Stochastic modeling of scrape-off layer fluctuations

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Bursts in single point measurements correspond to traversing blobs.
1. Stochastic model of data time series

2. Comparison to experimental measurements

3. Conclusions
Superpose uncorrelated pulses to model data time series

Superposition of $K$ pulses in a time interval $[0 : T]$

$$
\Phi_K(t) = \sum_{k=1}^{K(T)} A_k \phi \left( \frac{t - t_k}{\tau_d} \right)
$$

where $k$ labels a pulse and

- $A_k$ denotes the pulse amplitude
- $t_k$ denotes pulse arrival time
- $\phi$ denotes a pulse shape
- $\tau_d$ denotes pulse duration time

Intermittency parameter: $\gamma = \tau_d / \tau_w$
Pulses arrive uncorrelated and form a Poisson process

Choose distribution for all random variables
- \( P_K(K|T) \) gives the number of bursts in time interval \([0; T]\)
- \( P_A(A_k) \rightarrow \) distribution of pulse Amplitudes.
- \( P_t(t_k) \rightarrow \) distribution of pulse arrival times.

Consider a Poisson process:

1. Pulses arrive uncorrelated: \( P_t(t_k) = 1/T \)
2. Avg. rate of pulse arrival is \( 1/\tau_w \)

\[
P_K(K|T) = \exp \left( \frac{-T}{\tau_w} \right) \left( \frac{T}{\tau_w} \right)^K \frac{1}{K!}
\]

Exponentially distributed pulse amplitudes: \( \langle A \rangle P_A(A_k) = \exp \left( A_k / \langle A \rangle \right) \)

We often normalize the process as

\[
\tilde{\Phi} = \frac{\Phi - \langle \Phi \rangle}{\Phi_{\text{rms}}}
\]
Intermittency parameter governs pulse overlap

\[ \Phi(t) = \begin{cases} \gamma & t \geq 0 \\ 0 & t < 0 \end{cases} \]

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Model experimental data with double-exponential pulses

Experimental data is approximated by a double-exponential pulse shape

\[
\phi(\theta) = \Theta(-\theta) \exp\left(\frac{\theta}{\lambda}\right) + \Theta(\theta) \exp\left(-\frac{\theta}{1-\lambda}\right)
\]

In physical units: \( \theta = (t - t_k)/\tau_d, \tau_d \approx 10\mu s. \)

\( \lambda \) defines pulse asymmetry:

\[
\tau_r = \lambda \tau_d \\
\tau_f = (1 - \lambda) \tau_d
\]

Notation: \( I_n = \int_{-\infty}^{\infty} d\theta [\phi(\theta)]^n \)

Normalization: \( I_1 = 1 \)
Stochastic model of data time series

Correlation and power spectral density depend on pulse asymmetry

Correlation function of the pulse shape is given by

\[
\rho_\phi(\theta) = \frac{1}{l^2} \int_{-\infty}^{\infty} d\chi \phi(\chi) \phi(\chi + \theta)
\]

\[
= \frac{1}{1 - 2\lambda} \left[ (1 - \lambda) \exp \left( -\frac{|\theta|}{1 - \lambda} \right) - \lambda \exp \left( -\frac{|\theta|}{\lambda} \right) \right]
\]

Wiener-Khinchin theorem states that the power spectral density is the Fourier-transform of the autocorrelation function

\[
\sigma_\phi(\omega) = \int_{-\infty}^{\infty} d\theta \rho_\phi(\theta) \exp(-i\omega \theta)
\]

\[
= \frac{2}{[1 + (1 - \lambda)^2 \omega^2][1 + \lambda^2 \omega^2]}
\]


R. Kube et al. (UiT)
The mean of the process can be computed analytically

Averaging the process over all random variables and neglect finite box effects by extending time integration to the entire real axis:

\[
\langle \Phi_K \rangle = \int_{-\infty}^{\infty} dA_1 P_A(A_1) \int_{-\infty}^{\infty} \frac{dt_1}{T} \cdots \int_{-\infty}^{\infty} dA_K P_A(A_K) \int_{-\infty}^{\infty} \frac{dt_K}{T} \sum_{k=1}^{K} A_k \phi \left( \frac{t - t_k}{\tau_d} \right)
\]

\[= \frac{K}{T} \tau_d \langle A \rangle\]

Average over number of pulses \( K \):

\[\langle \Phi \rangle = \frac{\tau_d}{\tau_w} \langle A \rangle\]

Mean value of the process increases with pulse overlap and average pulse amplitude.
The variance can be computed analytically

\[ \langle \Phi_K^2 \rangle = \int_{-\infty}^{\infty} dA_1 P_A(A_1) \int_{-\infty}^{\infty} \frac{dt_1}{T} \cdots \int_{-\infty}^{\infty} dA_K P_A(A_K) \int_{-\infty}^{\infty} \frac{dt_K}{T} \]

\[ \sum_{k=1}^{K} A_k \phi \left( \frac{t - t_k}{\tau_d} \right) \sum_{l=1}^{K} A_l \phi \left( \frac{t - t_l}{\tau_d} \right) \]

This results in \( K(K - 1) \) terms with \( k \neq l \), \( K \) terms with \( k = l \).

\[ \langle \Phi_K^2 \rangle = \tau_d l_2 \langle A^2 \rangle \frac{K}{T} + \tau_d^2 l_1^2 \langle A \rangle^2 \frac{K(K - 1)}{T^2} \]

\[ \Rightarrow \langle \Phi^2 \rangle = \frac{\tau_d}{\tau_w} l_2 \langle A^2 \rangle + \langle \Phi \rangle^2 \]

where \( \langle K(K - 1) \rangle = \langle K \rangle^2 \) has been used.
Auto-correlation function is computed from $\langle \Phi(t)\Phi(t+k) \rangle$

$$R_\Phi(r) = \langle \Phi \rangle^2 + \Phi_{\text{rms}}^2 \rho_\phi \left( \frac{r}{\tau_d} \right)$$

$$= \langle \Phi \rangle^2 + \frac{\Phi_{\text{rms}}^2}{1 - 2\lambda} \left[ (1 - \lambda) \exp \left( -\frac{|r|}{(1 - \lambda)\tau_d} \right) - \lambda \exp \left( -\frac{|r|}{\tau_d} \right) \right]$$
Power spectral density

\[ \Omega_\Phi(\omega) = 2\pi \langle \Phi \rangle^2 \delta(\omega) + \Phi^2_{\text{rms}} \tau_d \sigma_\Phi(\tau_d \omega) \]

\[ = 2\pi \langle \Phi \rangle^2 \delta(\omega) + 2\Phi^2_{\text{rms}} \frac{\tau_d}{1 + (1 - \lambda)^2 \tau^2_d \omega^2} \left[ 1 + \lambda^2 \tau^2_d \omega^2 \right] \]

- \( \lambda = 0 \): Power law tail, \( \sim \omega^{-2} \)
- \( \lambda = 1/2 \): Power law tail, \( \sim \omega^{-4} \)
- Else: broken power law, curved spectrum.

For exponentially distributed amplitudes and exponential wave forms is the process Gamma distributed:

\[
\langle \Phi \rangle P_\Phi(\Phi) = \frac{\gamma}{\Gamma(\gamma)} \left( \frac{\gamma \Phi}{\langle \Phi \rangle} \right)^{\gamma - 1} \exp \left( - \frac{\gamma \Phi}{\langle \Phi \rangle} \right)
\]

SOL fluctuations measured in a density scan

- Ohmic L-mode plasma
- Lower single-null magnetic geometry
- Density varied from $\bar{n}_e/n_G = 0.12..0.62$
- Probe head dwelled at the limiter radius
- 4 electrodes with Mirror Langmuir probes
- Approximately 1s long data time series in steady state
Mirror Langmuir Probe allows fast $I_s$, $T_e$, and $V_f$ sampling

- MLP biases electrode to 3 voltages per microsecond.
- Voltage range is dynamically adjusted
- Probe current measured in each voltage state
- Fit input voltage and current is subject to 12pt smoothing (running average)
- Fit U-I characteristic on $(U,I)$ samples
- Largest error on $T_e$.
- Resolves fluctuations on $\mu$s time scale
Low density discharge, $\bar{n}_e/n_G = 0.12$

- Intermittent, large amplitude bursts in $I_s$.
- Bursts in $n_e$ and $T_e$ appear correlated
- Timescale approximately 25μs
- Irregular potential waveform
High density discharge, $\bar{n}_e/n_G = 0.62$

- Bursts appear more isolated
- Average density larger by factor of 10
- Average electron temperature approx. 8 eV
Ion saturation current histograms are well described by a Gamma distribution.

Electron temperature histograms are well described by a Gamma distribution.
PSD of $I_S$ shows broken power law

![Graph showing PSD of $I_S$ with broken power law]

Comparison to experimental measurements

\[ \text{PSD}(I_S) \]

$\nu/e/n_G = 0.12 : \tau_d = 15.91 \mu s \lambda = 0.0$

$\nu/e/n_G = 0.28 : \tau_d = 12.14 \mu s \lambda = 0.0$

$\nu/e/n_G = 0.59 : \tau_d = 15.64 \mu s \lambda = 0.0$
PSD of $T_e$ shows broken power law
$I_s$ shows exponential autocorrelation function.
$T_e$ shows exponential autocorrelation function
Bursts in $I_S$ are approximated by double-exponential waveform.

![Graph showing double-exponential waveform with different parameters for $\bar{n}_e/n_G$ and $\tau_d$, $\lambda$.]
Comparison to experimental measurements

Bursts in $T_e$ are approximated by double-exponential waveform

$$\bar{T}_e(t)/I_s(0) > 2.5$$

$\bar{n}_e/n_G = 0.12 : \tau_d = 17.31\mu s, \lambda = 0.4$

$\bar{n}_e/n_G = 0.28 : \tau_d = 14.16\mu s, \lambda = 0.4$

$\bar{n}_e/n_G = 0.59 : \tau_d = 12.24\mu s, \lambda = 0.4$
Time between bursts in $I_S$ signal is exponentially distributed

Exponential distribution describes the time between events in a Poisson process.

PDF($\omega$)

\[ \frac{\bar{n}_e}{n_G} = 0.12 : \omega = 233.7 \mu s \]
\[ \frac{\bar{n}_e}{n_G} = 0.28 : \omega = 169.4 \mu s \]
\[ \frac{\bar{n}_e}{n_G} = 0.59 : \omega = 171.3 \mu s \]
Time between bursts in $T_e$ signal is exponentially distributed

\begin{figure}
\centering
\includegraphics[width=\textwidth]{pdf_t_w.png}
\caption{PDF of $\tau_w$ for different $\bar{n}_e/n_G$ values.}
\end{figure}

- $\bar{n}_e/n_G = 0.12 : \tau_w = 279.6 \mu s$
- $\bar{n}_e/n_G = 0.28 : \tau_w = 240.7 \mu s$
- $\bar{n}_e/n_G = 0.59 : \tau_w = 198.1 \mu s$
Comparison to experimental measurements

**Burst amplitude distribution - Isat**

![Graph showing burst amplitude distribution]

- $\bar{n}_e/n_G = 0.12 : A = 1.0$
- $\bar{n}_e/n_G = 0.28 : A = 1.1$
- $\bar{n}_e/n_G = 0.59 : A = 2.1$
Comparison to experimental measurements

Burst amplitude distribution - Te

![Graph showing burst amplitude distribution with lines for different ne/nG values: ne/nG = 0.12, ne/nG = 0.28, ne/nG = 0.59 with corresponding A values of 0.8 and 1.8.]

\[ P_A(A) \]

\[ n_e/n_G = 0.12 : A = 0.8 \]
\[ n_e/n_G = 0.28 : A = 0.8 \]
\[ n_e/n_G = 0.59 : A = 1.8 \]
Conclusions
Overview of estimated parameters

<table>
<thead>
<tr>
<th></th>
<th>$\bar{n}_e/n_G$</th>
<th>$\gamma$ (PDF)</th>
<th>$\gamma\left(\Phi_{\text{rms}}/\langle\Phi\rangle\right)$</th>
<th>$\tau_d$ (PSD)</th>
<th>$\tau_d, \mathcal{R}$</th>
<th>$\tau_d$ (CA)</th>
<th>$\tau_w$</th>
<th>$\langle A \rangle$</th>
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</thead>
<tbody>
<tr>
<td>$I_s$</td>
<td>0.12</td>
<td>2.68</td>
<td>8.0</td>
<td>15.0 $\mu$s</td>
<td>15.0 $\mu$s</td>
<td>13.2 $\mu$s</td>
<td>234 $\mu$s</td>
<td>1.0</td>
</tr>
<tr>
<td>$I_s$</td>
<td>0.28</td>
<td>1.60</td>
<td>5.7</td>
<td>12.1 $\mu$s</td>
<td>11.3 $\mu$s</td>
<td>10.3 $\mu$s</td>
<td>169 $\mu$s</td>
<td>1.1</td>
</tr>
<tr>
<td>$I_s$</td>
<td>0.59</td>
<td>0.68</td>
<td>4.4</td>
<td>15.6 $\mu$s</td>
<td>12.8 $\mu$s</td>
<td>8.24 $\mu$s</td>
<td>171 $\mu$s</td>
<td>2.1</td>
</tr>
<tr>
<td>$T_e$</td>
<td>0.12</td>
<td>11.82</td>
<td>25</td>
<td>15.4 $\mu$s</td>
<td>14.9 $\mu$s</td>
<td>17.3 $\mu$s</td>
<td>280 $\mu$s</td>
<td>0.8</td>
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<tr>
<td>$T_e$</td>
<td>0.28</td>
<td>6.07</td>
<td>13</td>
<td>13.2 $\mu$s</td>
<td>12.6 $\mu$s</td>
<td>14.2 $\mu$s</td>
<td>241 $\mu$s</td>
<td>0.8</td>
</tr>
<tr>
<td>$T_e$</td>
<td>0.59</td>
<td>0.75</td>
<td>4.6</td>
<td>23.4 $\mu$s</td>
<td>16.7 $\mu$s</td>
<td>12.2 $\mu$s</td>
<td>198 $\mu$s</td>
<td>1.8</td>
</tr>
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Conclusions

<table>
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<th>Theory</th>
<th>Experimental data</th>
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<td>Process is Gamma distributed</td>
<td>$I_\text{s}$ and $T_\text{e}$ time series are Gamma distributed</td>
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<tr>
<td>Pulses arrive uncorrelated</td>
<td>Waiting time between bursts in $I_\text{s}$ and $T_\text{e}$ is exponential distributed</td>
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<td>Exponential distributed pulse amplitude</td>
<td>Burst amplitudes in $I_\text{s}$ and $T_\text{e}$ are expon. distributed</td>
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<td>Double-exponential pulse shape</td>
<td>PSD, autocorrelation function and cond. avg. of $I_\text{s}$ and $T_\text{e}$ time series agree</td>
</tr>
</tbody>
</table>

- Less burst overlap at high densities
- Burst duration time changes little with $\bar{n}_\text{e}/n_\text{G}$.
- Burst amplitude increases with $\bar{n}_\text{e}/n_\text{G}$
Thank you for your attention.