Moment approach to plasma fluid/kinetic theory and closures

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Outline

- Kinetic equation vs. moment equations
  - Several low order moments
  - Fluid equations and closures
  - General moment expansion
  - (Irreducible vs. reducible tensorial polynomials)
  - Collision operator and its moments
  - General moment equations
- Classical (Braginskii) closure theory (high collisionality)
  - Convergence with increasing the number of moments
  - Effect of ion-electron collisions on the ion closures
- Parallel integral (nonlocal) closures for electrons and ions
  - Along a magnetic field line in a strong magnetic field or inhomogeneous along only one direction (planar symmetry)
  - For arbitrary collision length (mean free path)
  - For arbitrary temperature ratio and charge and mass numbers
- Applications
  - Radial heat flow due to magnetic field fluctuations
  - Electron parallel transport for arbitrary collisionality

Distribution function and several lowest order fluid moments

- The state of a plasma (ionized gas) is specified by the distribution function $f_a(t, \mathbf{x}, \mathbf{v})$

$$f_a(t, \mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v} = \text{number of particles in } d\mathbf{x} d\mathbf{v} \text{ in the phase space}$$

- Several lowest order fluid moments of $\mathbf{w}_a = \mathbf{v} - \mathbf{V}_a$:

$$\text{Density } n_a = \int d\mathbf{v} f_a$$

$$\text{Flow velocity } \mathbf{V}_a = n_a^{-1} \int d\mathbf{v} \mathbf{v} f_a$$

$$\text{Temperature } T_a = n_a^{-1} \int d\mathbf{v} \frac{1}{3} m_a w_a^2 f_a$$

$$\text{Viscosity } \boldsymbol{\pi}_a = \int d\mathbf{v} m_a (\mathbf{w}_a \mathbf{w}_a - \frac{1}{3} w_a^2 \mathsf{I}) f_a$$

$$\left(\text{Pressure tensor } \mathsf{p}_a = \int d\mathbf{v} m_a \mathbf{w}_a \mathbf{w}_a f_a = p_a \mathsf{I} + \boldsymbol{\pi}_a, \qquad p_a = n_a T_a \right)$$

$$\text{Heat flow } \mathbf{h}_a = \int d\mathbf{v} \frac{1}{2} m_a w_a^2 \mathbf{w}_a f_a$$

Kinetic equation, fluid equations and closures

- The kinetic equation governs the evolution of the distribution function

$$\frac{\partial f_a}{\partial t} + \mathbf{v} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + \frac{q_a}{m_a}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \mathbf{v}} = C(f_a) \qquad \text{(KE)}$$

- Fluid equations are more amenable to analytic and computational study, but require closures

$$\int d\mathbf{v}\,\text{(KE)} \Rightarrow \frac{\partial n_a}{\partial t} + \nabla \cdot (n_a \mathbf{V}_a) = 0$$

$$\int d\mathbf{v} m_a \mathbf{v}\,\text{(KE)} \Rightarrow m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

$$\int d\mathbf{v} \frac{1}{2} m_a w_a^2\,\text{(KE)} \Rightarrow \frac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

where

$$d_a = \frac{\partial}{\partial t} + \mathbf{V}_a \cdot \nabla \qquad \text{(convective derivative)}$$

$$\mathbf{R}_a = \int d\mathbf{v} m_a \mathbf{v} C(f_a) \qquad \text{(collisional friction)}$$

$$Q_a = \int d\mathbf{v} \frac{1}{2} m_a w_a^2 C(f_a) \qquad \text{(collisional heating)}$$

- Express $\{\mathbf{h}_a, \boldsymbol{\pi}_a, \mathbf{R}_a, Q_a\}$ in terms of $\{n_a, T_a, \mathbf{V}_a\}$

$\Rightarrow$ Higher order moment equations $\Rightarrow$ Analytic calculation of asymptotic closures

General moment expansion [Chapman 1916 Enskog 1917 Grad 1963]
Irreducible tensorial Hermite polynomials

- Moment expansion: coefficients m_a^{lk} are symmetric traceless fluid moments

$$f(t,\mathbf{x},\mathbf{v}) = f^{\mathrm{M}} \sum_{lk} \mathsf{m}^{lk}(t,\mathbf{x}) \cdot \hat{\mathsf{p}}^{lk}$$

$$\mathsf{n}^{lk}(t,\mathbf{x}) \equiv n\mathsf{m}^{lk} = \int d\mathbf{v} \hat{\mathsf{p}}^{lk} f(t,\mathbf{x},\mathbf{v})$$

- $\hat{\mathsf{p}}^{lk}$'s are orthonormal, irreducible, tensorial polynomials and form a complete set

$$\int d\mathbf{v} \hat{\mathsf{p}}^{jp} \hat{\mathsf{p}}^{lk} \cdot \mathsf{m}^{lk} f^{\mathrm{M}} = \delta_{jl}\delta_{pk}\mathsf{n}^{jp}$$

$$\hat{\mathsf{p}}_a^{lk} = \frac{1}{\sqrt{\sigma_k^l}} \mathsf{p}_a^{lk}, \quad \mathsf{p}_a^{lk} = \mathsf{P}^l(\mathbf{c}_a) L_k^{(l+1/2)}(c_a^2)$$

$$f_a^{\mathrm{M}} = n_a \hat{f}_a^{\mathrm{M}} = \frac{n_a}{\pi^{3/2} v_{Ta}^3} e^{-c_a^2}$$

$$\mathbf{c}_a = \frac{\mathbf{w}_a}{v_{Ta}} = \frac{\mathbf{v} - \mathbf{V}_a}{v_{Ta}}, \quad v_{Ta} = \sqrt{\frac{2T_a}{m_a}}$$

Several low order moment equations (21 moments)

$\mathsf{p}^{lk} = \mathsf{P}^l(\mathbf{c})L_k^l(c^2)$	$L_k^l = L_k^{(l+\frac{1}{2})}$	n^{lk}	fluid moment equation	indep.
$\mathsf{P}^0 = 1$	$L_0^0 = 1$	n	density (n)	1
	$L_1^0 = \frac{3}{2} - c^2$	0	temperature (T)	1
$\mathsf{P}^1 = \mathbf{c}$	$L_0^1 = 1$	0	flow velocity $(\mathbf{V})$	3
	$L_1^1 = \frac{5}{2} - c^2$	n^{11}	heat flow $(\mathbf{h})$	3
	$L_2^1 = \frac{35}{8} - \frac{7}{2}c^2 + \frac{1}{2}c^4$	n^{12}	heat w. heat flow $(\mathbf{r})$	3
$\mathsf{P}^2 = \mathbf{cc} - \frac{c^2}{3}\mathsf{I}$	$L_0^2 = 1$	n^{20}	viscosity $(\boldsymbol{\pi})$	5
	$L_1^2 = \frac{7}{2} - c^2$	n^{21}	heat viscosity $(\boldsymbol{\theta})$	5

Comparison with reducible Hermite polynomials

[Grad 1949, Hirvijoki et al 2016 PoP, Pfefferlé et al 2017 PoP]

- Reducible tensorial Hermite polynomials

$$\bar{H}_{(i)}(\mathbf{w}; \frac{v_T^2}{2}) = e^{c^2} \left(-\frac{v_T^2}{2} \frac{\partial}{\partial \mathbf{v}} \right)^i e^{-c^2} = \frac{1}{f^{\mathrm{M}}} \left(\frac{v_T^2}{2} \frac{\partial}{\partial \mathbf{V}} \right)^i f^{\mathrm{M}}, \qquad \mathbf{c} = \frac{\mathbf{w}}{v_T} = \frac{\mathbf{v} - \mathbf{V}}{v_T}$$

$$\begin{aligned}
\bar{H}_{(0)}(\mathbf{w}; \frac{T}{m}) &= & 1 & & &= \mathsf{p}^{00} \\
\bar{H}_{(1)}(\mathbf{w}; \frac{T}{m}) &= & \mathbf{w} & & &= v_T \mathsf{p}^{10} \\
\bar{H}_{(2)}(\mathbf{w}; \frac{T}{m}) &= & \mathbf{ww} - \tfrac{T}{m}\mathsf{I} & & &= v_T^2 \left(\mathsf{p}^{20} - \frac{1}{3}\mathsf{p}^{01}\mathsf{I} \right) \\
\bar{H}_{(3)}(\mathbf{w}; \frac{T}{m}) &= & \mathbf{www} - \tfrac{T}{m}(\mathsf{I}\mathbf{w} + \mathsf{i}\mathbf{w}\mathsf{i} + \mathbf{w}\mathsf{I}) & & &= v_T^3 \left(\mathsf{p}^{30} - \frac{3}{5}\{\mathsf{p}^{11}\mathsf{I}\} \right)
\end{aligned}$$

- ⋆ Vector moments: $\bar{H}_{(1)}$, $\bar{H}_{(3)}$, $\bar{H}_{(5)}, \cdots$ vs. p^{10}, $\mathsf{p}^{11}(\mathbf{h})$, $\mathsf{p}^{12}, \cdots$
- ⋆ Rank-2 tensor moments: $\bar{H}_{(2)}$, $\bar{H}_{(4)}$, $\bar{H}_{(6)}, \cdots$ vs. $\mathsf{p}^{20}(\boldsymbol{\pi})$, p^{21}, $\mathsf{p}^{22}, \cdots$

Comparison with reducible Hermite polynomials (cont.)

[Grad 1949, Hirvijoki et al 2016 PoP, Pfefferlé et al 2017 PoP]

- Moment expansion

$$f_a(\mathbf{v}) = \sum_{i=0}^{\infty} \frac{\mathsf{M}_a^{(i)}}{i!} \cdot \frac{\bar{H}_{(i)}(\mathbf{w})}{(v_{Ta}^2/2)^i} f_a^{\mathrm{M}} = \sum_{i=0}^{\infty} \frac{\mathsf{M}_a^{(i)}}{i!} \cdot \nabla_a^i f_a^{\mathrm{M}}, \quad \nabla_a \equiv \frac{\partial}{\partial \mathbf{V}_a}$$

- Collisional moments

$$\int d\mathbf{v} \bar{H}_{(k)}(\mathbf{w}_a) C(f_a, f_b) = \sum_{i,j=0}^{\infty} \sum_{l=0}^{k+i} a_{(k)(i)}^{(l)} \frac{\mathsf{M}_a^{(i)}}{i!} \cdot \nabla_a^l \frac{\mathsf{M}_b^{(j)}}{j!} \cdot \nabla_b^j \int d\mathbf{v} C(f_a^{\mathrm{M}}, f_b^{\mathrm{M}})$$

  - ⋆ $\bar{H}_{(i)}$: Practical calculation is difficult for higher rank
  - ⋆ p^{lk}: $\mathsf{P}^l(\hat{\mathbf{v}})[\to l(l+1)]$ and speed polynomials v^{l+2k} $\Rightarrow$ Explicit formula

- Moments of the kinetic equation

  - ⋆ $\bar{H}_{(i)}$: General formulation is difficult
  - ⋆ p^{lk}: An infinite hierarchy is established

Collision operator and collisional moments

- Landau (Fokker-Planck) collision operator → integro-differential equation

$$C(f_a, f_b) = \frac{\gamma_{ab}}{2}\frac{\partial}{\partial \mathbf{v}} \cdot \int d\mathbf{v}' \mathsf{U} \cdot \left[\frac{1}{m_a}\frac{\partial f_a(\mathbf{v})}{\partial \mathbf{v}} f_b(\mathbf{v}') - f_a(\mathbf{v}) \frac{1}{m_b}\frac{\partial f_b(\mathbf{v}')}{\partial \mathbf{v}'} \right]$$

where $\mathsf{U} = \frac{u^2 \mathsf{I} - \mathbf{u}\mathbf{u}}{u^3}$, $\mathbf{u} = \mathbf{v} - \mathbf{v}'$, $\gamma_{ab} = \frac{q_a^2 q_b^2 \ln \Lambda_{ab}}{4\pi\epsilon_0^2 m_a}$.

- Linearized collision operator

$$\int d\mathbf{v} \mathsf{p}_a^{jp} C\left(f_a^{\mathrm{M}} \mathsf{m}_a^{lk} \cdot \mathsf{p}_a^{lk}, f_b^{\mathrm{M}}\right) = \sigma_j A_{ab}^{jp,lk} \mathsf{m}_a^{lk}$$

$$\int d\mathbf{v} \mathsf{p}_a^{jp} C\left(f_a^{\mathrm{M}}, f_b^{\mathrm{M}} \mathsf{m}_b^{nq} \cdot \mathsf{p}_b^{nq}\right) = \sigma_j B_{ab}^{jp,lk} \mathsf{m}_b^{lk}$$

  - ⋆ Computed for several lowest orders (21 or 29 moments) only
    Braginskii 1957, Hirshman 1977, Balescu 1988
  - ⋆ Explicit formulas $A_{ab}^{jp,lk}$ and $B_{ab}^{jp,lk}$ are derived [Ji & Held 2006 2008 PoP]
    for arbitrary temperature and mass ratios

- Nonlinear collision terms [Ji & Held 2009 PoP]

General moment equations [Ji & Held 2006 2008 2009 PoP]

- Landau (Fokker-Planck) kinetic equation

$$\frac{\partial f_a}{\partial t} + \mathbf{v}\cdot\nabla f_a + \frac{q_a}{m_a}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f_a}{\partial \mathbf{v}} = \sum_b C(f_a, f_b)$$

- Moment expansion

$$f_a(t,\mathbf{x},\mathbf{v}) = f_a^{\mathrm{M}} \sum_{lk} \mathsf{m}_a^{lk}(t,\mathbf{x})\cdot \hat{\mathsf{p}}_a^{lk}$$

- General moment equations $\int d\mathbf{v}\hat{\mathsf{p}}_a^{jp}$(kinetic eq.) $\Rightarrow$

$$d_a \mathsf{n}_a^{jp} + \Omega_a \mathbf{b}\breve{\times}\mathsf{n}_a^{jp} + \{\hat{\Xi}^j(d_a \ln T) + \hat{U}_{\mathrm{c}}^j \nabla\cdot\mathbf{V} + \hat{U}_{\mathrm{l}}^j(\underline{\nabla}\mathbf{V})\cdot + \hat{U}_{\mathrm{r}}^j(\nabla\underline{\mathbf{V}})\cdot\}_{pk}\mathsf{n}_a^{jk}$$
$$+\{v_T\hat{\Psi}^{j\pm}\nabla + v_T\hat{\Phi}^{j\pm}\nabla\ln T + v_T^{-1}\hat{\Theta}^{j\pm}\mathbf{a}_a\}_{pk}(\cdot)\mathsf{n}_a^{j\pm1,k} + \hat{U}_{pk}^{j\pm}\nabla\mathbf{V}(:)\mathsf{n}_a^{j\pm2,k}$$
$$= (\hat{C}_{aa}^{jpk} + \hat{A}_{ab}^{jpk})\mathsf{n}_a^{jk} + \hat{B}_{ab}^{jpk}\mathsf{n}_b^{jk} + C_{ab}^{(2)jp}$$

where $d_a \equiv \partial_t + \mathbf{V}_a\cdot\nabla$ and $\mathbf{a}_a \equiv \frac{q_a}{m_a}(\mathbf{E}+\mathbf{V}_a\times\mathbf{B}) - d_a\mathbf{V}_a$

▶ Farewell to the velocity variable $\mathbf{v}$

Moment equations for closures

- Maxwellian moment $(n_a, \mathbf{V}_a, T_a)$ equations

$$(0,0): \quad d_a n_a + n_a \nabla \cdot \mathbf{V}_a = 0$$

$$(0,1): \quad \tfrac{3}{2} n_a d_a T_a + n_a T_a \nabla \cdot \mathbf{V}_a + \nabla \cdot \mathbf{h}_a + \nabla \mathbf{V}_a : \boldsymbol{\pi}_a = Q_a$$

$$(1,0): \quad m_a n_a d_a \mathbf{V}_a - n_a q_a (\mathbf{E} + \mathbf{V}_a \times \mathbf{B}) + \nabla p_a + \nabla \cdot \boldsymbol{\pi}_a = \mathbf{R}_a$$

- Non-Maxwellian moment equations $(j,p) \neq (0,0), (0,1), (1,0)$

$$\hat{\mathsf{L}}_a \mathsf{n}_a + \Omega_a \mathbf{b} \breve{\times} \mathsf{n}_a = (\hat{C}_{aa} + \hat{A}_{ab}) \mathsf{n}_a + \mathsf{G}_a + \hat{B}_{ab} \mathsf{n}_b + \hat{C}_a^{(2)}$$

where $\mathsf{n}_a = (\mathsf{n}_a^{02}, \mathsf{n}_a^{03}, \cdots, \mathsf{n}_a^{11}, \mathsf{n}_a^{12}, \cdots, \mathsf{n}_a^{20}, \mathsf{n}_a^{21}, \cdots, \cdots)^{\mathrm{T}}$,

$$\mathsf{G}_a^1 = \begin{pmatrix} \frac{\sqrt{5}}{2} \frac{n_a v_{Ta}}{T_a} \nabla T_a + \delta_{ae} \sqrt{2} a_{\mathrm{ei}}^{110} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \delta_{ae} a_{\mathrm{ei}}^{120} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \delta_{ae} a_{\mathrm{ei}}^{130} \frac{n_e}{\tau_{\mathrm{ei}}} \frac{\mathbf{V}_{\mathrm{ei}}}{v_{Te}} \\ \vdots \end{pmatrix}, \quad \mathsf{G}_a^2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} n_a \mathsf{W}_a \\ 0 \\ 0 \\ \vdots \end{pmatrix},$$

$a_{\mathrm{ei}}^{1p0} = -\sqrt{\frac{3(p+\frac{1}{2})!}{(2p+3)p!(\frac{1}{2})!}}$ and $\mathsf{W} = \nabla \mathbf{V} + (\nabla \mathbf{V})^{\mathrm{T}} - \frac{2}{3} \mathsf{I} \nabla \cdot \mathbf{V}$

- Solve the non-Maxwellian moment equations to express $\{\mathbf{h}_a, \boldsymbol{\pi}_a, \mathbf{R}_a, Q_a\}$ in terms of $\{n_a, T_a, \mathbf{V}_a\}$

21 moment $(n, T, \mathbf{V}, \mathsf{n}^{11}, \mathsf{n}^{12}, \mathsf{n}^{20}, \mathsf{n}^{21})$ equations

[Here $d_t = \partial_t + \mathbf{V}\cdot\nabla,\ \dot\nabla = v_T\nabla,\ \mathbf{a} = \frac{q}{m}(\mathbf{E}+\mathbf{V}\times\mathbf{B}) - d_t\mathbf{V}$**]**

- $d_t\mathsf{n}^{11} + \frac{3}{2}d_t\ln T\mathsf{n}^{11} + \Omega\mathbf{b}\times\mathsf{n}^{11} - \sqrt{\frac{2}{3}}\dot\nabla\mathsf{n}^{02} + \dot\nabla\cdot(-\sqrt{\frac{2}{5}}\mathsf{n}^{20} + \sqrt{\frac{7}{5}}\mathsf{n}^{21})$
$-\frac{\sqrt5}{2}n\dot\nabla\ln T - 2\sqrt{\frac{2}{3}}\dot\nabla\ln T\mathsf{n}^{02} + \dot\nabla\ln T\cdot(-\frac{9}{\sqrt{10}}\mathsf{n}^{20} + 2\sqrt{\frac{7}{5}}\mathsf{n}^{21}) + 2\sqrt{\frac{2}{5}}\frac{1}{v_T}\mathbf{a}\cdot\mathsf{n}^{20}$
$+\frac{7}{5}\nabla\cdot\mathbf{V}\mathsf{n}^{11} + \frac{7}{5}\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{11} + \frac{2}{5}\nabla\underline{\mathbf{V}}\cdot\mathsf{n}^{11} - 2\sqrt{\frac{3}{5}}\nabla\mathbf{V}:\mathsf{n}^{30} = \hat{A}^{11,00}n + \hat{C}^{11,11}\mathsf{n}^{11} + \hat{C}^{11,12}\mathsf{n}^{12}$
- $d_t\mathsf{n}^{12} + d_t\ln T(-\sqrt{7}\mathsf{n}^{11} + \frac{5}{2}\mathsf{n}^{12}) + \Omega\mathbf{b}\times\mathsf{n}^{12} + \dot\nabla(\sqrt{\frac{7}{6}}\mathsf{n}^{02} - \mathsf{n}^{03}) + \dot\nabla\cdot(-\frac{2}{\sqrt5}\mathsf{n}^{21} + \frac{3}{\sqrt5}\mathsf{n}^{22})$
$-\sqrt{\frac{14}{3}}\frac{1}{v_T}\mathbf{a}\mathsf{n}^{02} + \frac{4}{\sqrt5}\frac{1}{v_T}\mathbf{a}\cdot\mathsf{n}^{21} + \dot\nabla\ln T(5\sqrt{\frac{7}{6}}\mathsf{n}^{02} - 3\mathsf{n}^{03}) + \dot\nabla\ln T\cdot(\sqrt{\frac{14}{5}}\mathsf{n}^{20} - \frac{13}{\sqrt5}\mathsf{n}^{21} + \frac{9}{\sqrt5}\mathsf{n}^{22})$
$+\nabla\cdot\mathbf{V}(-\frac{2\sqrt7}{5}\mathsf{n}^{11} + \frac{9}{5}\mathsf{n}^{12}) + \underline{\nabla}\mathbf{V}\cdot(-\frac{2\sqrt7}{5}\mathsf{n}^{11} + \frac{9}{5}\mathsf{n}^{12}) + \nabla\underline{\mathbf{V}}\cdot(-\frac{2\sqrt7}{5}\mathsf{n}^{11} + \frac{4}{5}\mathsf{n}^{12})$
$+\nabla\mathbf{V}:(4\sqrt{\frac{3}{35}}\mathsf{n}^{30} - 6\sqrt{\frac{6}{35}}\mathsf{n}^{31}) = \hat{A}^{12,00}n + \hat{C}^{12,11}\mathsf{n}^{11} + \hat{C}^{12,12}\mathsf{n}^{12}$
- $d_t\mathsf{n}^{20} + d_t\ln T\mathsf{n}^{20} + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{20} - \sqrt{\frac{2}{5}}\dot\nabla\mathsf{n}^{11} + \sqrt{\frac{3}{2}}\dot\nabla\cdot\mathsf{n}^{30}$
$-\frac{3}{\sqrt{10}}\dot\nabla\ln T\mathsf{n}^{11} + \frac{3}{2}\sqrt{\frac{3}{2}}\nabla\ln T\cdot\mathsf{n}^{30} + \nabla\cdot\mathbf{V}\mathsf{n}^{20} + 2\underline{\nabla}\mathbf{V}\cdot\mathsf{n}^{20} + \sqrt{2}n\overline{\nabla\mathbf{V}} = \hat{C}^{20,20}\mathsf{n}^{20} + \hat{C}^{20,21}\mathsf{n}^{21}$
- $d_t\mathsf{n}^{21} + d_t\ln T(-\sqrt{\frac{7}{2}}\mathsf{n}^{20} + 2\mathsf{n}^{21}) + \Omega\mathbf{b}\breve{\times}\mathsf{n}^{21} + \dot\nabla(\sqrt{\frac{7}{5}}\mathsf{n}^{11} - \frac{2}{\sqrt5}\mathsf{n}^{12}) + \dot\nabla\cdot(-\sqrt{\frac{3}{7}}\mathsf{n}^{30} + 3\sqrt{\frac{3}{14}}\mathsf{n}^{31}$
$-2\sqrt{\frac{7}{5}}\frac{1}{v_T}\mathbf{a}\mathsf{n}^{11} + 2\sqrt{\frac{3}{7}}\frac{1}{v_T}\mathbf{a}\cdot\mathsf{n}^{30} + \dot\nabla\ln T(\frac{7}{2}\sqrt{\frac{7}{5}}\mathsf{n}^{11} - \sqrt5\mathsf{n}^{12}) + \dot\nabla\ln T\cdot(-6\sqrt{\frac{3}{7}}\mathsf{n}^{30} + \frac{15}{2}\sqrt{\frac{3}{14}}\mathsf{n}^{31})$
$+\nabla\cdot\mathbf{V}(-\sqrt{\frac{2}{7}}\mathsf{n}^{20} + \frac{9}{7}\mathsf{n}^{21}) + \underline{\nabla}\mathbf{V}\cdot(-2\sqrt{\frac{2}{7}}\mathsf{n}^{20} + \frac{18}{7}\mathsf{n}^{21}) + \nabla\underline{\mathbf{V}}\cdot(-2\sqrt{\frac{2}{7}}\mathsf{n}^{20} + \frac{4}{7}\mathsf{n}^{21})$
$-2\sqrt{\frac{14}{15}}\overline{\nabla\mathbf{V}}\mathsf{n}^{02} - 2\sqrt{\frac{6}{7}}\nabla\mathbf{V}:\mathsf{n}^{40} = \hat{C}^{21,20}\mathsf{n}^{20} + \hat{C}^{21,21}\mathsf{n}^{21}$

$\hat{C}_a^{11,11} = -\frac{1}{\tau_{aa}}(\frac{2\sqrt2}{5} + z_a\frac{13}{10}),\ \hat{C}_a^{11,12} = -\frac{1}{\tau_{aa}}(\frac{3}{5}\sqrt{\frac{2}{7}} + z_a\frac{69}{20\sqrt7}),\ \hat{C}_a^{12,12} = -\frac{1}{\tau_{aa}}(\frac{9}{7\sqrt2} + z_a\frac{433}{280})$

$\hat{C}_a^{20,20} = -\frac{1}{\tau_{aa}}(\frac{3\sqrt2}{5} + z_a\frac{6}{5}),\ \hat{C}_a^{20,21} = -\frac{1}{\tau_{aa}}(\frac{9}{10\sqrt7} + z_a\frac{9}{5}\sqrt{\frac{2}{7}}),\ \hat{C}_a^{12,12} = -\frac{1}{\tau_{aa}}(\frac{41}{28\sqrt2} + z_a\frac{51}{35})$

$\hat{A}_a^{11,00} = -\frac{z_a}{\tau_{aa}}\frac{3}{\sqrt5}\frac{\mathbf{V}_{ei}}{v_{Te}},\ \hat{A}_a^{12,00} = -\frac{z_a}{\tau_{ee}}\frac{3}{2}\sqrt{\frac{5}{7}}\frac{\mathbf{V}_{ei}}{v_{Te}},\ z_e = Z,\ z_i = 0,\ \mathbf{V}_{ei} = \mathbf{V}_e - \mathbf{V}_i$

Braginskii closures for high collisionality (1957, 1965)

$\mathrm{Kn} \sim |\partial_t|/\nu_{\mathrm{col}} \sim \lambda_C/|\nabla^{-1}| \ll 1$

$$\begin{pmatrix} -x\mathbf{b}\times \mathsf{m}^{11} \\ -x\mathbf{b}\times \mathsf{m}^{12} \\ -x\mathbf{b}\times \mathsf{m}^{13} \\ \vdots \end{pmatrix} = \begin{pmatrix} c^1_{11} & c^1_{12} & c^1_{13} & \cdots \\ c^1_{21} & c^1_{22} & c^1_{23} & \cdots \\ c^1_{31} & c^1_{32} & c^1_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \mathsf{m}^{11} \\ \mathsf{m}^{12} \\ \mathsf{m}^{13} \\ \vdots \end{pmatrix} + \begin{pmatrix} \frac{\sqrt{5}}{2}\hat{\nabla}\ln T + \sqrt{2}a^{101}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \sqrt{2}a^{102}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \sqrt{2}a^{103}_{\mathrm{ei}}\frac{\mathbf{V}_{\mathrm{ei}}}{v_{T\mathrm{e}}} \\ \vdots \end{pmatrix}$$

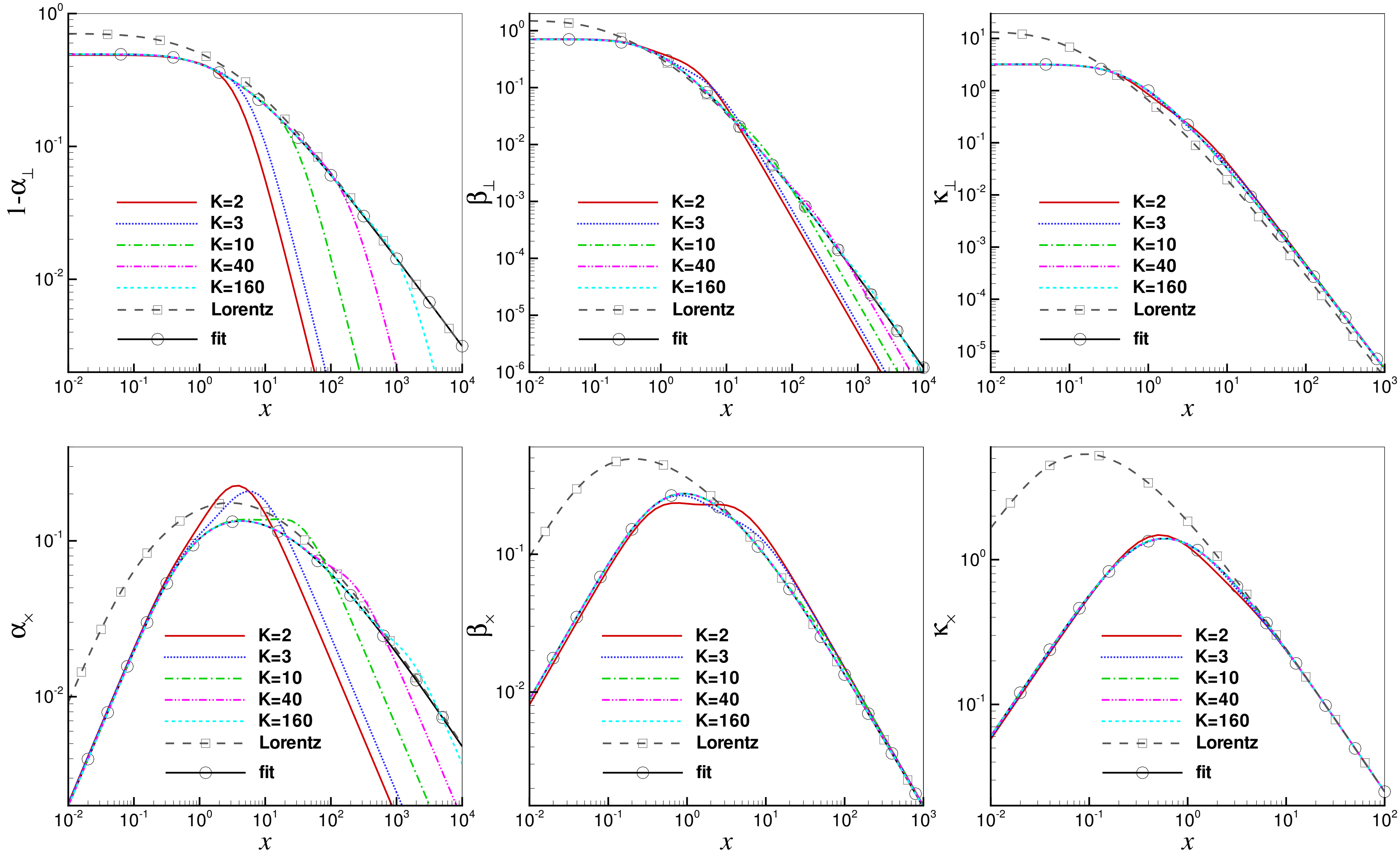
$$\mathsf{m}^{1p}_{\perp} = \sum_{q=1}^{K}(x^2+c^2)^{-1}_{pq}\left(x\mathbf{g}^{1q}_{\times} - \sum_{k=1}^{K} c_{qk}\mathbf{g}^{1k}_{\perp}\right),\ \text{where } \mathbf{g}_{\times} = \mathbf{b}\times\mathbf{g}$$

$$\begin{aligned} \mathbf{R}_{\mathrm{e}} &= -\alpha_{\parallel}\mathbf{V}_{\mathrm{ei}\parallel} - \alpha_{\perp}\mathbf{V}_{\mathrm{ei}\perp} + \alpha_{\times}\mathbf{V}_{\mathrm{ei}\times} - \beta_{\parallel}\nabla_{\parallel}T_{\mathrm{e}} - \beta_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \beta_{\times}\nabla_{\times}T_{\mathrm{e}} \\ \mathbf{h}_{\mathrm{e}} &= \beta_{\parallel}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\parallel} + \beta_{\perp}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\perp} + \beta_{\times}T_{\mathrm{e}}\mathbf{V}_{\mathrm{ei}\times} - \kappa^{\mathrm{e}}_{\parallel}\nabla_{\parallel}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\perp}\nabla_{\perp}T_{\mathrm{e}} - \kappa^{\mathrm{e}}_{\times}\nabla_{\times}T_{\mathrm{e}} \end{aligned}$$

where $x = |\Omega_{\mathrm{e}}|\tau_{\mathrm{ei}}$, $\alpha_A = \hat{\alpha}_A\frac{m_{\mathrm{e}}n_{\mathrm{e}}}{\tau_{\mathrm{ei}}}$, $\beta_A = \hat{\beta}_A n_{\mathrm{e}}$, $\kappa^{\mathrm{e}}_A = \hat{\kappa}^{\mathrm{e}}_A\frac{n_{\mathrm{e}}T_{\mathrm{e}}\tau_{\mathrm{ei}}}{m_{\mathrm{e}}}$ $(A = \parallel, \times, \perp)$

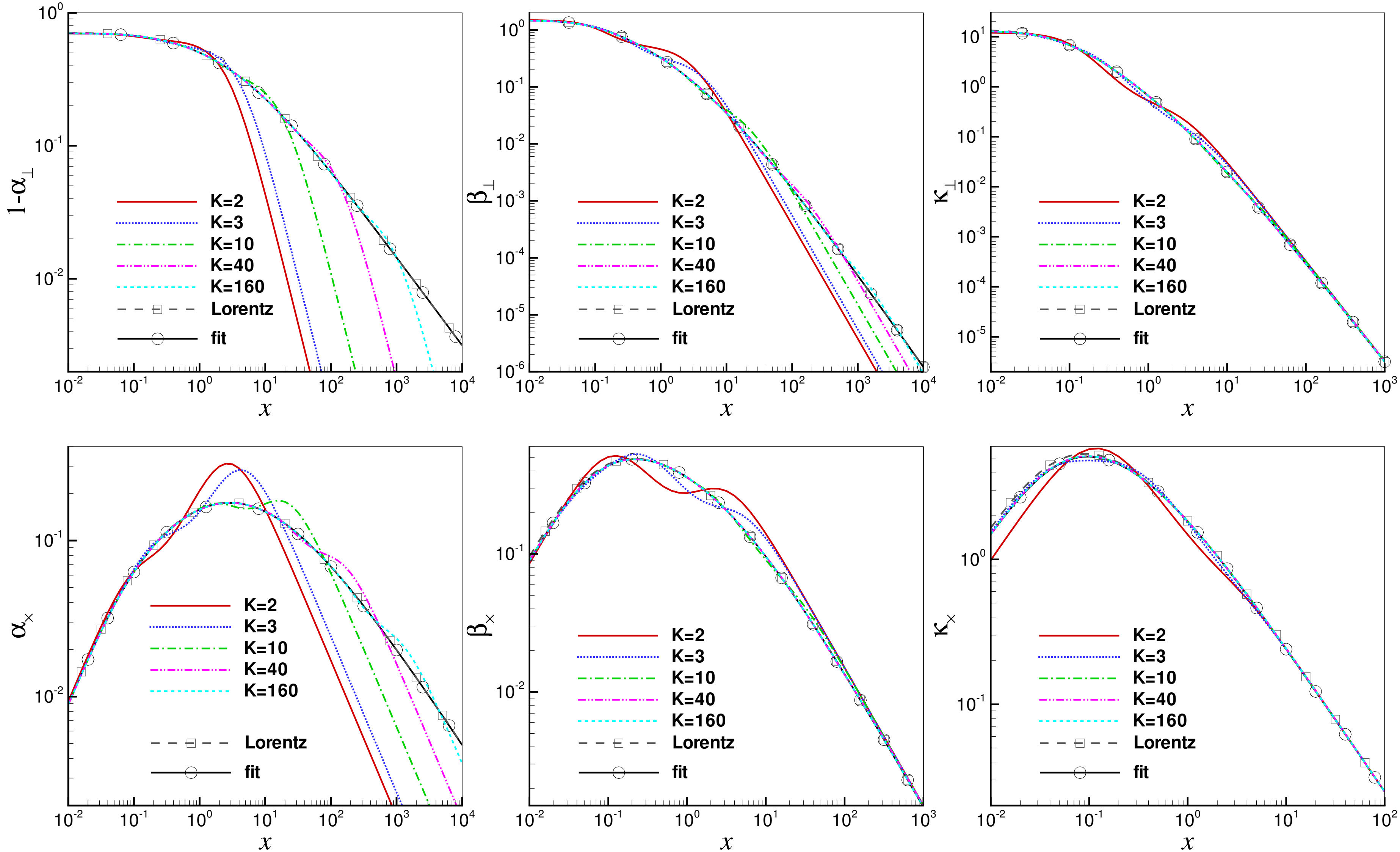
- Braginskii used two ($K = 2$) moments m^{11} (heat flux $\mathbf{h}$) and m^{12} (m^{20} and m^{21} for viscosity, not shown here)
- Improving Braginskii closures
  - ⋆ For electrons: convergence with increasing the number of moments
    - ◇ $\alpha_{\times} \sim x^{-1}$ and $\beta_{\perp} \sim x^{-2}$ never converge as $x \to \infty$
    - ◇ Epperlein and Haines 1986: $\alpha_{\times} \sim x^{-2/3}$, $\beta_{\perp} \sim x^{-5/3}$
  - ⋆ For ions: ion-electron collisions are not ignorable, $C_{\mathrm{i}} = C_{\mathrm{ii}} + C_{\mathrm{ie}}$

Electron closures for high collisionality for $Z = 1$
Convergence with increasing moments



$c_\mathrm{e} = c_\mathrm{ee} + Z a_\mathrm{ei}$ $\Rightarrow$ Lorentz gas is not a good approximation for a finite $x = |\Omega_\mathrm{e}|\tau_\mathrm{ei}$

Electron closures for high collisionality for $Z = 100$ Convergence with increasing moments



$c_{\mathrm{e}} = c_{\mathrm{ee}} + Z a_{\mathrm{ei}}$ $\Rightarrow$ Lorentz gas approximation is accurate for arbitrary $x = |\Omega_{\mathrm{e}}|\tau_{\mathrm{ei}}$

Electron closures for high collisionality (fitted coefficients)

	Braginskii	Epperlein & Haines (1986)	Ji & Held (2013)
$1-\hat{\alpha}_\perp$	$\frac{\alpha_1' x^2+\alpha_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\alpha_1' x+\alpha_0'}{x^2+a_1' x+a_0'}$	$\frac{\alpha_1' x+\alpha_0'}{x^{\frac{5}{3}}+a_2' x^{\frac{4}{3}}+a_1' x+a_0'}$
$\hat{\alpha}_\times$	$\frac{x(\alpha_1'' x^2+\alpha_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\alpha_1'' x+\alpha_0'')}{(x^3+a_2'' x^2+a_1'' x+a_0'')^{8/9}}$	$\frac{x(\alpha_1'' x+\alpha_0'')}{x^{\frac{8}{3}}+a_4'' x^{\frac{7}{3}}+a_3'' x^{\frac{6}{3}}+a_2'' x^{\frac{5}{3}}+a_1'' x+a_0''}$
$\hat{\beta}_\perp$	$\frac{\beta_1' x^2+\beta_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\beta_1' x+\beta_0'}{(x^3+b_2' x^2+b_1' x+b_0')^{8/9}}$	$\frac{\beta_1' x+\beta_0'}{x^{\frac{8}{3}}+b_4' x^{\frac{7}{3}}+b_3' x^{\frac{6}{3}}+b_2' x^{\frac{5}{3}}+b_1' x+b_0'}$
$\hat{\beta}_\times$	$\frac{x(\beta_1'' x^2+\beta_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\beta_1'' x+\beta_0'')}{x^3+b_2'' x^2+b_1'' x+b_0''}$	$\frac{x(\beta_1'' x+\beta_0'')}{x^3+b_4'' x^{\frac{7}{3}}+b_3'' x^{\frac{6}{3}}+b_2'' x^{\frac{5}{3}}+b_1'' x+b_0''}$
$\hat{\kappa}_\perp$	$\frac{\gamma_1' x^2+\gamma_0'}{x^4+\delta_1 x^2+\delta_0}$	$\frac{\gamma_1' x+\gamma_0'}{x^3+c_2' x^2+c_1' x+c_0'}$	$\frac{\gamma_1' x+\gamma_0'}{x^3+c_4' x^{\frac{7}{3}}+c_3' x^{\frac{6}{3}}+c_2' x^{\frac{5}{3}}+c_1' x+c_0'}$
$\hat{\kappa}_\times$	$\frac{x(\gamma_1'' x^2+\gamma_0'')}{x^4+\delta_1 x^2+\delta_0}$	$\frac{x(\gamma_1'' x+\gamma_0'')}{x^3+c_2'' x^2+c_1'' x+c_0''}$	$\frac{x(\gamma_1'' x+\gamma_0'')}{x^3+c_4'' x^{\frac{7}{3}}+c_3'' x^{\frac{6}{3}}+c_2'' x^{\frac{5}{3}}+c_1'' x+c_0''}$
Error		less than 15 %	less than 1%
$Z=$	$1,2,3,4,\infty$	$1-8,10,12,14,20,30,60,\infty$	arbitrary function of Z

Ion closures for high collisionality

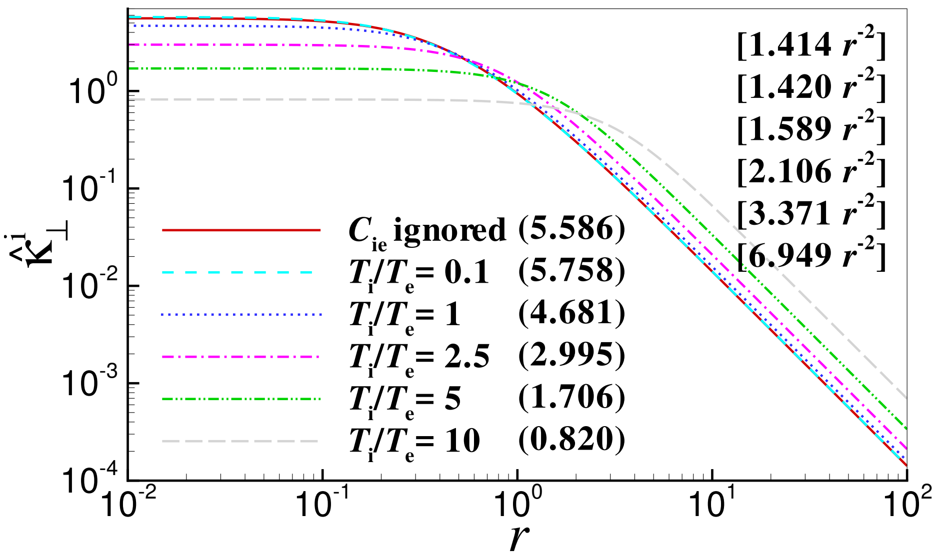
[Ji & Held 2015 PoP]

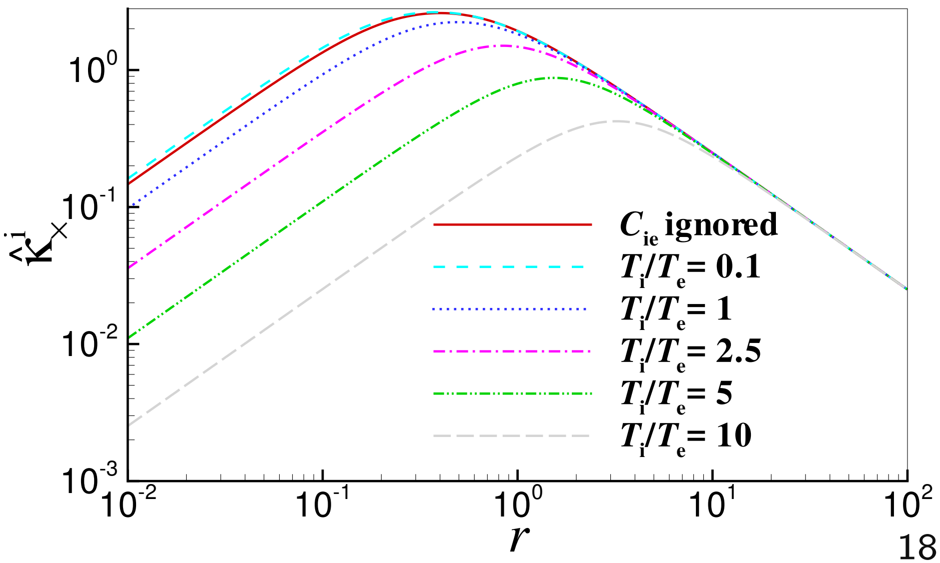
- Ion-electron collisions were ignored in $C_\mathrm{i}^1 = \frac{1}{\tau_\mathrm{ii}}(c_\mathrm{ii}^1 + \frac{1}{Z} a_\mathrm{ie}^1)$ before

$$c_\mathrm{ii} = \begin{pmatrix} \frac{2\sqrt{2}}{5} & \frac{3}{5}\sqrt{\frac{2}{7}} & \frac{1}{4}\sqrt{\frac{3}{7}} \\ \frac{3}{5}\sqrt{\frac{2}{7}} & \frac{9}{7\sqrt{2}} & \frac{103\sqrt{3}}{280} \\ \frac{1}{4}\sqrt{\frac{3}{7}} & \frac{103\sqrt{3}}{280} & \frac{5657}{3360\sqrt{2}} \end{pmatrix}$$

$$a_\mathrm{ie}^1 = \begin{pmatrix} 3\sqrt{\frac{\mu}{\theta^3}} & 0 & 0 \\ 2\sqrt{7}(\sqrt{\frac{\mu}{\theta}} - \sqrt{\frac{\mu}{\theta^3}}) & 5\sqrt{\frac{\mu}{\theta^3}} & 0 \\ 0 & 3\sqrt{6}(\sqrt{\frac{\mu}{\theta}} - \sqrt{\frac{\mu}{\theta^3}}) & 7\sqrt{\frac{\mu}{\theta^3}} \end{pmatrix}, \ \mu = \frac{m_\mathrm{e}}{m_\mathrm{i}}, \ \theta = \frac{T_\mathrm{e}}{T_\mathrm{i}}$$

- With the ion-electron collision terms kept $(r = \Omega_\mathrm{i}\tau_\mathrm{ii})$

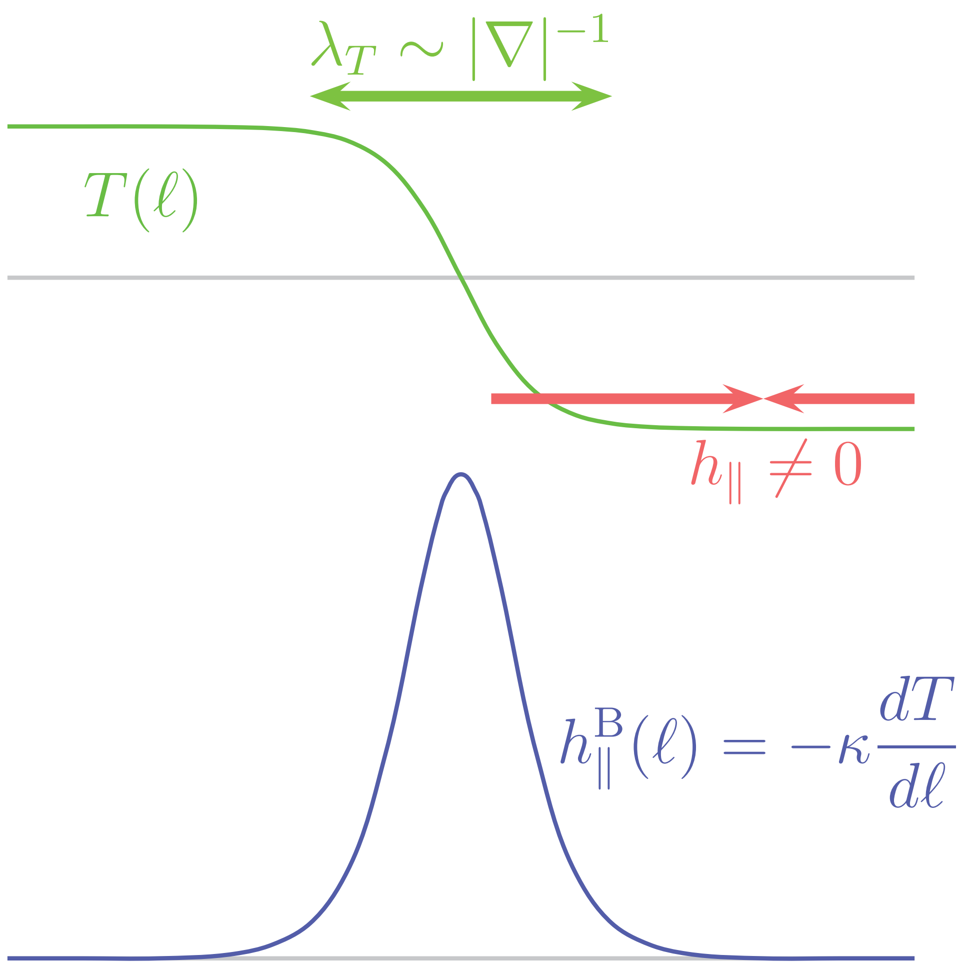




Generalizing Braginskii's parallel closures to low collisionality

- High collisionality (improved)
  - ⋆ Convergence
    with increasing moments
  - ⋆ Ion-electron collision effects
- Low or arbitrary collisionality
  - ⋆ Hammett & Perkins 1993
  - ⋆ Chang & Callen 1992
  - ⋆ Held et al 2001
- For capturing kinetic effects,
  Krook or Lorentz type
  model collision operators adopted

⇒ Exact collisional moments



21 moment $(n, T, \mathbf{V}, \mathsf{n}^{11}, \mathsf{n}^{12}, \mathsf{n}^{20}, \mathsf{n}^{21})$ equations revisited $\nabla(\cdot)\mathsf{n}^{lk}$ terms kept for parallel closures for low collisionality

- $d_t \mathsf{n}^{11} + \frac{3}{2} d_t \ln T \mathsf{n}^{11} + \Omega \mathbf{b} \times \mathsf{n}^{11} - \sqrt{\frac{2}{3}} \dot{\nabla} \mathsf{n}^{02} + \dot{\nabla} \cdot (-\sqrt{\frac{2}{5}} \mathsf{n}^{20} + \sqrt{\frac{7}{5}} \mathsf{n}^{21})$
$-\frac{\sqrt{5}}{2} n \dot{\nabla} \ln T - 2\sqrt{\frac{2}{3}} \dot{\nabla} \ln T \mathsf{n}^{02} + \dot{\nabla} \ln T \cdot (-\frac{9}{\sqrt{10}} \mathsf{n}^{20} + 2\sqrt{\frac{7}{5}} \mathsf{n}^{21}) + 2\sqrt{\frac{2}{5}} \frac{1}{v_T} \mathbf{a} \cdot \mathsf{n}^{20}$
$+\frac{7}{5} \nabla \cdot \mathbf{V} \mathsf{n}^{11} + \frac{7}{5} \mathsf{n}^{11} \cdot \nabla \mathbf{V} \cdot + \frac{2}{5} \nabla \mathbf{V} \cdot \mathsf{n}^{11} - 2\sqrt{\frac{3}{5}} \nabla \mathbf{V} : \mathsf{n}^{30} = \hat{A}^{11,00} n + \hat{C}^{11,11} \mathsf{n}^{11} + \hat{C}^{11,12} \mathsf{n}^{12}$
- $d_t \mathsf{n}^{12} + d_t \ln T(-\sqrt{7} \mathsf{n}^{11} + \frac{5}{2} \mathsf{n}^{12}) + \Omega \mathbf{b} \times \mathsf{n}^{12} + \dot{\nabla}(\sqrt{\frac{7}{6}} \mathsf{n}^{02} - \mathsf{n}^{03}) + \dot{\nabla} \cdot (-\frac{2}{\sqrt{5}} \mathsf{n}^{21} + \frac{3}{\sqrt{5}} \mathsf{n}^{22})$
$+ \cdots + \nabla \cdot \mathbf{V}(-\frac{2\sqrt{7}}{5} \mathsf{n}^{11} + \frac{9}{5} \mathsf{n}^{12}) + (-\frac{2\sqrt{7}}{5} \mathsf{n}^{11} + \frac{9}{5} \mathsf{n}^{12}) \cdot \nabla \mathbf{V} \cdot + \nabla \mathbf{V} \cdot (-\frac{2\sqrt{7}}{5} \mathsf{n}^{11} + \frac{4}{5} \mathsf{n}^{12})$
$+\nabla \mathbf{V} : (4\sqrt{\frac{3}{35}} \mathsf{n}^{30} - 6\sqrt{\frac{6}{35}} \mathsf{n}^{31}) = \hat{A}^{12,00} n + \hat{C}^{12,11} \mathsf{n}^{11} + \hat{C}^{12,12} \mathsf{n}^{12}$
- $d_t \mathsf{n}^{20} + d_t \ln T \mathsf{n}^{20} + \Omega \mathbf{b} \breve{\times} \mathsf{n}^{20} - \sqrt{\frac{2}{5}} \dot{\nabla} \mathsf{n}^{11} + \sqrt{\frac{3}{2}} \dot{\nabla} \cdot \mathsf{n}^{30}$
$-\frac{3}{\sqrt{10}} \dot{\nabla} \ln T \mathsf{n}^{11} + \frac{3}{2}\sqrt{\frac{3}{2}} \nabla \ln T \cdot \mathsf{n}^{30} + \nabla \cdot \mathbf{V} \mathsf{n}^{20} + 2\mathsf{n}^{20} \cdot \nabla \mathbf{V} + \sqrt{2} n \overline{\nabla \mathbf{V}} = \hat{C}^{20,20} \mathsf{n}^{20} + \hat{C}^{20,21} \mathsf{n}^{21}$
- $d_t \mathsf{n}^{21} + d_t \ln T(-\sqrt{\frac{7}{2}} \mathsf{n}^{20} + 2\mathsf{n}^{21}) + \Omega \mathbf{b} \breve{\times} \mathsf{n}^{21} + \dot{\nabla}(\sqrt{\frac{7}{5}} \mathsf{n}^{11} - \frac{2}{\sqrt{5}} \mathsf{n}^{12}) + \dot{\nabla} \cdot (-\sqrt{\frac{3}{7}} \mathsf{n}^{30} + 3\sqrt{\frac{3}{14}} \mathsf{n}^{31}$
$-2\sqrt{\frac{7}{5}} \frac{1}{v_T} \mathbf{a} \mathsf{n}^{11} + 2\sqrt{\frac{3}{7}} \frac{1}{v_T} \mathbf{a} \cdot \mathsf{n}^{30} + \dot{\nabla} \ln T(\frac{7}{2}\sqrt{\frac{7}{5}} \mathsf{n}^{11} - \sqrt{5} \mathsf{n}^{12}) + \dot{\nabla} \ln T \cdot (-6\sqrt{\frac{3}{7}} \mathsf{n}^{30} + \frac{15}{2}\sqrt{\frac{3}{14}} \mathsf{n}^{31})$
$+\nabla \cdot \mathbf{V}(-\sqrt{\frac{2}{7}} \mathsf{n}^{20} + \frac{9}{7} \mathsf{n}^{21}) + (-2\sqrt{\frac{2}{7}} \mathsf{n}^{20} + \frac{18}{7} \mathsf{n}^{21}) \cdot \nabla \mathbf{V} \cdot + \nabla \mathbf{V} \cdot (-2\sqrt{\frac{2}{7}} \mathsf{n}^{20} + \frac{4}{7} \mathsf{n}^{21})$
$-2\sqrt{\frac{14}{15}} \overline{\nabla \mathbf{V}} \mathsf{n}^{02} - 2\sqrt{\frac{6}{7}} \nabla \mathbf{V} : \mathsf{n}^{40} = \hat{C}^{21,20} \mathsf{n}^{20} + \hat{C}^{21,21} \mathsf{n}^{21}$

For $\lambda_C / |\nabla_\perp^{-1}| \lesssim 1$ instead of $\lambda_C / |\nabla_\perp^{-1}| \ll 1$ (Braginskii),

- Mikhailovskii and Tsypin 1971 kept some $\dot{\nabla} \ln T$ and $\mathbf{a}$ terms and n^{02} and n^{03} equations but no higher order moments n^{30}, n^{31}, $\cdots$
- Catto and Simakov 2014 calculated up to f_2 in the drift ordering

$\Rightarrow$ May be a good approximation for transverse closures but not for parallel

Parallel moment equations for closures (in matrix form)

$$[\psi]\partial_\eta[n]+\{\partial_\eta \ln B[\Psi_B]+[\Phi](\partial_\eta \ln T)+\hat{E}_\|[\Theta]\}[n]=[c]\bar{n}_\|+[g]$$

- Notation simplified $n^{lk}=\bar{n}_\|^{lk}$, $d\eta=d\ell/\lambda_\mathrm{C}$, $\lambda_\mathrm{C}=v_T\tau$

$$\begin{bmatrix} 0 & \psi^0 & & & \\ \tilde{\psi}^0 & 0 & \psi^1 & & \\ & \tilde{\psi}^1 & 0 & \psi^2 & \\ & & \tilde{\psi}^2 & 0 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}\begin{bmatrix}\partial_\eta\vec{n}^0\\ \partial_\eta\vec{n}^1\\ \partial_\eta\vec{n}^2\\ \partial_\eta\vec{n}^3\\ \vdots\end{bmatrix}=\begin{bmatrix}c^0\vec{n}^0\\ c^1\vec{n}^1\\ c^2\vec{n}^2\\ c^3\vec{n}^3\\ \vdots\end{bmatrix}+\begin{bmatrix}0\\ \vec{g}^1\\ \vec{g}^2\\ 0\\ \vdots\end{bmatrix}$$

where $\hat{V}_{\mathrm{ei}\|}=\frac{\mathbf{b}\cdot(\mathbf{V}_\mathrm{e}-\mathbf{V}_\mathrm{i})}{v_T}$, $W_\|=\mathbf{bb}:\mathsf{W}$, $(\mathsf{W})_{\alpha\beta}=\partial_\alpha V_\beta+\partial_\beta V_\alpha-\frac{2}{3}\delta_{\alpha\beta}\nabla\cdot\mathbf{V}$,

$$\vec{n}^0=\begin{pmatrix}n^{02}\\ n^{03}\\ n^{04}\\ \vdots\end{pmatrix},\ \vec{n}^1=\begin{pmatrix}n^{11}\\ n^{12}\\ n^{13}\\ \vdots\end{pmatrix},\ \vec{n}^{l\geq 2}=\begin{pmatrix}n^{l0}\\ n^{l1}\\ n^{l2}\\ \vdots\end{pmatrix}$$

$$\vec{g}^1=\begin{pmatrix}\sqrt{2}Za_\mathrm{ei}^{110}n\hat{V}_{\mathrm{ei}\|}+\frac{\sqrt{5}}{2}\frac{n}{T}\frac{dT}{d\eta}\\ \sqrt{2}Za_\mathrm{ei}^{120}n\hat{V}_{\mathrm{ei}\|}\\ \sqrt{2}Za_\mathrm{ei}^{130}n\hat{V}_{\mathrm{ei}\|}\\ \vdots\end{pmatrix},\ \vec{g}^2=\begin{pmatrix}-\frac{\sqrt{3}}{2}n\tau_\mathrm{ee}W_\|\\ 0\\ 0\\ \vdots\end{pmatrix}$$

Solving the linear ODE system

- Linear system of ordinary differential equations

$$[\Psi]\frac{d[n]}{d\eta} = [c][n] + [g] \;\Rightarrow\; \frac{d[n]}{d\eta} = [\Psi^{-1}c][n] + [\Psi^{-1}g]$$

- Diagonalize using the eigensystem $[\Psi^{-1}c][W_A] = k_A[W_A]$

$$n_A(\eta) = \sum_D \int^\eta \underbrace{\sum_{BC} W_{AB} W^{-1}_{BC} \Psi^{-1}_{CD}}_{\gamma^B_{AD}} e^{k_B(\eta-\eta')} g_D(\eta')d\eta'$$

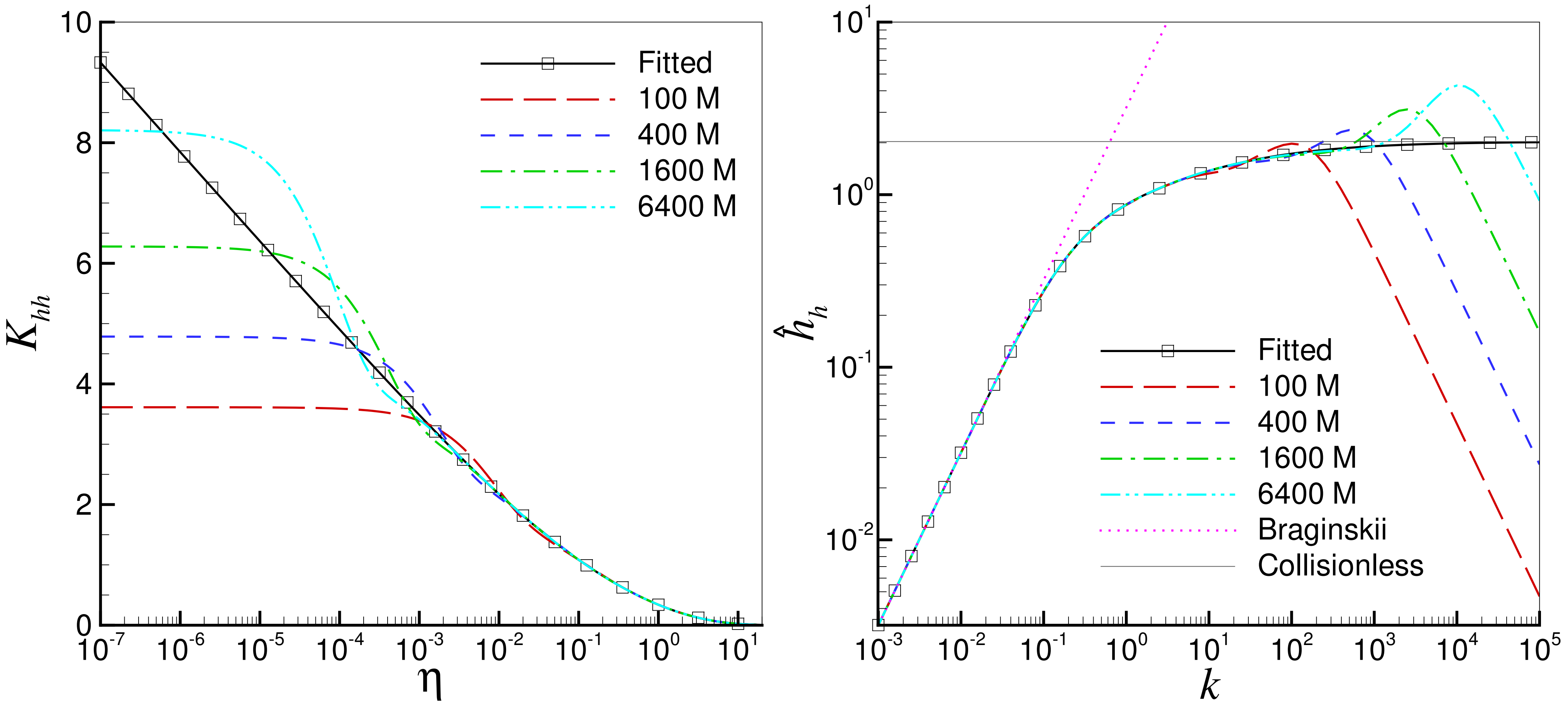
- Integral form of closures $n_A(\eta) = \sum_D \int_{-\infty}^{\infty} K_{AD}(\eta-\eta') g_D(\eta')d\eta'$

where Kernel $K_{AD}(\eta>0) = \sum_{\{B|k_B<0\}} \gamma^B_{AD} e^{k_B \eta}$ (too many terms)

- For sinusoidal drives $(k_g \ell = \frac{2\pi}{\lambda_g}\ell = 2\pi\frac{\lambda_C}{\lambda_g}\eta = k\eta)$

$$\int_{-\infty}^{\infty} K_{AD}(\eta-\eta')\cos k\eta' d\eta' = \begin{cases} \sum_{B=1}^{N} \frac{-\gamma^B_{AD} k_B}{k_B^2+k^2}\cos k\eta, & AD = hh, hR, RR, \pi\pi \\ \sum_{B=1}^{N} \frac{\gamma^B_{AD} k}{k_B^2+k^2}\sin k\eta, & AD = h\pi, R\pi \end{cases}$$

Convergence test
Heat flow (n^{11}) for $T = T_0(1 + \epsilon_T \sin k\eta)$, $k = 2\pi\lambda_C/\lambda_T$



- SSPX, 1600 M, better agreement with temperature measurements [Ji et al 2009 PoP]
- SOL, 900 M, capturing kinetic effects and more efficient than PIC [Omotani & Dudson 2013 PPCF]

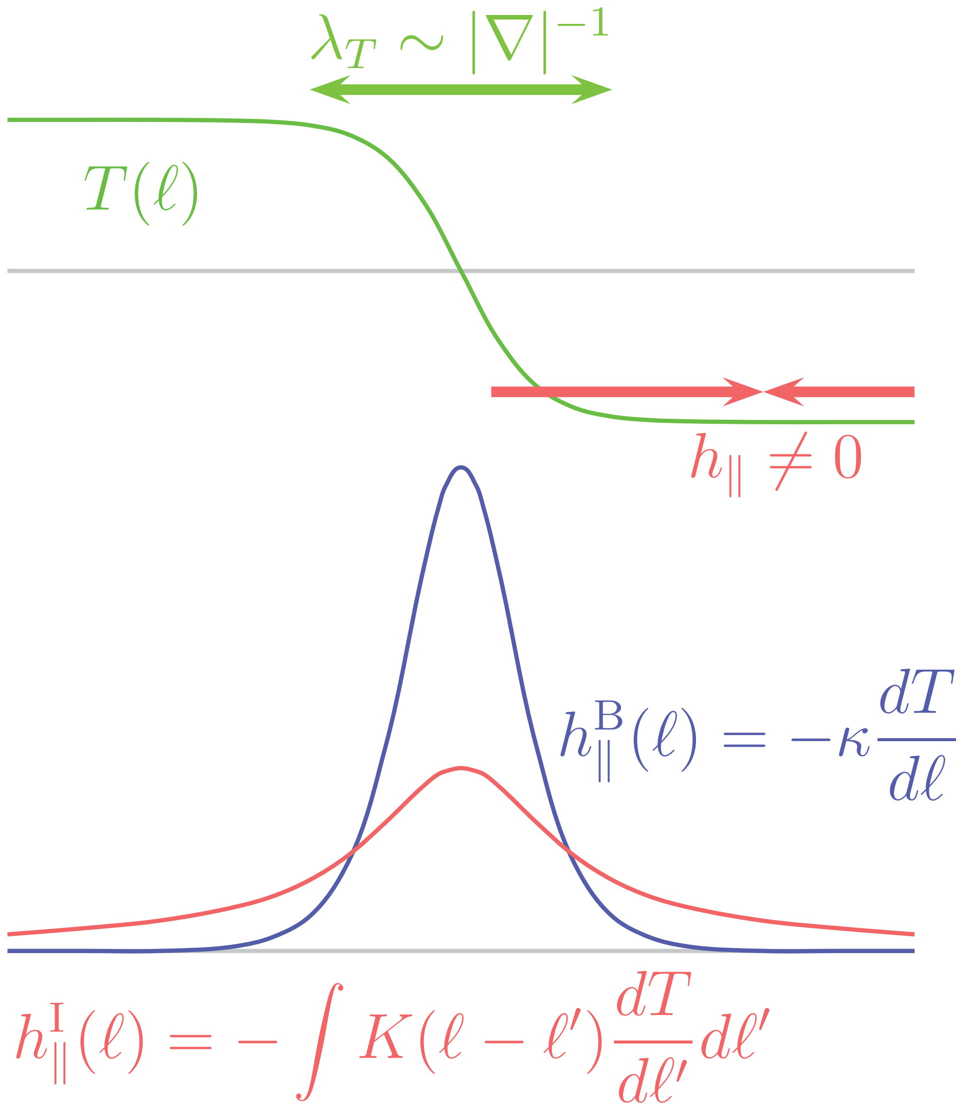
Parallel heat flow: Braginskii vs. integral closures

- Integral closures
  - ⋆ For convergence, lower collisionality requires more moments
  - ⋆ For $k = \lambda_C / |\nabla^{-1}| \sim 100$, 6400 moments

$$K(\eta) = \sum_{\{B|k_B<0\}}^{6400} \gamma_B e^{k_B \eta}$$

$\Rightarrow$ Find simple fitted functions

- ⋆ For the collisionless limit, solve a kinetic equation with $C(f) = -\nu(f - f^{\mathrm{M}})$ and take the $\nu \to 0$ limit



Obtaining fitted kernel functions

- Fitted to the 6400 moment solution in the convergent regime

$$\text{error}\lesssim 1\ \% \quad \text{for} \quad k = 2\pi\frac{\lambda_C}{\lambda_g} \lesssim 100 \text{ (nearly collisionless)}$$

- In the $z = \dfrac{\ell}{\lambda_C} \to 0$ limit, let kernels behave asymptotically collisionless

Collisional limit [Braginskii]	Collisionless limit [Ji et al 2013 PoP]
$h_\parallel = -3.20\frac{nT\tau_{ee}}{m}\partial_\parallel T + 0.703nTV_{ei\parallel}$	$h_\parallel(\eta) = \frac{9}{5\pi^{3/2}}n_0 v_0 \int_{-\infty}^{\infty} d\eta' \frac{T_1(\eta')}{\eta-\eta'} - \frac{2}{5}p_0 V_\parallel(\eta)$
$R_\parallel = -0.703n\partial_\parallel T - 0.504\frac{mn}{\tau_{ei}}V_{ei\parallel}$	$R_\parallel = 0$
$\pi_\parallel = -0.978nT\tau_{ee}\frac{3}{4}W_\parallel$	$\pi_\parallel(\eta) = -\frac{2}{5}n_0 T_1(\eta) + \frac{4}{5\sqrt{\pi}}\frac{p_0}{v_0}\int_{-\infty}^{\infty} d\eta' \frac{V_\parallel(\eta')}{\eta-\eta'}$

- Asymptotic behavior for $\eta \ll 1$

$$\begin{aligned} K_{hh}(\eta) &\approx -\frac{18}{5\pi^{3/2}}(\ln|\eta| + \gamma_h) \\ K_{h\pi}(\eta) &\approx \frac{1}{5} \\ K_{\pi\pi}(\eta) &\approx -\frac{4}{5\pi^{1/2}}(\ln|\eta| + \gamma_\pi) \end{aligned}$$

Integral (nonlocal) parallel closures with fitted kernels

[Ji et al 2014 2016 2017 PoP]

- Closure n_A responding to g_B:

$$n_{AB}(\eta) = \int d\eta' K_{AB}(\eta - \eta') g_B(\eta'), \quad A, B = h, R, \pi$$

$$h_\parallel(\eta) = -\frac{1}{2} T v_T \int d\eta' K_{hh} \frac{n}{T} \frac{dT}{d\eta'} + T v_T \int d\eta' K_{hR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - T v_T \int d\eta' K_{h\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

$$R_\parallel(\eta) = -\frac{mn}{\tau_{\mathrm{ei}}} V_{\mathrm{ei}\parallel} + \frac{m v_T}{\tau_{\mathrm{ei}}} \int d\eta' [-K_{Rh} \frac{n}{2T} \frac{dT}{d\eta'} + K_{RR} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - K_{R\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)]$$

$$\pi_\parallel(\eta) = -T \int d\eta' K_{\pi h} \frac{n}{T} \frac{dT}{d\eta'} + 2T \int d\eta' K_{\pi R} Z n \frac{V_{\mathrm{ei}\parallel}}{v_T} - T \int d\eta' K_{\pi\pi} (\frac{3}{4} n \tau_{\mathrm{ee}} W_\parallel)$$

- Fitted kernel functions

$$K_{AB}(\eta) = -[d + a \exp(-b\eta^c)] \ln[1 - \alpha \exp(-\beta \eta^\gamma)]$$

	a	b	c	d	α	β	γ
K_{hh}	-5.32	0.170	0.646	6.87	1	2.02	0.417
K_{hR}	6.37	5.12	0.160	0.100	1	1	0.583
$K_{h\pi}$	-0.229	2.26	0.594	0.363	0.775	1.49	0.478
K_{RR}	245	8.06	0.147	0.432	1	3.40	0.347
$K_{R\pi}$	-0.226	3.21	0.678	0.696	1	3.40	0.347
$K_{\pi\pi}$	0.724	0.932	0.654	0.195	1	1.60	0.491

Parallel closures in k space (for single harmonic drives)

- Fourier transform (FT) of integral closures: $\tilde{n}_{AB}(k) = \tilde{K}_{AB}(k)\tilde{g}_B(k)$

$$\tilde{h}_\parallel = -\frac{1}{2}n_0 v_0 \tilde{K}_{hh} ik\tilde{T}_1 + p_0 \tilde{K}_{hR}\tilde{u}_{\mathrm{ei}} - p_0 \tilde{K}_{h\pi} ik\tilde{u}$$

$$\tilde{R}_\parallel = -\frac{1}{2}\frac{m v_0}{\tau_{\mathrm{ee}}}\frac{n_0}{T_0}\tilde{K}_{Rh} ik\tilde{T}_1 - \frac{m n_0}{\tau_{\mathrm{ee}}}\left(1 - \tilde{K}_{RR}\right)\tilde{u}_{\mathrm{ei}} - \frac{m n_0}{\tau_{\mathrm{ee}}}\tilde{K}_{R\pi} ik\tilde{u}$$

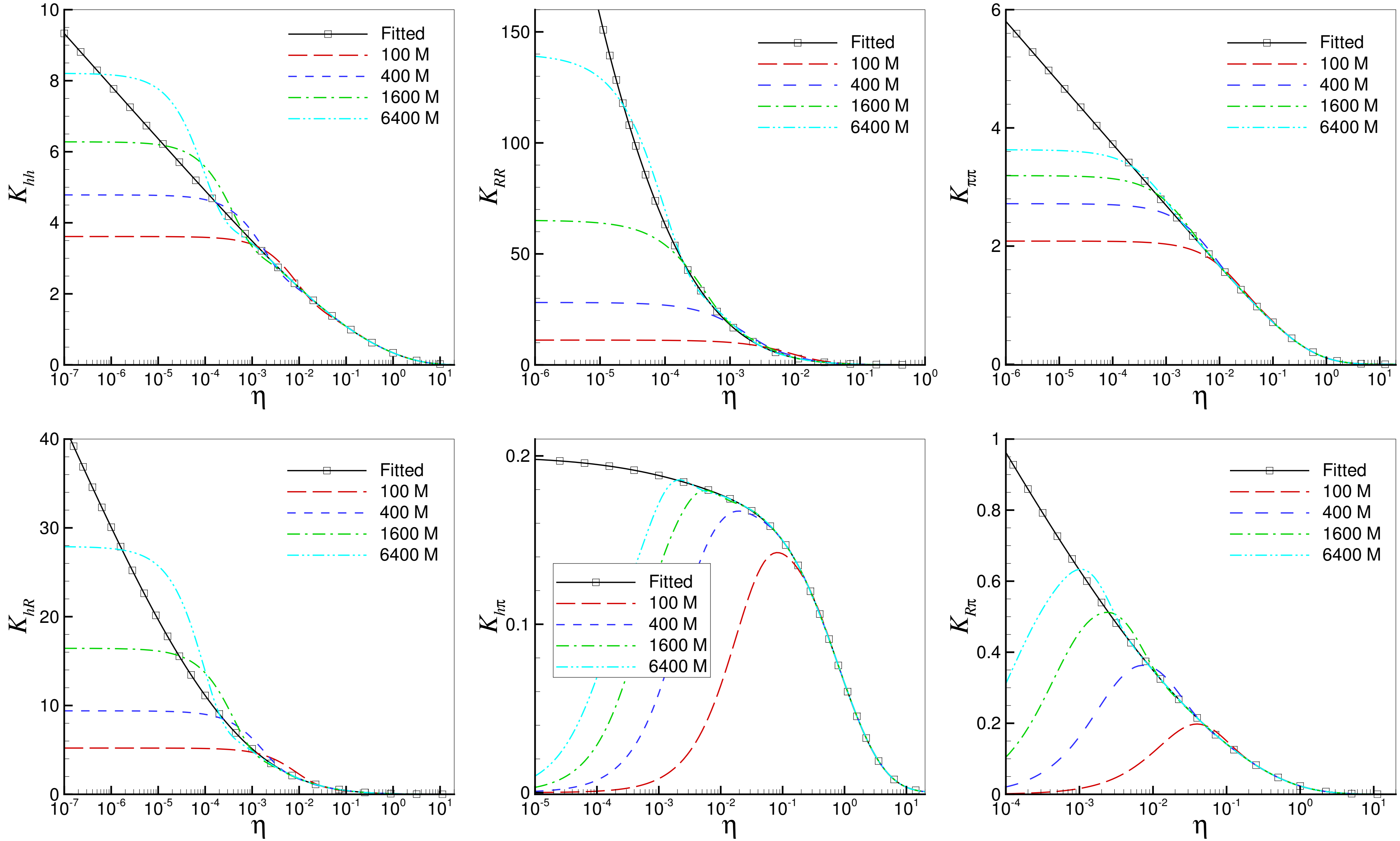
$$\tilde{\pi}_\parallel = -n_0 \tilde{K}_{\pi h} ik\tilde{T}_1 + 2\frac{p_0}{v_0}\tilde{K}_{\pi R}\tilde{u}_{\mathrm{ei}} - \frac{p_0}{v_0}\tilde{K}_{\pi\pi} ik\tilde{u}$$

- Fitted kernels in k space

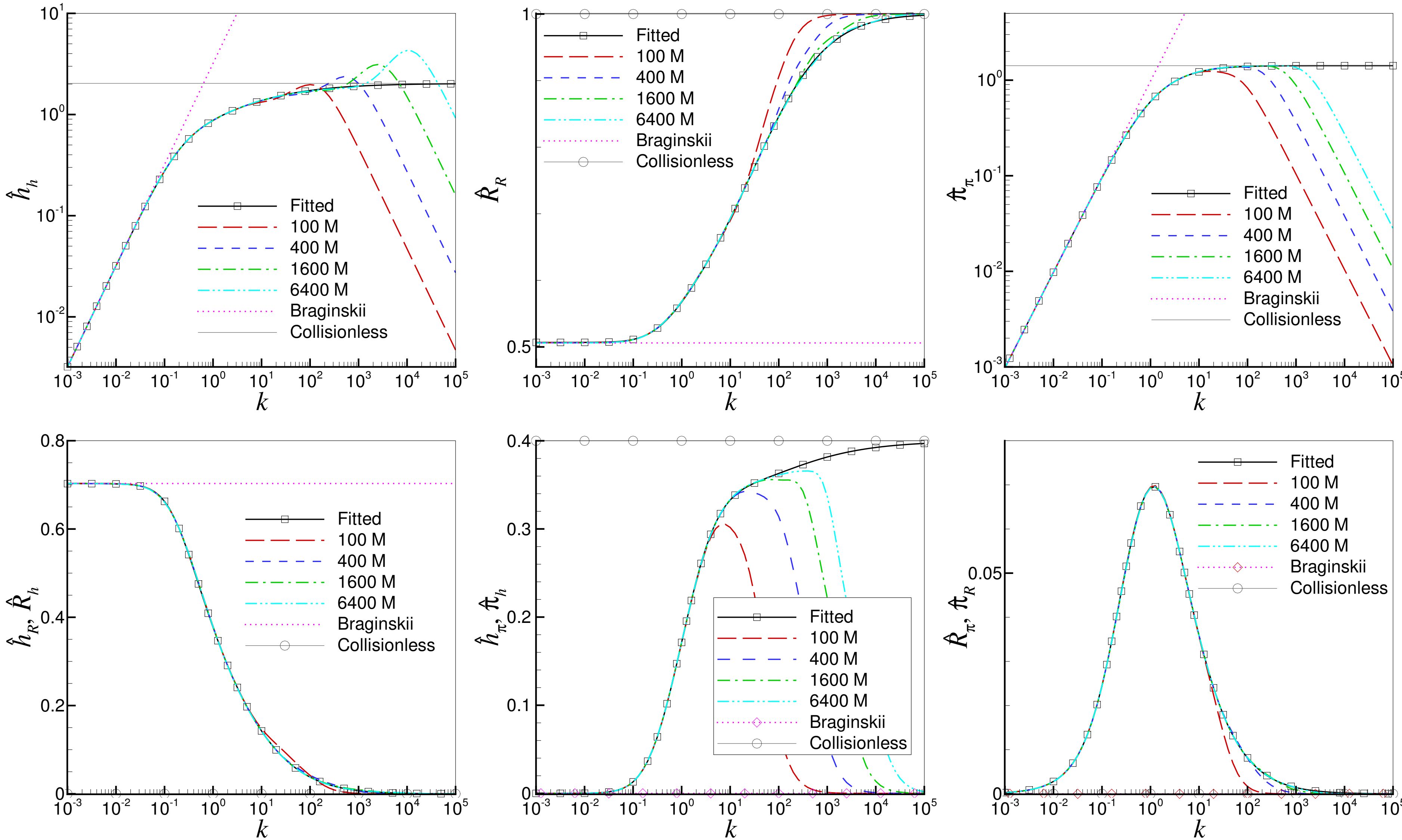
$$\tilde{K}_{AB} = \frac{a k^\alpha}{1 + d_1 k^\delta + d_2 k^{2\delta} + d_3 k^{3\delta} + d_4 k^{4\delta} + d_5 k^{5\delta} + d_6 k^{6\delta}}$$

	a	α	δ	d_1	d_2	d_3	d_4	d_5	d_6	err.
$\tilde{K}_{hh}$	3.20	0	1/6	0.675	-2.65	1.30	1.97	-0.279	1.58	2.9%
$\tilde{K}_{hR}$	0.703	0	1/6	0	0.837	-4.94	8.43	-4.33	0.824	2.7%
$i\tilde{K}_{h\pi}$	1.83	1	1/3	1.497	-9.36	21.7	-12.9	4.11	4.575	1.3%
$\tilde{K}_{RR}$	0.494	0	1/6	0.0648	-0.431	0.670	-0.198	0.0347	0	1.0%
$i\tilde{K}_{R\pi}$	0.284	1	1/3	0.387	-2.67	6.71	-3.39	2.06	0	1.5%
$\tilde{K}_{\pi\pi}$	0.978	0	1/6	0	0.337	-1.37	1.35	-0.375	0.690	0.8%

Fitted kernel functions

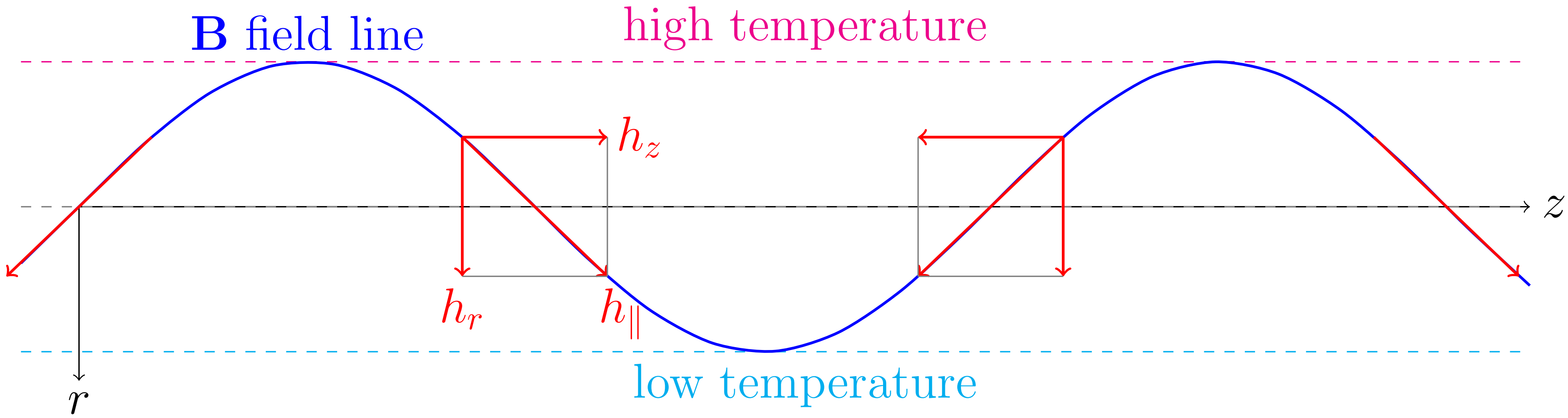


Closures responding to sinusoidal drives (kernels in k space)



A1. Radial heat flow due to a sinusoidal magnetic field line

[Ji et al 2016 PPCF]



- Magnetic field fluctuation introduces temperature fluctuation
$$r = r_0 + r_\delta \sin \frac{2\pi}{\lambda} z \to T = T_0 + T_\delta \sin \frac{2\pi}{\lambda} z \text{ where } T_\delta = \frac{dT_0}{dr} r_\delta$$
- For small fluctuation $B_\delta \ll B_0$ (RMP $B_\delta / B_0 \sim 10^{-4}$)
$$\langle h_r \rangle = -\frac{1}{8\pi} n_0 v_T \hat{h}(k) \lambda \left(\frac{B_\delta}{B_0} \right)^2 \frac{dT_0}{dr}, \qquad \chi_{\text{eff}} = \frac{1}{8\pi} v_T \hat{h}(k) \lambda \left(\frac{B_\delta}{B_0} \right)^2$$
- Collisionless limit $\lim_{k \to \infty} \hat{h}(k) \approx 2.03 \approx 2$

  Rechester-Rosenbluth $\chi_{\text{eff}}^{\text{RR}} = \pi R v_T \left(\frac{B_\delta}{B_0} \right)^2$ ($2\pi R$ is the period of the system)

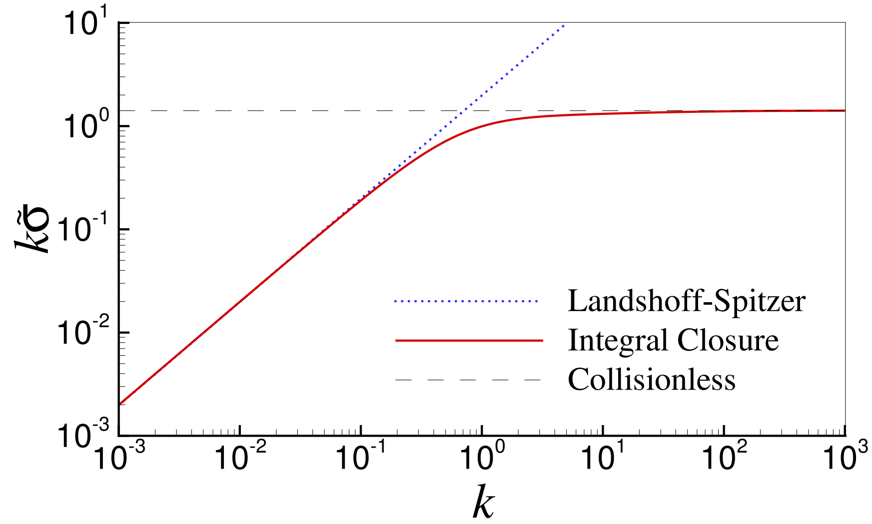
$$\frac{\chi_{\text{eff}}}{\chi_{\text{eff}}^{\text{RR}}} = \frac{1}{4\pi^2} \frac{\lambda}{R} = \frac{1}{2\pi n} \text{ for } \lambda = \frac{2\pi R}{n}$$

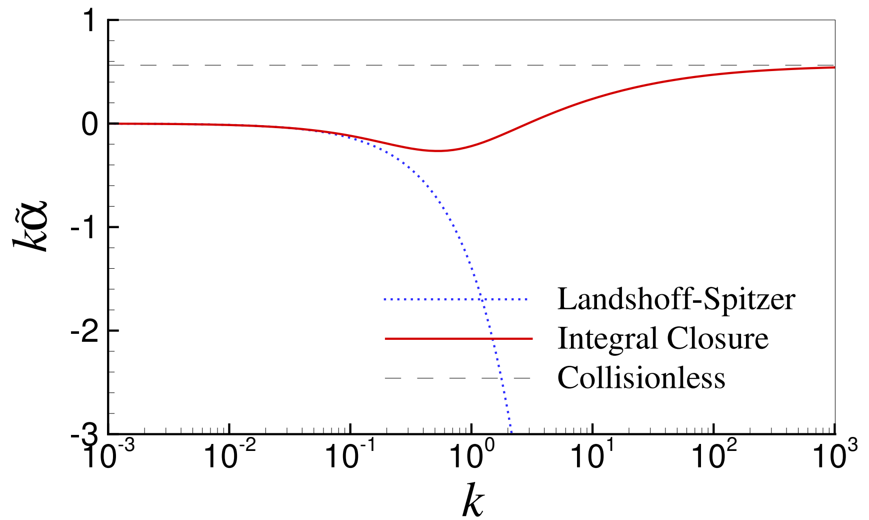
A2. Electron parallel transport for arbitrary collisionality

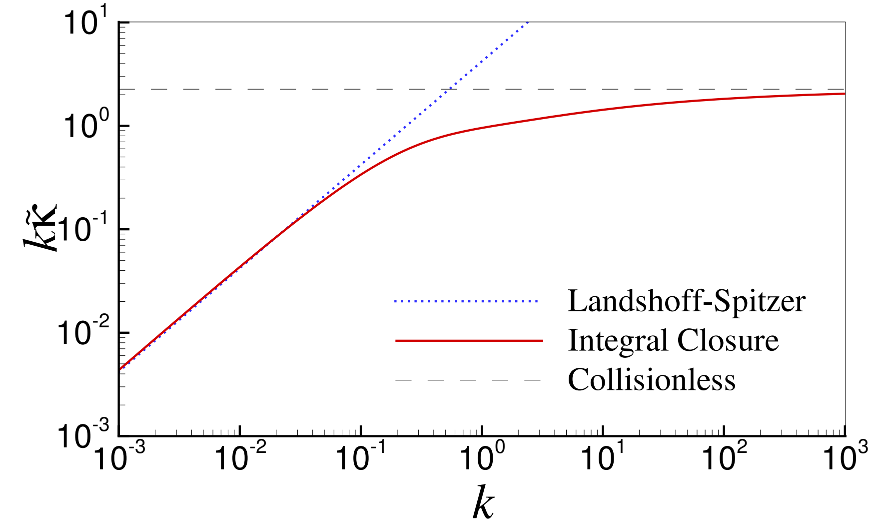
[Ji et al 2017 PoP]

- Momentum balance equation with integral closures $ne\tilde{E}_\parallel + ik_\parallel \tilde{p} + ik_\parallel \tilde{\pi}_\parallel = \tilde{R}_\parallel$
- Transport relations in k space

$$
\begin{aligned}
\tilde{J}_\parallel &= \frac{n_0 e^2 \tau_{\text{ei}}}{m} \tilde{\sigma}_\parallel \tilde{E}_\parallel - \frac{n_0 e \tau_{\text{ei}}}{m} \tilde{\alpha}_\parallel ik\tilde{T} \\
\tilde{h}_\parallel &= \frac{n_0 e T_0 \tau_{\text{ei}}}{m} \tilde{\alpha}_\parallel \tilde{E}_\parallel - \frac{n_0 T_0 \tau_{\text{ei}}}{m} \tilde{\kappa}_\parallel ik\tilde{T}
\end{aligned}
$$







⋆ Magnetic reconnection time $\dfrac{\tau_{\text{rec}}}{\tau_{\text{rec}}^{\text{Spitzer}}} = \dfrac{\sigma_\parallel}{\sigma_\parallel^{\text{Spitzer}}} \xrightarrow{k\to\infty} \dfrac{1}{1.4k} = \dfrac{1}{8.8}\dfrac{\lambda}{\lambda_{\text{C}}}$

Summary and future work

- Perfected Braginskii closure theory for high collisionality $(\mathrm{Kn} \ll 1)$
  - Obtained convergent solutions
  - Included ion-electron collision effects for the ion closures
  - Arbitrary ion charge number Z and Hall parameter x
- Integral (nonlocal) parallel closures
  - Arbitrary collisionality
  - Electron for $1 \leq Z \leq 10$ [Ji et al 2016 PoP]
    - Being extended to $Z > 10$
  - Ion for $T_\mathrm{i}/T_\mathrm{e} \lesssim 10$ and arbitrary Z [Ji et al 2017 PoP]
- Parallel closures with inhomogeneous coupling terms
  $[\psi]\partial_\parallel[n] + \{\partial_\parallel \ln B[\Psi_B] + [\Phi](\partial_\parallel \ln T) + \hat{E}_\parallel[\Theta]\}[n] = [c][n] + [g]$
  - Fourier method
  - $\partial_\parallel \ln B$: neoclassical effects
- Impurity transport: multiple ion species
- Partially ionized plasmas: ionization, recombination, and charge exchange operators

Thanks

and

enjoy the Moments!