_{ss'(k+1)}$ are found

- Introduce the variables

$$\Sigma_{ss'} = \sqrt{\sigma_s^2 + \sigma_{s'}^2}, \qquad \boldsymbol{\Delta}_{ss'} = (\boldsymbol{V}_s - \boldsymbol{V}_{s'})/\sqrt{2}\Sigma_{ss'}$$

- The drag- and diffusion-related coefficients are given by

$$\boldsymbol{R}_{ss'(k+1)} = \partial_{\boldsymbol{\Delta}_{ss'}} \sum_{i,j=0}^{\infty} \sum_{l=0}^{i+k} \frac{(-1)^j \sigma_s^{k+l-i}}{(\sqrt{2}\Sigma_{ss'})^{l+j+2}} \frac{\boldsymbol{c}_{s(i)}}{i!} \frac{\boldsymbol{c}_{s'(j)}}{j!} \bar{\boldsymbol{a}}^{(l)}_{(i)(k)} \partial^{(l)}_{\boldsymbol{\Delta}_{ss'}} \partial^{(j)}_{\boldsymbol{\Delta}_{ss'}} \Phi(|\boldsymbol{\Delta}_{ss'}|), \tag{8}$$

$$\boldsymbol{D}_{ss'(k+1)} = \partial_{\boldsymbol{\Delta}_{ss'}} \partial_{\boldsymbol{\Delta}_{ss'}} \sum_{i,j=0}^{\infty} \sum_{l=0}^{i+k-1} \frac{(-1)^j \sigma_s^{k-1+l-i}}{(\sqrt{2}\Sigma_{ss'})^{l+j+1}} \frac{\boldsymbol{c}_{s(i)}}{i!} \frac{\boldsymbol{c}_{s'(j)}}{j!} \bar{\boldsymbol{a}}^{(l)}_{(i)(k-1)} \partial^{(l)}_{\boldsymbol{\Delta}_{ss'}} \partial^{(j)}_{\boldsymbol{\Delta}_{ss'}} \Psi(|\boldsymbol{\Delta}_{ss'}|). \tag{9}$$

- The functions $\Phi(z)$ and $\Psi(z)$ are RMJT potentials for unitary Gaussians

$$\Phi(z) = \frac{\mathrm{erf}(z)}{z}, \quad \Psi(z) = \left(z + \frac{1}{2z}\right)\mathrm{erf}(z) + \frac{e^{-z^2}}{\sqrt{\pi}}.$$

The ten-moment equations is an illustrative toy model

Series truncated after second non-vanishing expansion coefficients $c_{s(2)}$

- Distribution functions are written as

$$\frac{f_s(\boldsymbol{v})}{n_s} = \left(\frac{m_s}{2\pi T_s}\right)^{3/2} e^{-\frac{m_s}{2T_s}(\boldsymbol{v}-\boldsymbol{V}_s)^2}\left[1 + \frac{m_s}{2T_s}(\boldsymbol{v}-\boldsymbol{V}_s)^{(2)} : \left(\frac{\boldsymbol{P}_s}{p_s} - \boldsymbol{I}\right)\right].$$

- Fluid quantities are $\rho_s = m_s n_s$, $\boldsymbol{K}_s = m_s n_s \boldsymbol{V}_s$, $\boldsymbol{\Pi}_s = \boldsymbol{P}_s + m_s n_s \boldsymbol{V}_s \boldsymbol{V}_s$.
- A closed set of conservative equations is obtained, 10 per species
- The closing relation is

$$Q^{ijk} = m_s \int d\boldsymbol{v} v^i v^j v^k f_s = 3\ \mathsf{Sym}(m_s n_s V_s^i V_s^j V_s^k + P_s^{ij} V_s^k)$$

Note: viscosity is implicitly embedded, but not thermal conductivity (requires at least 13 moments)

Ten moments hyperbolic equations are given explicitly

- Continuity equation

$$\frac{\partial \rho_s}{\partial t} + \partial_{\boldsymbol{x}} \cdot \boldsymbol{K}_s = 0$$

- Momentum equation

$$\frac{\partial \boldsymbol{K}_s}{\partial t} + \partial_{\boldsymbol{x}} \cdot \boldsymbol{\Pi}_s - \frac{e_s}{m_s}(\rho_s \boldsymbol{E} + \boldsymbol{K}_s \times \boldsymbol{B}) = \sum_{s'} c_{ss'} \mu_{ss'} \boldsymbol{R}_{ss'(1)}$$

- Evolution equation for the stress tensor

$$\frac{\partial \Pi_s^{ij}}{\partial t} + \frac{\partial}{\partial x^k} \left(\frac{\Pi_s^{ij} K_s^k + \Pi_s^{jk} K_s^i + \Pi_s^{ki} K_s^j}{\rho_s} - 2 \frac{K_s^i K_s^j K_s^k}{\rho_s^2} \right)$$
$$- \frac{e_s}{m_s}(E^i K_s^j + B^m \varepsilon_{i\ell m} \Pi_s^{j\ell} + \text{transpose})$$
$$= \sum_{s'} c_{ss'} [m_s^{-1} D^{ij}_{ss'(2)} + \mu_{ss'}(R^{ij}_{ss'(2)} + V_s^i R^j_{ss'(1)}) + \text{transpose}].$$

The collisional moments are analytic and satisfy conservation laws

$$
\begin{aligned}
\boldsymbol{R}_{ss'(1)} &= \frac{1}{2\Sigma_{ss'}^2}\partial_{\boldsymbol{\Delta}_{ss'}}\mathcal{O}_{ss'}[\Phi](\Delta_{ss'}),\\
\boldsymbol{D}_{ss'(2)} &= \frac{1}{\sqrt{2}\Sigma_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\mathcal{O}_{ss'}[\Psi](\Delta_{ss'}),\\
\boldsymbol{R}_{ss'(2)} &= \frac{1}{\sqrt{2}\Sigma_{ss'}}\left[\frac{\sigma_s^2}{2\Sigma_{ss'}^2}\partial_{\boldsymbol{\Delta}_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\mathcal{O}_{ss'}[\Phi](\Delta_{ss'})\right.\\
&\quad\left.+\left(\frac{\boldsymbol{c}_{s(2)}}{2\Sigma_{ss'}^2}\cdot\partial_{\boldsymbol{\Delta}_{ss'}}\right)\partial_{\boldsymbol{\Delta}_{ss'}}\left(1+\frac{\boldsymbol{c}_{s'(2)}}{4\Sigma_{ss'}^2}:\partial_{\boldsymbol{\Delta}_{ss'}}^2\right)\Phi(\Delta_{ss'})\right].
\end{aligned}
$$

- The scalar differential operator $\mathcal{O}_{ss'}$ is given by

$$
\begin{aligned}
\mathcal{O}_{ss'} \equiv 1 &+ \left(\frac{\boldsymbol{c}_{s(2)}+\boldsymbol{c}_{s'(2)}}{4\Sigma_{ss'}^2}\right):\partial_{\boldsymbol{\Delta}_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\\
&+\left(\frac{\boldsymbol{c}_{s(2)}}{4\Sigma_{ss'}^2}:\partial_{\boldsymbol{\Delta}_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\right)\left(\frac{\boldsymbol{c}_{s'(2)}}{4\Sigma_{ss'}^2}:\partial_{\boldsymbol{\Delta}_{ss'}}\partial_{\boldsymbol{\Delta}_{ss'}}\right).
\end{aligned}
$$

- These collisional terms satisfy the momentum and energy conservation laws exactly (follows immediately from symmetry of $\mathcal{O}_{ss'}$ and properties of special functions Φ and Ψ)

(Now) a small parameter expansion can be safely performed

For example, the **extended Ohm's law** for quasi-neutral ion-electron plasma is

$$\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=\frac{\boldsymbol{F}_{ei}}{ne}+\frac{\boldsymbol{J}\times\boldsymbol{B}-\nabla\cdot\mathbb{P}_e}{ne}+\frac{m_e}{ne^2}\left[\frac{\partial\boldsymbol{J}}{\partial t}+\nabla\cdot\left(\boldsymbol{V}\boldsymbol{J}+\boldsymbol{J}\boldsymbol{V}-\frac{\boldsymbol{J}\boldsymbol{J}}{ne}\right)\right],$$

- Collisional momentum exchange rate provides electrical resistivity

$$\frac{\boldsymbol{F}_{ei}}{ne}=\eta_0\,\mathbb{R}\cdot\boldsymbol{J}$$

- Nonlinear resistivity tensor at first order in $\Delta^2=\frac{m_e}{ne^2}\frac{J^2}{2p_e}\ll 1$

$$\mathbb{R}\approx\mathbb{I}-\frac{3}{5}\left(\frac{\mathbb{P}_e}{p_e}-\mathbb{I}\right)+\frac{3\Delta^2}{7}\left(\frac{\mathbb{P}_e}{p_e}-\mathbb{I}\right)-\frac{3\Delta^2}{5}\,\mathbb{I}\left[1-\frac{5}{7}\left(\frac{\mathbb{P}_e}{p_e}-\mathbb{I}\right):\frac{\boldsymbol{J}\boldsymbol{J}}{J^2}\right]$$

- Spitzer value correctly appears at lowest order

$$\eta_0=\frac{4}{3}\frac{\sqrt{2\pi m_e}e^2\ln\Lambda}{(4\pi\varepsilon_0)^2T_e^{3/2}}$$

Conclusions

- Exact analytic expression for moments of the Landau collision operator (no linearisation nor reduced form)
- Compact and programmable $\Rightarrow$ systematic higher-rank moment equations
- Transport equations remain strictly advective: simpler to address numerically than advective-diffusive kind from CE, especially in the context of shocks.

FYI, the detailed mathematical derivation can be found on arXiv[1]:
Exact collisional plasma fluid theories, https://arxiv.org/abs/1701.08037

Thank you for your attention

[1] Comments and feedback will be very much appreciated