Parametric Instability, Inverse Cascade, and the $1/f$ Range of Solar-Wind Turbulence

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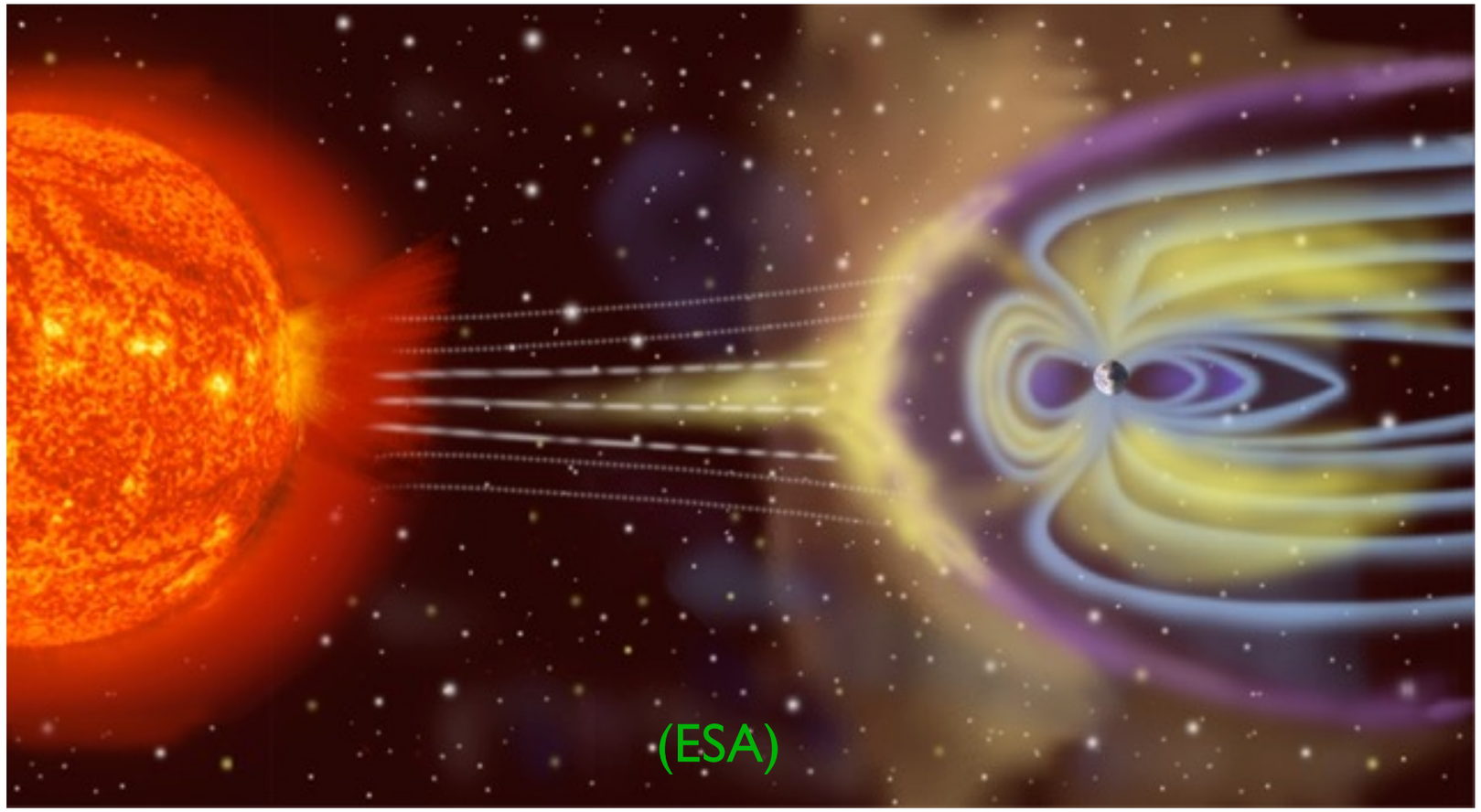
Outline

- Background on the Solar Wind
- Parametric instability and weak turbulence theory
- Nonlinear evolution of the parametric instability when slow waves are strongly damped
- Predictions

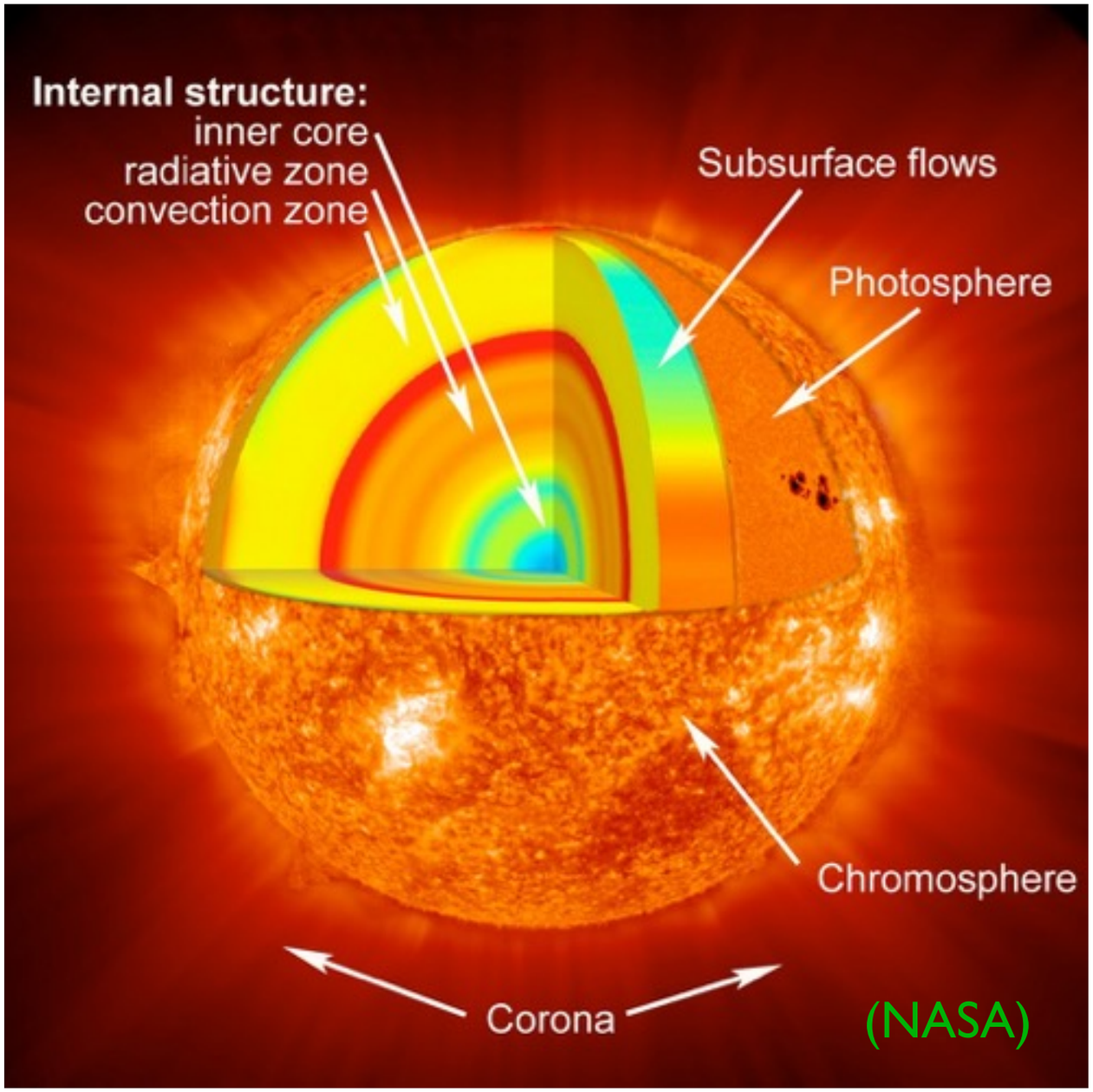
What is the Solar Wind?



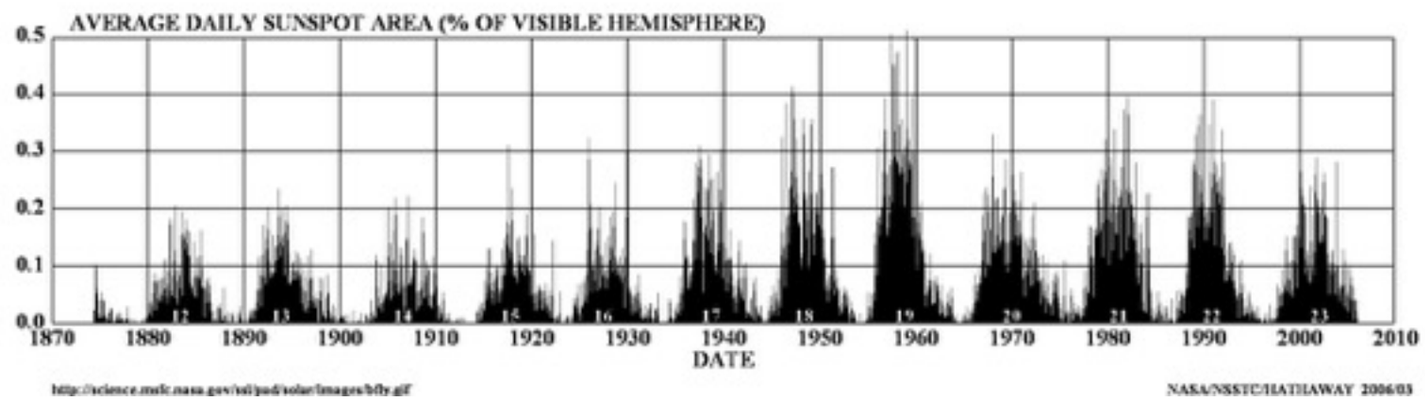
- Quasi-continuous, radial outflow of particles from the Sun
- Fast (300 - 800 km/s), hot and dilute (10^5 K, 5 cm^{-3} at 1 AU)
- Plasma: behaves like a fluid, but it generates and is in turn influenced by electromagnetic fields

The Solar Wind in Relation to the Sun

- it is the extension of the solar atmosphere
- represents about 10^{-6} of the energy output of the Sun
- mass loss rate is about 10^{-14} M_{sun} yr^{-1}



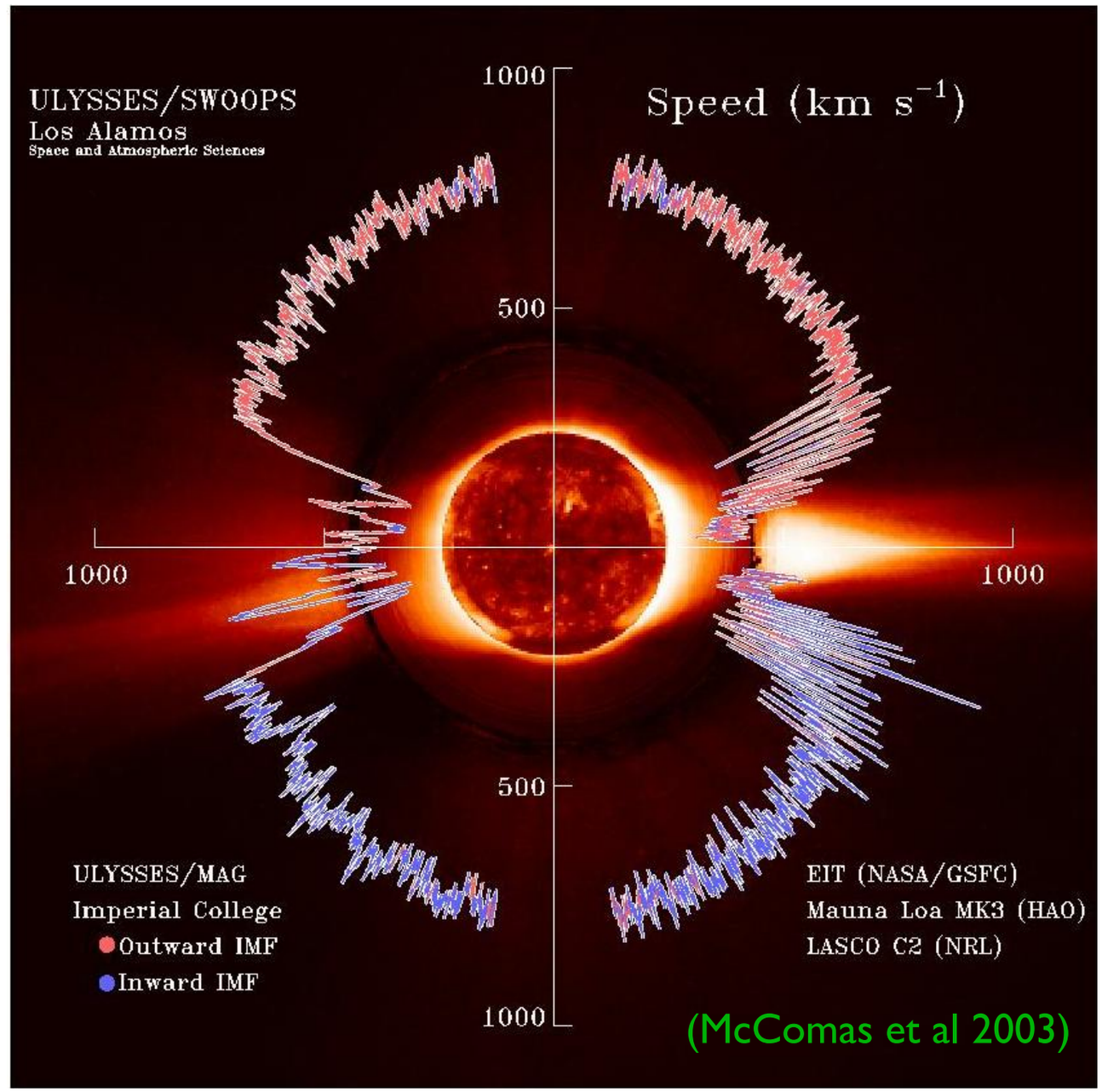
Solar-Wind Structure Depends on the Solar Cycle



- “Solar minimum” - very few sunspots.
- “Solar maximum” - many sunspots, solar flares, and coronal mass ejections.

3D Structure Near Solar Minimum

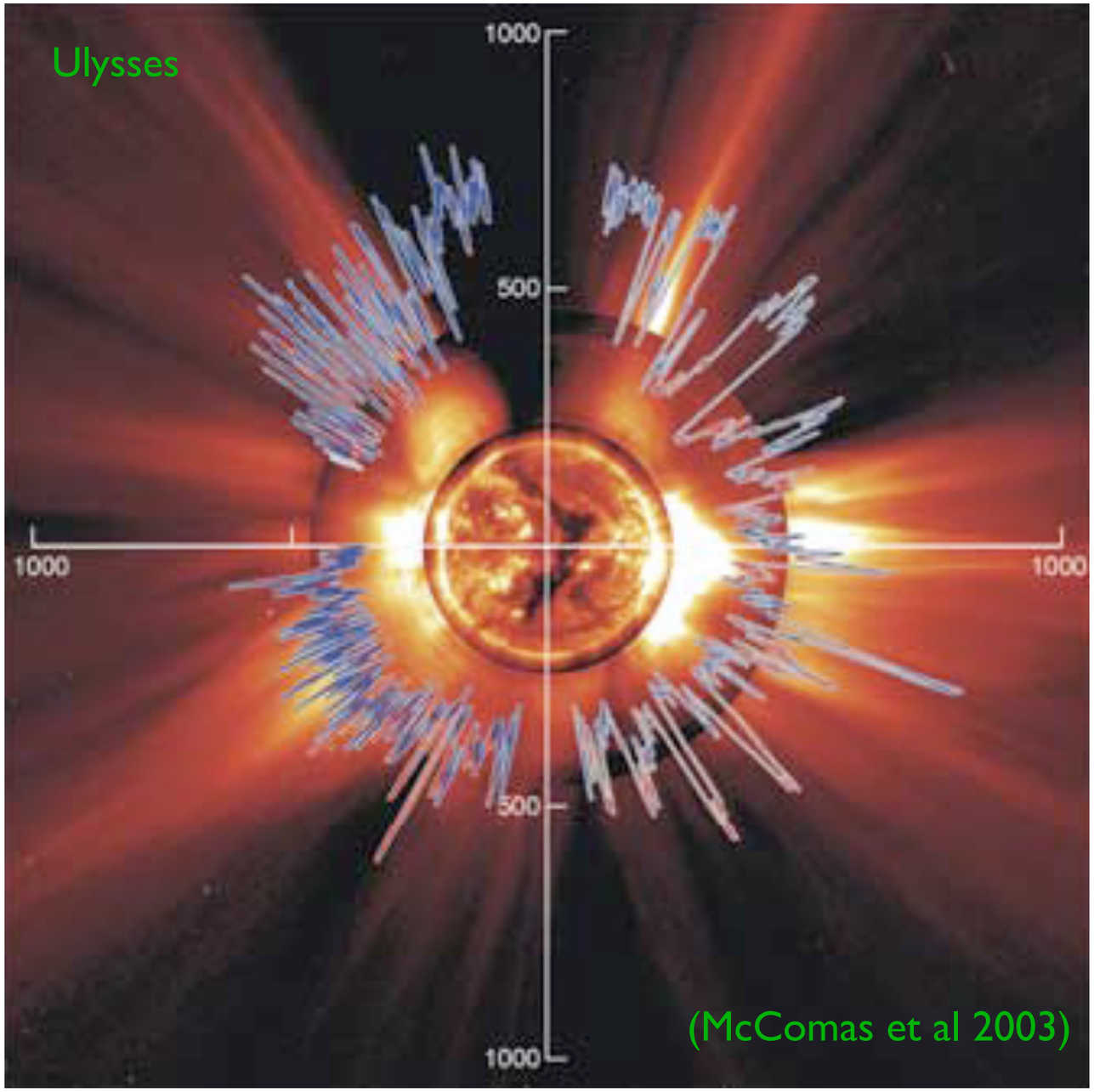
- Fast wind (700-800 km/s) is the basic state of the flow near solar minimum
- Fast wind emanates primarily from open-field-line regions near the poles - “polar coronal holes”
- Slow wind (300-500 km/s) is confined to low latitudes (less so at solar maximum)



(McComas et al 2003)

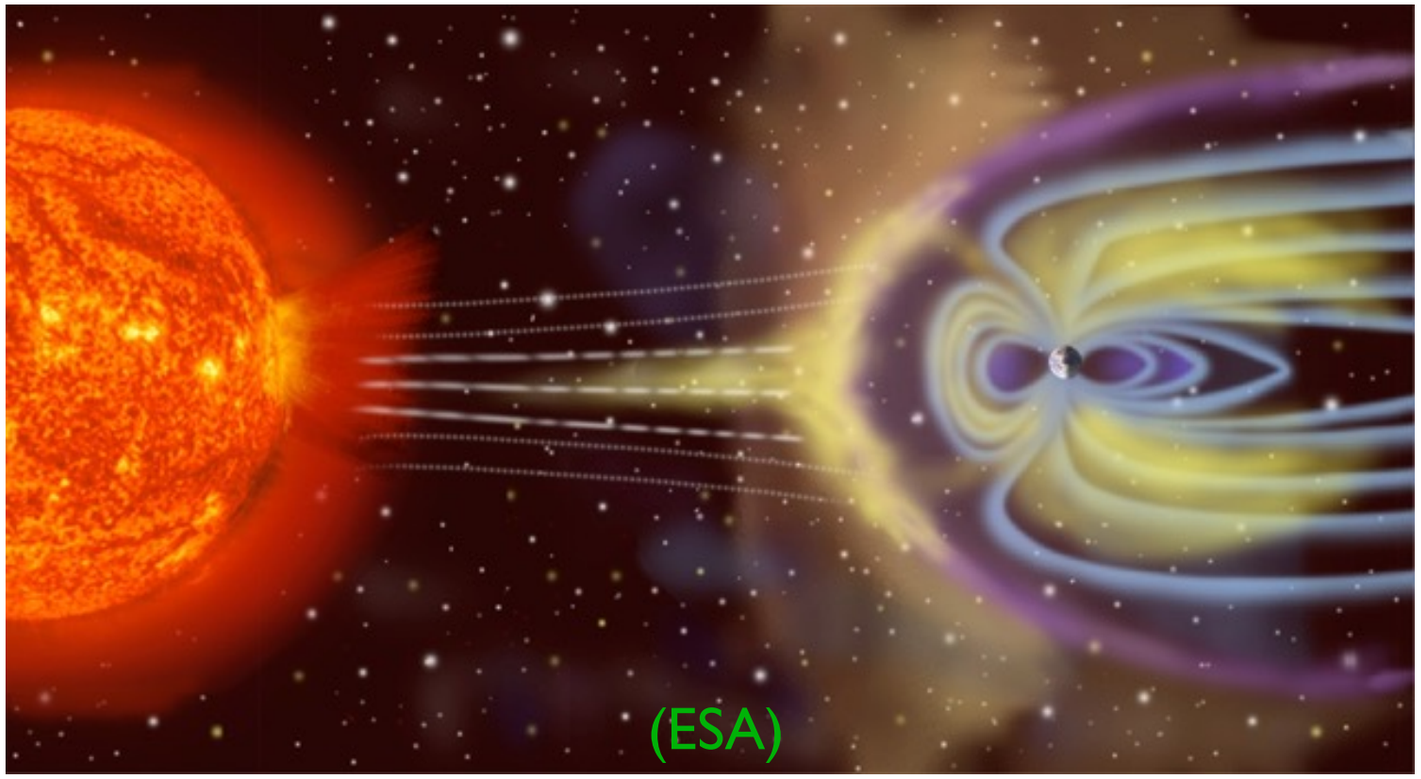
3D Structure Near Solar Maximum

- much more complex
- alternates between fast and slow wind at virtually all heliographic latitudes



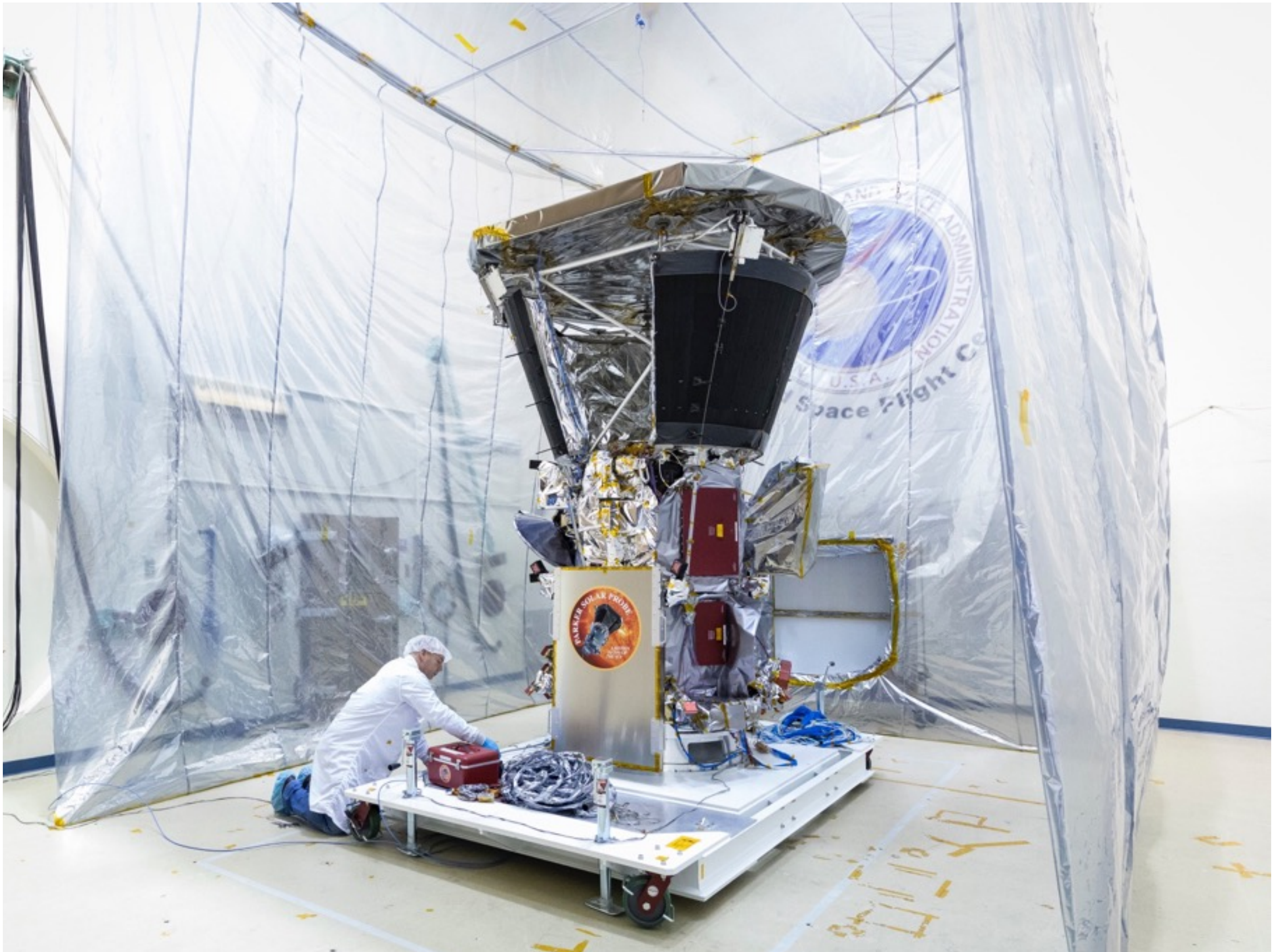
(McComas et al 2003)

Why Study the Solar Wind?



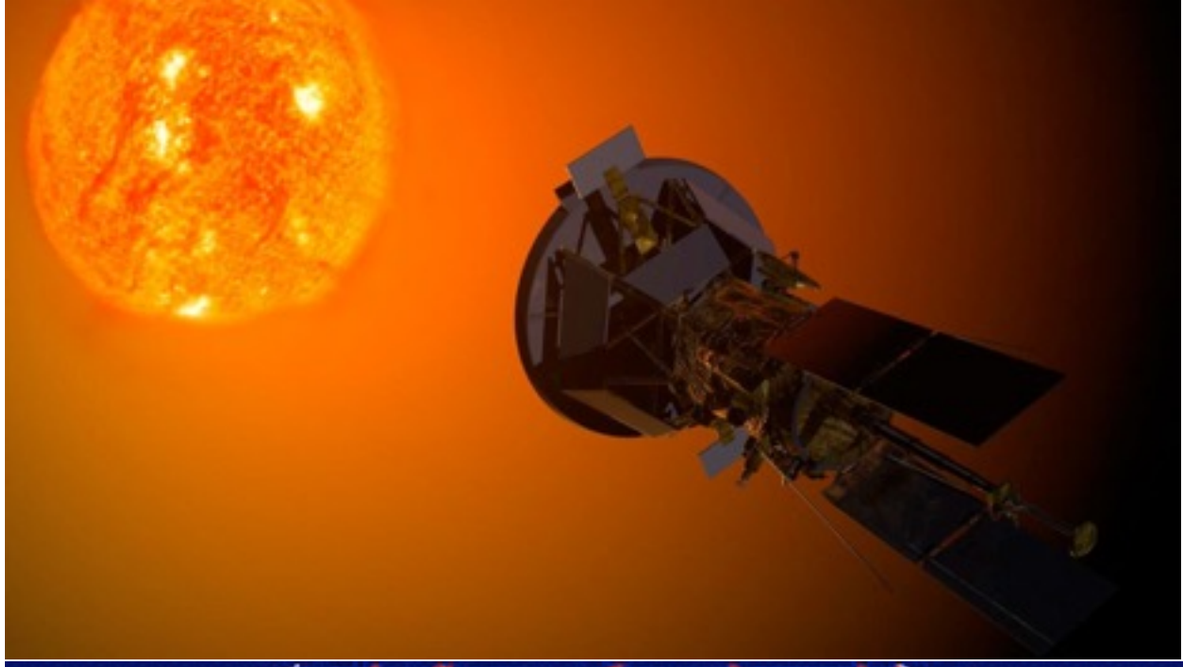
- It is a fundamental property of the Sun and a model for other astrophysical outflows.
- It is the backdrop for much of space physics - propagation of energetic particles, shock waves, space weather.
- Laboratory for plasma physics.

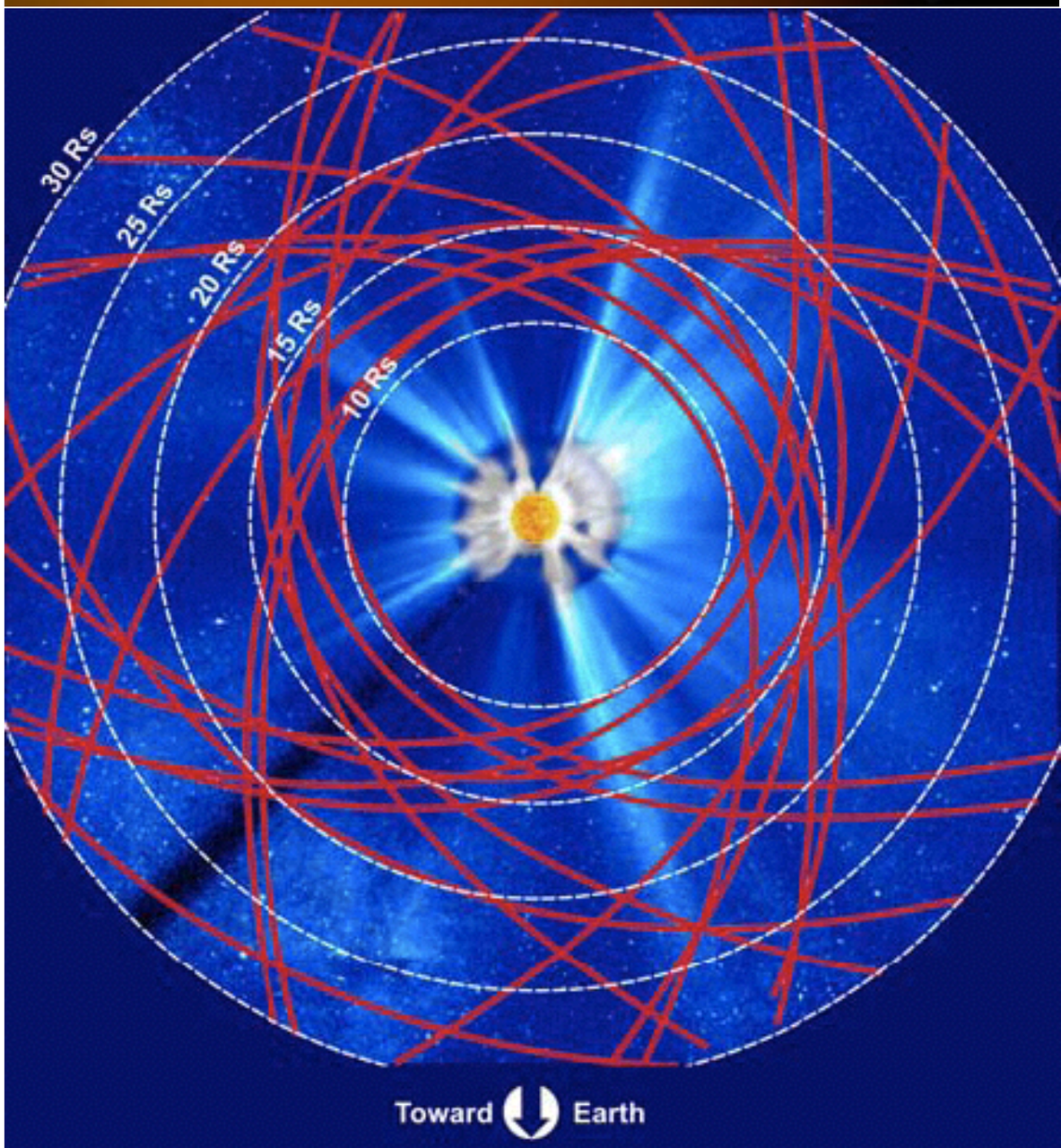
NASA's Parker Solar Probe



Parker Solar Probe

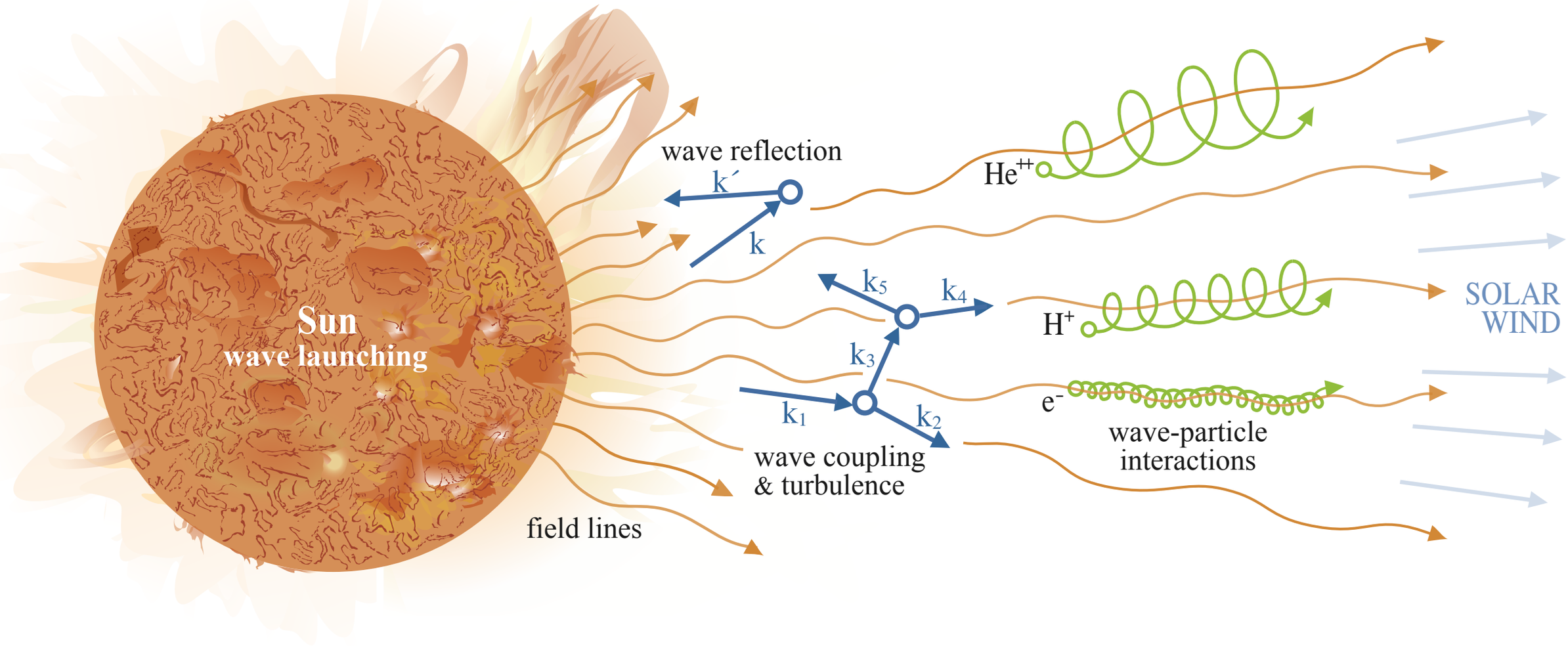
- Several passes to within 10 solar radii of Sun.
- First in situ measurements ever of the solar-wind acceleration region.
- Will measure *E, B,* ***u****, T, f*(*v*), energetic particles.
- Could also provide important insights into other astrophysical systems.





Coronal Heating and Solar-Wind Acceleration by Waves

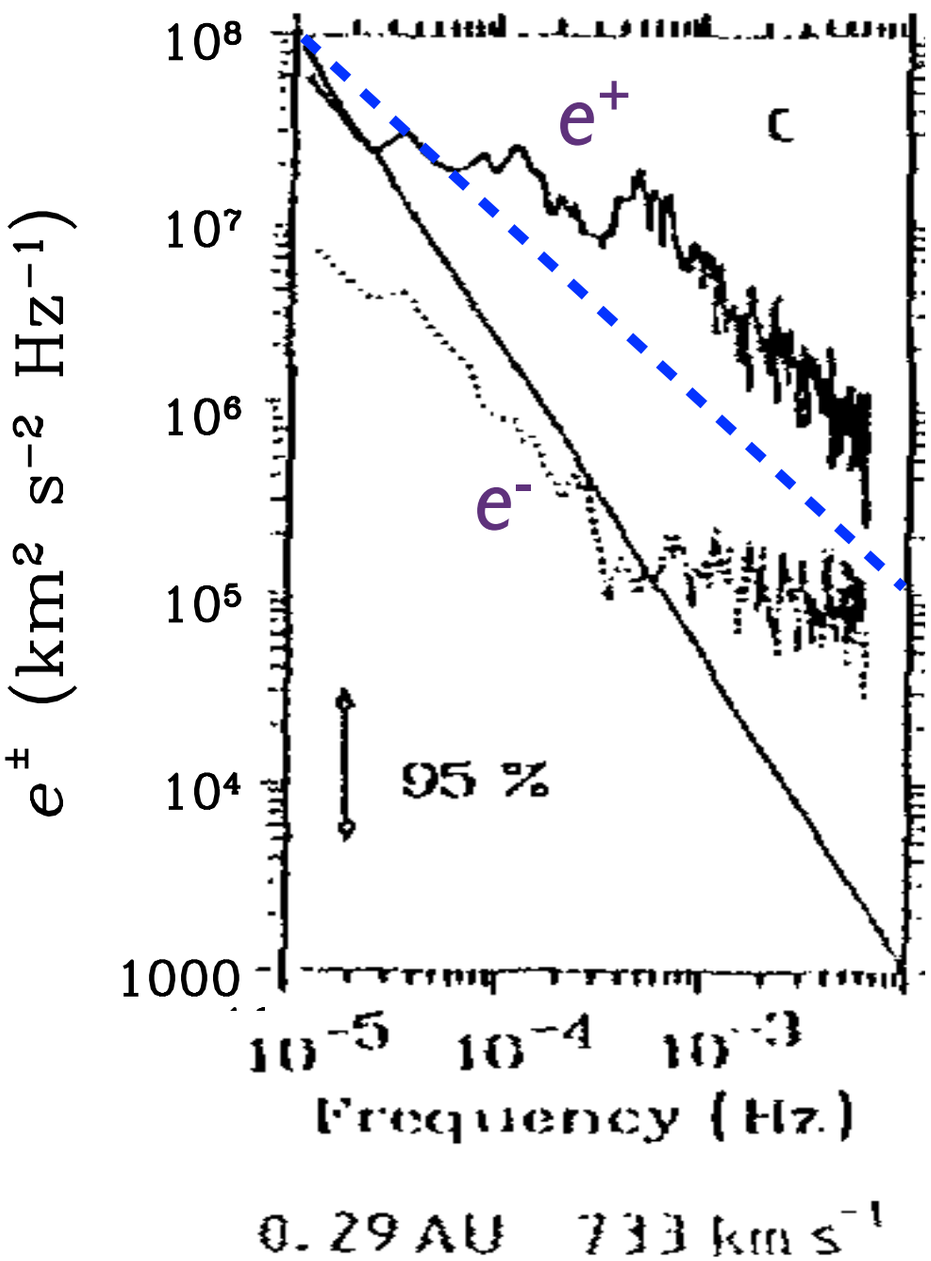
(Parker 1965, Coleman 1968, Velli et al 1989, Zhou & Matthaeus 1989, Cranmer et al 2007)



- The Sun launches Alfven waves, which transport energy outwards
- The waves become turbulent, which causes wave energy to ‘cascade’ from long wavelengths to short wavelengths
- Short-wavelength waves dissipate, heating the plasma. This increases the thermal pressure, which, along with the wave pressure, accelerates the solar wind.

In Situ Measurement of Magnetic Power Spectrum in Fast Solar Wind

Helios Measurements



Tu & Marsch (1995)

Outward-propagating Alfven waves have a $1/f$ frequency spectrum at $f > (1 \text{ hour})^{-1}$

Parametric Instability at Low β

- An outward-propagating Alfven wave (AW) decays into an outward-propagating slow magnetosonic wave ("slow wave") and an inward-propagating AW.
- This talk will focus on fast solar wind at $r < 0.3$ AU.
- β is assumed small. Not great at 0.3 AU, but better at smaller r, and possibly still relevant at 0.3 AU.

Small Selection of Previous Work on the Parametric Instability

- Linear stability analysis in MHD (Oraevskii & Galeev 1963, Goldstein 1978, Inhester 1990, Hollweg 1994)
- Linear stability analysis in drift-kinetic theory (Inhester 1990)
- MHD simulations (Malara et al 2000, del Zanna et al 2001, Shi et al 2017)
- Hybrid simulations (Terasawa et al 1986, Vasquez 1994, Araneda et al 2008, Maneva et al 2013)
- Parametric instability in the presence of strong slow-wave damping (Cohen & Dewar 1974)
- Temperature anisotropy and arc polarization (Tenerani & Velli 2017)

Weak Turbulence Theory?

- δv_{rms} ~ 60 km/s in fast wind at 0.4 AU (Bavassano et al 2000)
- v_A ~ 150 km/s in fast wind at 0.3 AU (Marsch et al 1982, 1990)
- $v_A \sim 1/r$ ⟶ v_A ~ 120 km/s in fast wind at 0.4 AU
- $\omega_{nl}/\omega_{linear} \sim (\delta v_{rms}/v_A)^2 \sim 1/4$ at 0.4 AU. Smaller at smaller *r.*
- ⟶ weak turbulence theory is perhaps useful
- resonant interactions between 3 AWs involve a zero-frequency mode, violate weak-turbulence ordering
- parametric instability does not involve a zero-frequency mode.

Weak Compressible MHD Turbulence at Low Beta

- Perturbation theory to describe wave-wave interactions. ($\omega_{\text{nonlinear}} << \omega_{\text{linear}}$)
- Add collisionless damping terms post facto. (Strong slow-wave damping.)
- Resonant 3-wave interactions:
  - $\omega_k = \omega_p + \omega_q$
  - $\vec{k} = \vec{p} + \vec{q}$
- AAA interactions = interactions among 3 Alfvén waves
- FFF interactions = interactions among 3 fast waves
- AFF interactions - 2 fast waves and 1 Alfvén wave
- $A_k^{\pm}$ = 3D power spectrum of Alfvén waves propagating in $\pm z$ direction.
- $S_k^{\pm}$ = 3D power spectrum of slow waves propagating in $\pm z$ direction.
- F_k = 3D power spectrum of fast waves propagating in **k** direction.

$$\vec{B}_0 = B_0 \hat{z}$$

Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

$$\frac{\partial S_k^{\pm}}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big[\delta(q_z)4k_\perp^2\overline{m}^2\left(A_q^+ + A_q^-\right)\left(S_p^{\pm}-S_k^{\pm}\right) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-$$
$$+ \delta(p_z+q)k_z^2\overline{l}^2\left(A_p^+F_q + A_p^-F_{-q}\right) + \delta(p_z-q)k_z^2\overline{l}^2\left(A_p^+F_{-q}+A_p^-F_q\right)\Big] - 2\gamma_{s,k}^{\pm}S_k^{\pm},$$

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_{-q}+p_zF_{-q}A_k^+ + qA_p^-A_k^+\right)$$
$$+ \delta(k_z+p_z-q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_q + p_zF_qA_k^+ - qA_p^-A_k^+\right) + \delta(k_z-p+q)k_zM_{pk-q}\left(k_zF_pF_{-q} - pF_{-q}A_k^+ + qF_pA_k^+\right)$$
$$+ \delta(q-k_z)p_zA_k^+\left[2(k_z+p_z)F_q + p_zq\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_zA_k^+\left[2(k_z+p_z)F_{-q}+p_zq\frac{\partial F_{-q}}{\partial q}\right]$$
$$+ \varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q}-A_k^+\right) + \delta(p_z)m^2\left(A_q^+ - A_k^+\right)$$
$$+ \delta(k_z+q_z)m^2\left(A_q^- - A_k^+\right)\Big] + \delta(q_z+k_z)4k_z^2A_k^+\frac{\partial}{\partial q_z}\left(q_zA_q^-\right)\Big\} - 2\gamma_{a,k}^+A_k^+,$$

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q-F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_qF_k - qF_{-p}F_k\right)\Big]$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+A_q^- - p_zA_q^-F_k + q_zA_p^+F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+F_q - p_zF_qF_k - qA_p^+F_k\right)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^-F_q + p_zF_qF_k - qA_p^-F_k\right) + \delta(k-q)k^{-3}p_zF_k\left[k_z\frac{\partial}{\partial q}\left(q^4F_q\right) - k^2q_z\frac{\partial}{\partial q}\left(q^2F_q\right)\right]$$
$$+ \varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q-F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]$$
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Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

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perpendicular Alfven-wave cascade

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\boxed{\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right)} + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_{-q}+p_zF_{-q}A_k^+ + qA_p^-A_k^+\right)$$
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Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

passive-scalar mixing

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$$\boxed{+\varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)}\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q}-A_k^+\right) \boxed{+\delta(p_z)m^2\left(A_q^+ - A_k^+\right)}$$
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phase mixing

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q-F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_qF_k - qF_{-p}F_k\right)\Big]$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+A_q^- - p_zA_q^-F_k + q_zA_p^+F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+F_q - p_zF_qF_k - qA_p^+F_k\right)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^-F_q + p_zF_qF_k - qA_p^-F_k\right) + \delta(k-q)k^{-3}p_zF_k\left[k_z\frac{\partial}{\partial q}\left(q^4F_q\right) - k^2q_z\frac{\partial}{\partial q}\left(q^2F_q\right)\right]$$
$$+ \varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q-F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]$$
$$+ \delta(k-q_z)p_zF_k\left(2k_zA_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_zF_k\left(2k_zA_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k,$$

Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

$$\frac{\partial S_k^{\pm}}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big[\delta(q_z)4k_\perp^2\overline{m}^2\left(A_q^+ + A_q^-\right)\left(S_p^{\pm}-S_k^{\pm}\right) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-$$
$$+ \delta(p_z+q)k_z^2\overline{l}^2\left(A_p^+F_q + A_p^-F_{-q}\right) + \delta(p_z-q)k_z^2\overline{l}^2\left(A_p^+F_{-q}+A_p^-F_q\right)\Big] - 2\gamma_{s,k}^{\pm}S_k^{\pm},$$

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_{-q}+p_zF_{-q}A_k^+ + qA_p^-A_k^+\right)$$
$$+ \delta(k_z+p_z-q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_q + p_zF_qA_k^+ - qA_p^-A_k^+\right) + \delta(k_z-p+q)k_zM_{pk-q}\left(k_zF_pF_{-q} - pF_{-q}A_k^+ + qF_pA_k^+\right)$$
$$+ \delta(q-k_z)p_zA_k^+\left[2(k_z+p_z)F_q + p_zq\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_zA_k^+\left[2(k_z+p_z)F_{-q}+p_zq\frac{\partial F_{-q}}{\partial q}\right]$$
$$+ \varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q}-A_k^+\right) + \delta(p_z)m^2\left(A_q^+ - A_k^+\right)$$
$$+ \delta(k_z+q_z)m^2\left(A_q^- - A_k^+\right)\Big] + \delta(q_z+k_z)4k_z^2A_k^+\frac{\partial}{\partial q_z}\left(q_zA_q^-\right)\Big\} - 2\gamma_{a,k}^+A_k^+,$$

"radial" fast-wave cascade

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\boxed{9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q-F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_qF_k - qF_{-p}F_k\right)\Big]}$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+A_q^- - p_zA_q^-F_k + q_zA_p^+F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+F_q - p_zF_qF_k - qA_p^+F_k\right)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^-F_q + p_zF_qF_k - qA_p^-F_k\right) + \delta(k-q)k^{-3}p_zF_k\left[k_z\frac{\partial}{\partial q}\left(q^4F_q\right) - k^2q_z\frac{\partial}{\partial q}\left(q^2F_q\right)\right]$$
$$+ \varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q-F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]$$
$$+ \delta(k-q_z)p_zF_k\left(2k_zA_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_zF_k\left(2k_zA_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k,$$

Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

parametric instability

$$\frac{\partial S_k^{\pm}}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big[\delta(q_z)4k_\perp^2\overline{m}^2\left(A_q^+ + A_q^-\right)\left(S_p^{\pm}-S_k^{\pm}\right) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \boxed{\delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-}$$
$$+ \delta(p_z+q)k_z^2\overline{l}^2\left(A_p^+F_q + A_p^-F_{-q}\right) + \delta(p_z-q)k_z^2\overline{l}^2\left(A_p^+F_{-q}+A_p^-F_q\right)\Big] - 2\gamma_{s,k}^{\pm}S_k^{\pm},$$

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_{-q}+p_zF_{-q}A_k^+ + qA_p^-A_k^+\right)$$
$$+ \delta(k_z+p_z-q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_q + p_zF_qA_k^+ - qA_p^-A_k^+\right) + \delta(k_z-p+q)k_zM_{pk-q}\left(k_zF_pF_{-q} - pF_{-q}A_k^+ + qF_pA_k^+\right)$$
$$+ \delta(q-k_z)p_zA_k^+\left[2(k_z+p_z)F_q + p_zq\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_zA_k^+\left[2(k_z+p_z)F_{-q}+p_zq\frac{\partial F_{-q}}{\partial q}\right]$$
$$+ \boxed{\varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)}\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q}-A_k^+\right) + \delta(p_z)m^2\left(A_q^+ - A_k^+\right)$$
$$\times\ \boxed{+\ \delta(k_z+q_z)m^2\left(A_q^- - A_k^+\right)} + \boxed{\delta(q_z+k_z)4k_z^2A_k^+\frac{\partial}{\partial q_z}\left(q_zA_q^-\right)}\Big\} - 2\gamma_{a,k}^+A_k^+,$$

parametric instability

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q-F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_qF_k - qF_{-p}F_k\right)\Big]$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+A_q^- - p_zA_q^-F_k + q_zA_p^+F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+F_q - p_zF_qF_k - qA_p^+F_k\right)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^-F_q + p_zF_qF_k - qA_p^-F_k\right) + \delta(k-q)k^{-3}p_zF_k\left[k_z\frac{\partial}{\partial q}\left(q^4F_q\right) - k^2q_z\frac{\partial}{\partial q}\left(q^2F_q\right)\right]$$
$$+ \varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q-F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]$$
$$+ \delta(k-q_z)p_zF_k\left(2k_zA_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_zF_k\left(2k_zA_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k,$$

Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

$$\cancel{\frac{\partial S_k^{\pm}}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big[\delta(q_z)4k_\perp^2\overline{m}^2\left(A_q^+ + A_q^-\right)\left(S_p^{\pm} - S_k^{\pm}\right) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-}$$

$$\cancel{+\,\delta(p_z+q)k_z^2\overline{l}^2\left(A_p^+F_q + A_p^-F_{-q}\right) + \delta(p_z-q)k_z^2\overline{l}^2\left(A_p^+F_{-q} + A_p^-F_q\right)\Big] - 2\gamma_{s,k}^{\pm}S_k^{\pm},}$$

$$\begin{aligned}
\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{&\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_{-q} + p_zF_{-q}A_k^+ + qA_p^-A_k^+\right)\\
&+\delta(k_z+p_z-q)k_z\Lambda_{q-pk}\left(k_zA_p^-F_q + p_zF_qA_k^+ - qA_p^-A_k^+\right) + \delta(k_z-p+q)k_zM_{pk-q}\left(k_zF_pF_{-q} - pF_{-q}A_k^+ + qF_pA_k^+\right)\\
&+\delta(q-k_z)p_zA_k^+\left[2(k_z+p_z)F_q + p_zq\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_zA_k^+\left[2(k_z+p_z)F_{-q} + p_zq\frac{\partial F_{-q}}{\partial q}\right]\\
&\cancel{+\,\varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q} - A_k^+\right) + \delta(p_z)m^2\left(A_q^+ - A_k^+\right)}\\
&\cancel{+\,\delta(k_z+q_z)m^2\left(A_q^- - A_k^+\right)\Big]} + \boxed{\delta(q_z+k_z)4k_z^2A_k^+\frac{\partial}{\partial q_z}\left(q_zA_q^-\right)}\Big\} - 2\gamma_{a,k}^+A_k^+,
\end{aligned}$$

parametric instability when slow waves are strongly damped

$$\begin{aligned}
\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{&9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q - F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_qF_k - qF_{-p}F_k\right)\Big]\\
&+\delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+A_q^- - p_zA_q^-F_k + q_zA_p^+F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+F_q - p_zF_qF_k - qA_p^+F_k\right)\\
&+\delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^-F_q + p_zF_qF_k - qA_p^-F_k\right) + \delta(k-q)k^{-3}p_zF_k\left[k_z\frac{\partial}{\partial q}\left(q^4F_q\right) - k^2q_z\frac{\partial}{\partial q}\left(q^2F_q\right)\right]\\
&+\varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q - F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]\\
&+\delta(k-q_z)p_zF_k\left(2k_zA_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_zF_k\left(2k_zA_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k,
\end{aligned}$$

Wave Kinetic Equations for Weak MHD Turbulence

(Chandran 2008)

$$\frac{\partial S_k^{\pm}}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big[\delta(q_z)4k_\perp^2\overline{m}^2\left(A_q^+ + A_q^-\right)\left(S_p^{\pm} - S_k^{\pm}\right) + \delta(p-q)k_z^2 l^2 F_p F_{-q} + \delta(p_z-q_z)k_z^2 l^2 A_p^+ A_q^-$$
$$+ \delta(p_z+q_z)k_z^2\overline{l}^2\left(A_p^+ F_q + A_p^- F_{-q}\right) + \delta(p_z-q_z)k_z^2\overline{l}^2\left(A_p^+ F_{-q} + A_p^- F_q\right)\Big] - 2\gamma_{s,k}^{\pm} S_k^{\pm},$$

$$\frac{\partial A_k^+}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{\delta(q_z)8(k_\perp n\overline{m})^2 A_q^-\left(A_p^+ - A_k^+\right) + \delta(k_z+p_z+q)k_z\Lambda_{q-pk}\left(k_z A_p^- F_{-q} + p_z F_{-q} A_k^+ + q A_p^- A_k^+\right)$$
$$+ \delta(k_z+p_z-q)k_z\Lambda_{q-pk}\left(k_z A_p^- F_q + p_z F_q A_k^+ - q A_p^- A_k^+\right) + \delta(k_z-p+q)k_z M_{pk-q}\left(k_z F_p F_{-q} - p F_{-q} A_k^+ + q F_p A_k^+\right)$$
$$+ \delta(q-k_z)p_z A_k^+\left[2(k_z+p_z)F_q + p_z q\frac{\partial F_q}{\partial q}\right] + \delta(q+k_z)p_z A_k^+\left[2(k_z+p_z)F_{-q} + p_z q\frac{\partial F_{-q}}{\partial q}\right]$$
$$+ \varepsilon^{-2}k_z^2\left(S_p^+ + S_p^-\right)\Big[\delta(q-k_z)\overline{m}^2\left(F_q - A_k^+\right) + \delta(q+k_z)\overline{m}^2\left(F_{-q} - A_k^+\right) + \delta(p_z)m^2\left(A_q^+ - A_k^+\right)$$
$$+ \delta(k_z+q_z)m^2\left(A_q^- - A_k^+\right)\Big] + \boxed{\delta(q_z+k_z)4k_z^2 A_k^+\frac{\partial}{\partial q_z}\left(q_z A_q^-\right)}\Big\} - 2\gamma_{a,k}^+ A_k^+,$$

parametric instability when slow waves are strongly damped

$$\frac{\partial F_k}{\partial t} = \frac{\pi}{4v_A}\int d^3p\, d^3q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\Big\{9\sin^2\theta\Big[\delta(k-p-q)kqF_p\left(F_q - F_k\right) + \delta(k+p-q)k\left(kF_{-p}F_q + pF_q F_k - qF_{-p}F_k\right)\Big]$$
$$+ \delta(k-p_z+q_z)k\Lambda_{kpq}\left(kA_p^+ A_q^- - p_z A_q^- F_k + q_z A_p^+ F_k\right) + \delta(k-p_z-q)kM_{kpq}\left(kA_p^+ F_q - p_z F_q F_k - qA_p^+ F_k\right)$$
$$+ \delta(k+p_z-q)kM_{-k-p-q}\left(kA_p^- F_q + p_z F_q F_k - qA_p^- F_k\right) + \delta(k-q)k^{-3}p_z F_k\left[k_z\frac{\partial}{\partial q}\left(q^4 F_q\right) - k^2 q_z\frac{\partial}{\partial q}\left(q^2 F_q\right)\right]$$
$$+ \varepsilon^{-2}k^2\left(S_p^+ + S_p^-\right)\Big[\delta(k-q)m^2\left(F_q - F_k\right) + \delta(k-q_z)\overline{m}^2\left(A_q^+ - F_k\right) + \delta(k+q_z)\overline{m}^2\left(A_q^- - F_k\right)\Big]$$
$$+ \delta(k-q_z)p_z F_k\left(2k_z A_q^+ + kp_z\frac{\partial A_q^+}{\partial q_z}\right) + \delta(k+q_z)p_z F_k\left(2k_z A_q^- - kp_z\frac{\partial A_q^-}{\partial q_z}\right)\Big\} - 2\gamma_{f,k}F_k,$$

Integrate the Wave Kinetic Equations over $k_\perp$

$$E^{\pm}(k_z, t) = \int dk_x dk_y A^{\pm}(k_x, k_y, k_z, t)$$

$$\frac{\partial E^+}{\partial t} = \frac{\pi}{v_{\mathrm{A}}} k_z^2 E^+ \frac{\partial}{\partial k_z}\left(k_z E^-\right)$$

$$\frac{\partial E^-}{\partial t} = \frac{\pi}{v_{\mathrm{A}}} k_z^2 E^- \frac{\partial}{\partial k_z}\left(k_z E^+\right)$$

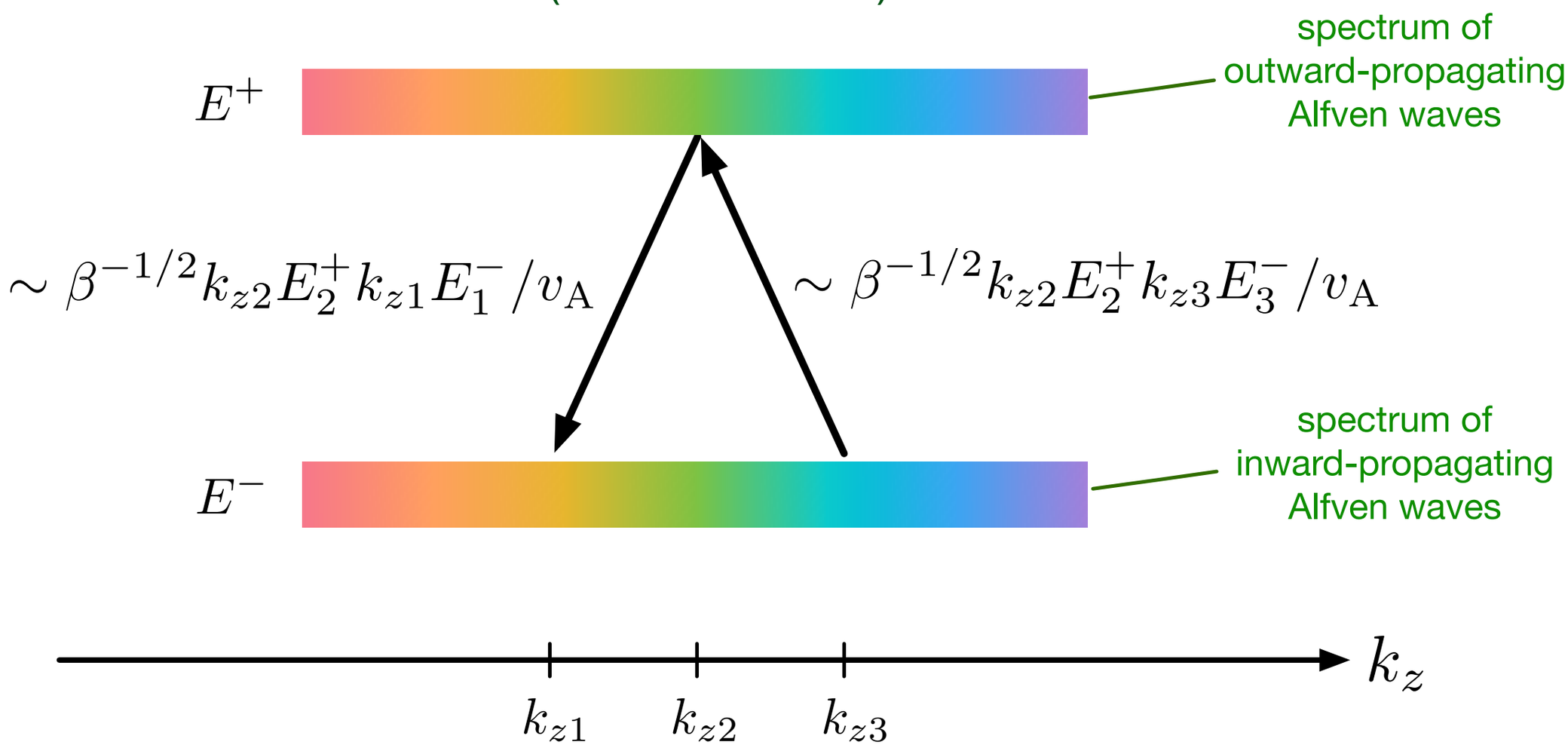
The wave kinetic equations allow for obliquely propagating waves, but these integrated equations depend only on the parallel wavenumber k_z and t.

Alfven Wave Frequency Decreases Slightly During Each Parametric Decay

$$
\begin{aligned}
k_z &= p_z + q_z \\
k_z v_{\mathrm{A}} &= p_z c_{\mathrm{s}} - q_z v_{\mathrm{A}} \\
k_z v_{\mathrm{A}} &= (k_z - q_z) c_{\mathrm{s}} - q_z v_{\mathrm{A}} \\
k_z (v_{\mathrm{A}} - c_{\mathrm{s}}) &= -q_z (v_{\mathrm{A}} + c_{\mathrm{s}}) \\
k_z \left(\frac{v_{\mathrm{A}} - c_{\mathrm{s}}}{v_{\mathrm{A}} + c_{\mathrm{s}}} \right) &= -q_z \qquad (\text{ Note: } c_{\mathrm{s}}/v_{\mathrm{A}} \sim \beta^{1/2} \ll 1) \\
k_z (1 - 2\beta^{1/2}) &= -q_z
\end{aligned}
$$

Why Do the Wave Kinetic Equations Have This Form?

(Chandran 2018)



$$\frac{\partial E_2^+}{\partial t} \sim \frac{\beta^{-1/2} k_{z2} E_2^+}{v_{\rm A}} \left(k_{z3} E_3^- - k_{z1} E_1^- \right)$$

$$k_{z3} - k_{z1} \sim \beta^{1/2} k_{z2}$$

$$\frac{\partial E_2^+}{\partial t} \sim \frac{\beta^{-1/2} k_{z2} E_2^+}{v_{\rm A}} \times \beta^{1/2} k_{z2} \frac{\partial}{\partial k_z} \left(k_z E^- \right) \sim \frac{k_{z2}^2 E_2^+}{v_{\rm A}} \frac{\partial}{\partial k_z} \left(k_z E^- \right)$$

Linear Limit

"Pump-wave" amplitude fixed ($E^+ = \text{constant}$):

$$\frac{\partial E^-}{\partial t} = \frac{\pi}{v_\mathrm{A}} k_z^2 E^- \frac{\partial}{\partial k_z}\left(k_z E^+\right)$$

$$\gamma \equiv \frac{\partial \ln E^-}{\partial t} = \frac{\pi}{v_\mathrm{A}} k_z^2 \frac{\partial}{\partial k_z}\left(k_z E^+\right)$$

E^- grows exponentially if $E^+ \propto k^{\alpha^+}$ with $\alpha^+ > -1$.

This result was found by Cohen & Dewar (1974) for parallel-propagating waves at low beta, assuming slow waves are strongly damped.

Conservation of Wave Quanta and Inverse Cascade

(Chandran 2018)

$$E^{\pm}(k_z,t) = \int dk_x dk_y A^{\pm}(k_x,k_y,k_z,t)$$

$$\boxed{\begin{aligned} \frac{\partial E^+}{\partial t} &= \frac{\pi}{v_{\rm A}} k_z^2 E^+ \frac{\partial}{\partial k_z}\left(k_z E^-\right) \\ \frac{\partial E^-}{\partial t} &= \frac{\pi}{v_{\rm A}} k_z^2 E^- \frac{\partial}{\partial k_z}\left(k_z E^+\right) \end{aligned}}$$

Divide previous two eqns by $k_z v_{\rm A}$ and add:

$$\frac{\partial}{\partial t}\left(\frac{E^+ + E^-}{k_z v_{\rm A}}\right) = \frac{\partial}{\partial k_z}\left(\frac{\pi k_z^2 E^+ E^-}{v_{\rm A}^2}\right) \longleftarrow$$

inverse cascade of wave quanta

$$\longrightarrow \int_{-\infty}^{\infty} dk_z \left(\frac{E^+ + E^-}{k_z v_{\rm A}}\right) = \text{ constant} \longleftarrow$$

conservation of wave quanta (wave action)

Exact Solutions to Wave Kinetic Equation

(Chandran 2018)

$$\frac{\partial E^+}{\partial t} = \frac{\pi}{v_{\rm A}} k_z^2 E^+ \frac{\partial}{\partial k_z}\left(k_z E^-\right)$$

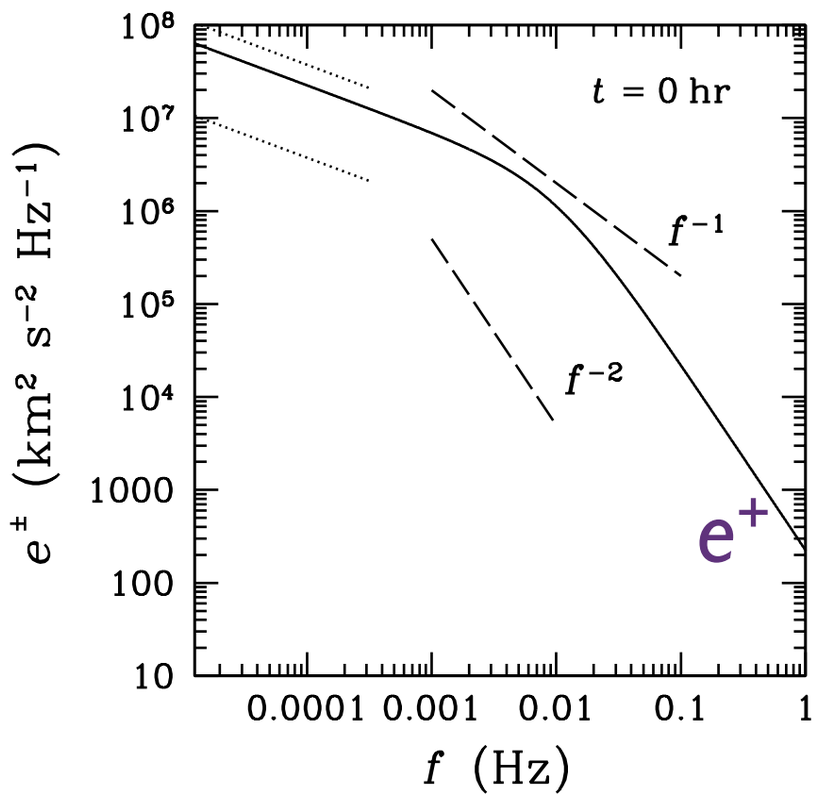
$$\frac{\partial E^-}{\partial t} = \frac{\pi}{v_{\rm A}} k_z^2 E^- \frac{\partial}{\partial k_z}\left(k_z E^+\right)$$

$$E^\pm(k_z, t) = \frac{c^\pm}{k_z},$$

$$E^\pm(k_z, t) = \frac{a^\pm(t)}{k_z^2} \qquad a^\pm(t) = \frac{a_0^\pm(a_0^\pm - a_0^\mp)}{a_0^\pm - a_0^\mp e^{-\pi(a_0^\pm - a_0^\mp)t/v_{\rm A}}},$$

(can also construct truncated versions of these solutions, and combinations of k_z^{-1} and k_z^{-2} solutions)

Numerical Solution of the Nonlinear Evolution



$$e^{\pm} = \frac{2\pi E^{\pm}}{U} = \text{frequency spectrum}$$

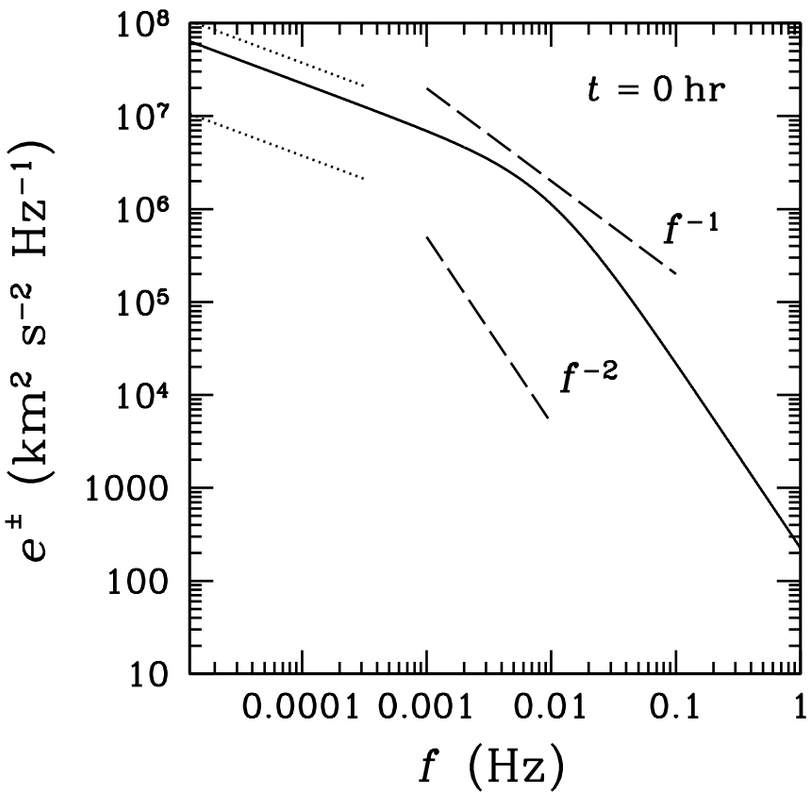
(via Taylor's hypothesis.
U = solar-wind speed = 733 km/s.)

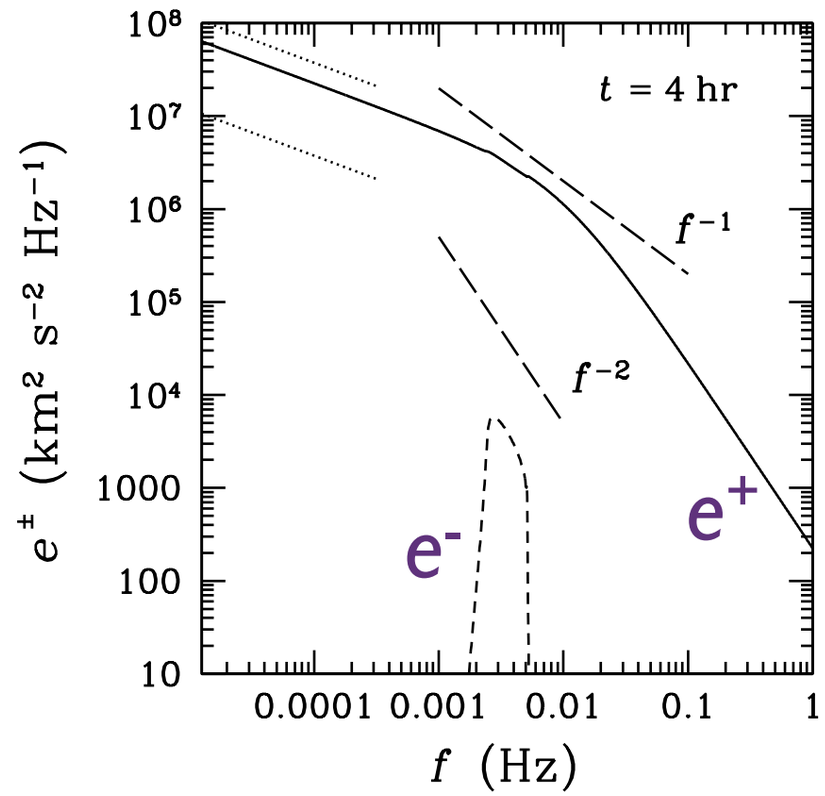
Alfven speed = 150 km/s. Initial dominant frequency (maximum of fe_f) is 0.01 Hz.

$$e^{+}(f, t = 0) = \frac{\sigma^{+}(f/f_0)^{-0.5}}{1 + (f/f_0)^{1.5}}$$

(Chandran 2018)

Numerical Solution of the Nonlinear Evolution

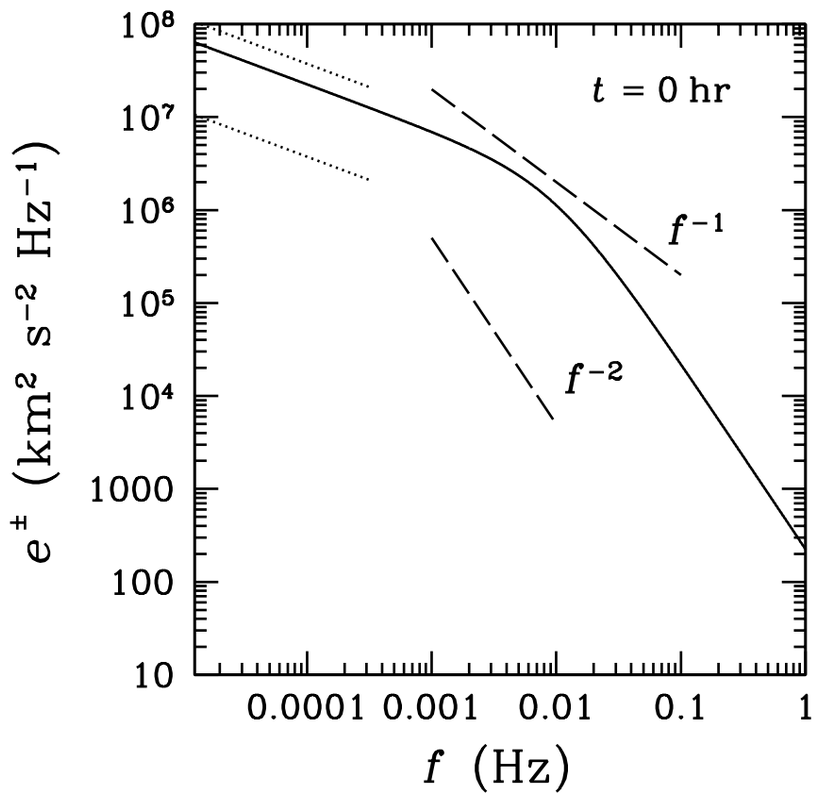


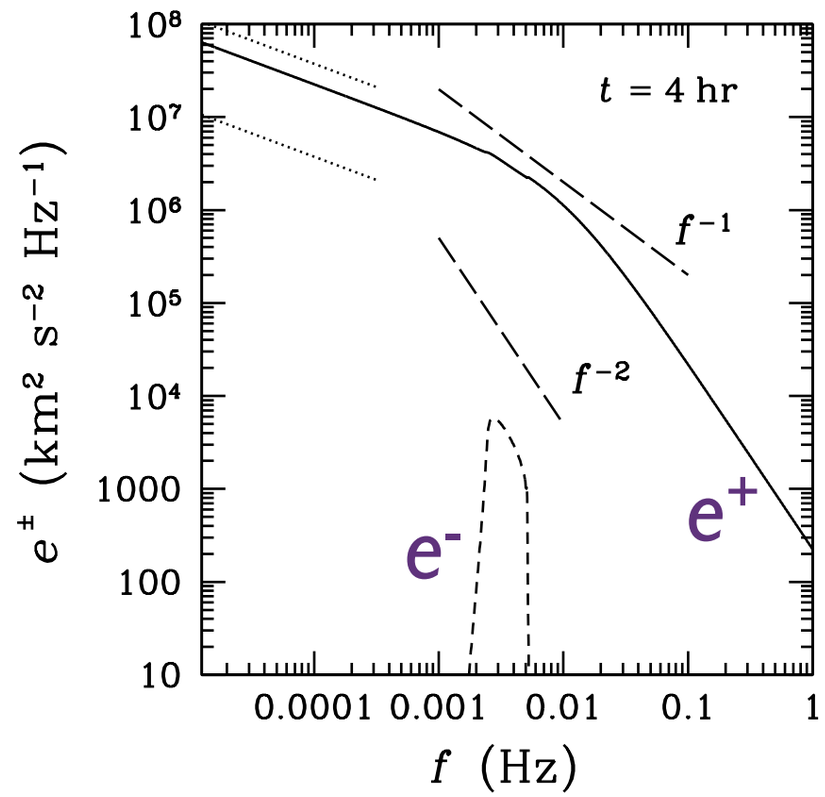


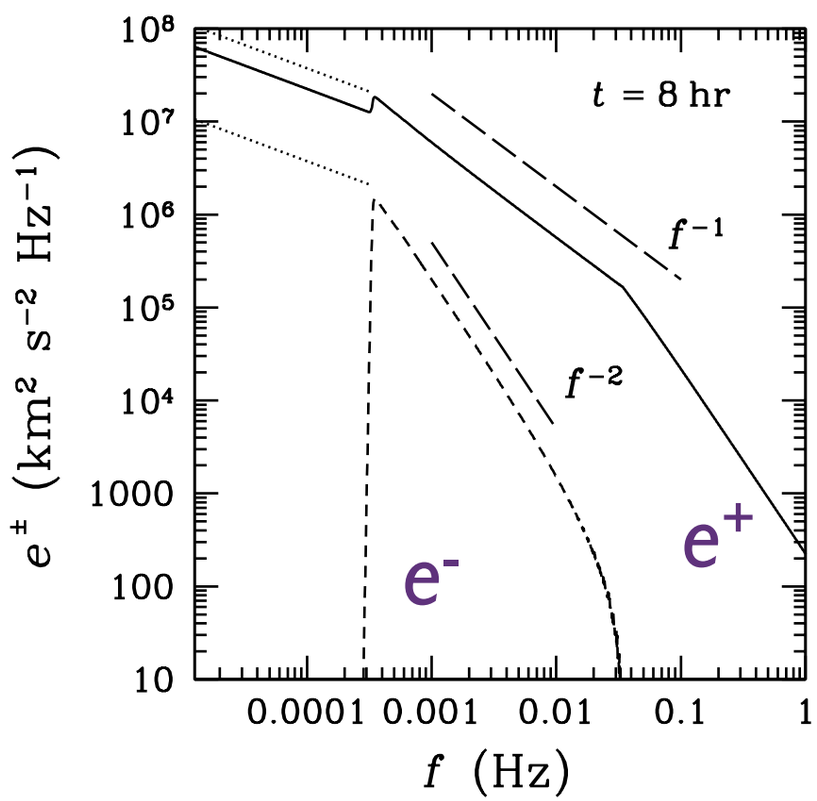
Linear stage: the inward waves grow fastest at the largest wavenumbers where the spectrum is flatter than 1/f.

(Chandran 2018)

Numerical Solution of the Nonlinear Evolution



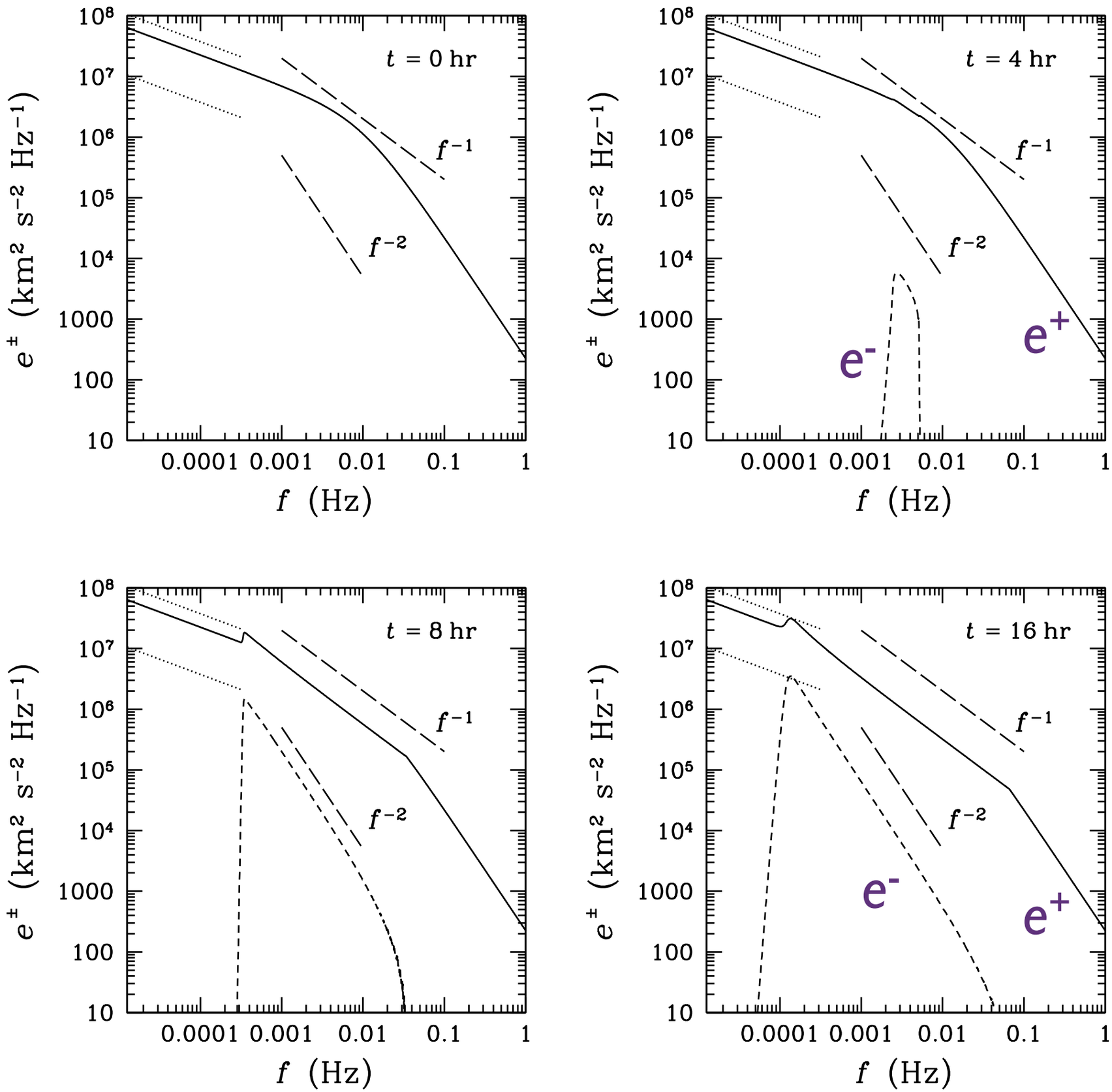




The outward-going "pump waves" acquire a 1/f spectrum in a form of quasilinear "flattening"

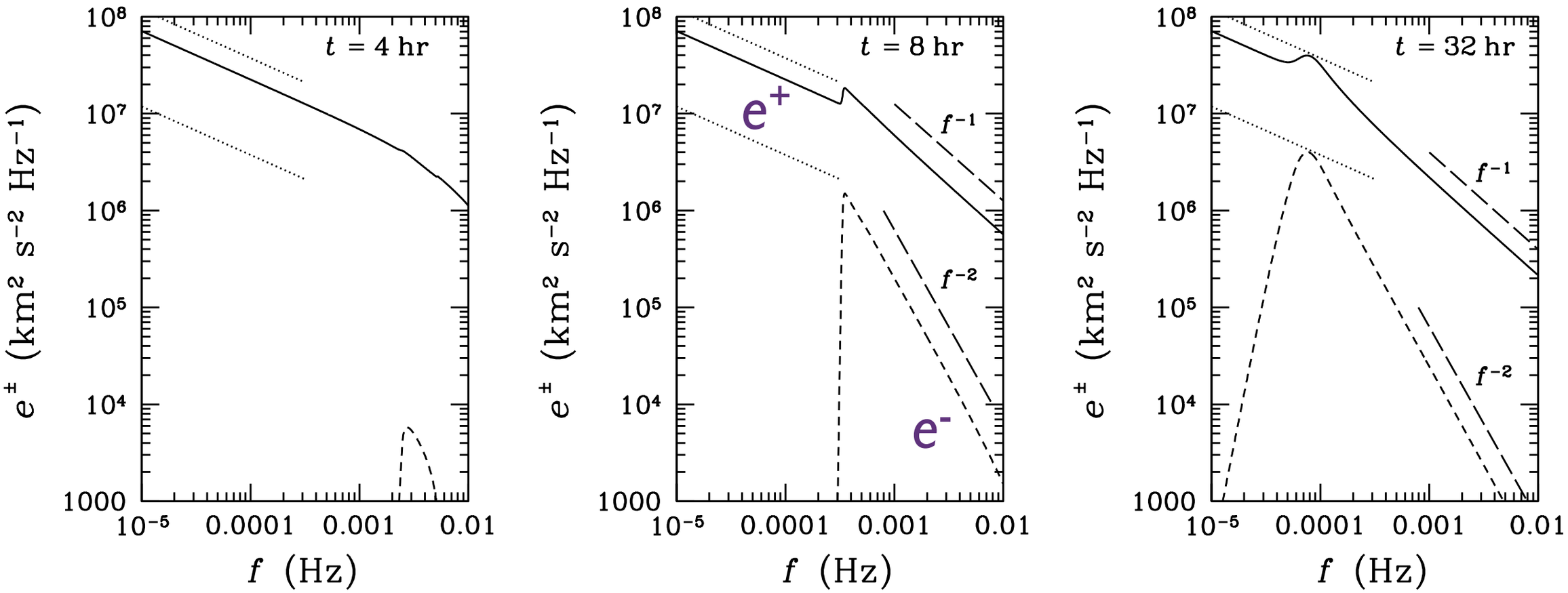
(Chandran 2018)

Numerical Solution of the Nonlinear Evolution



(Chandran 2018)

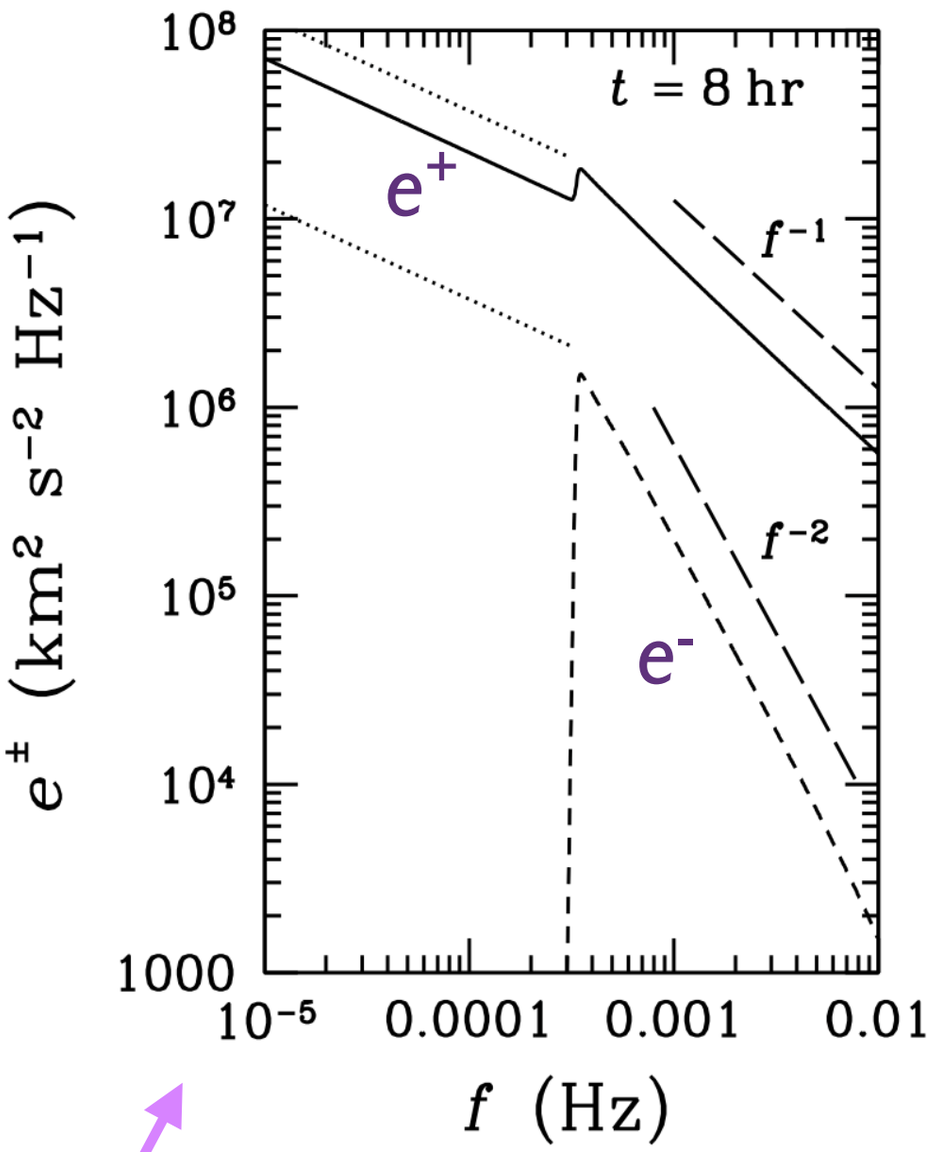
Same Simulation, Plotted over a Smaller Frequency Range, out to 32 Hrs



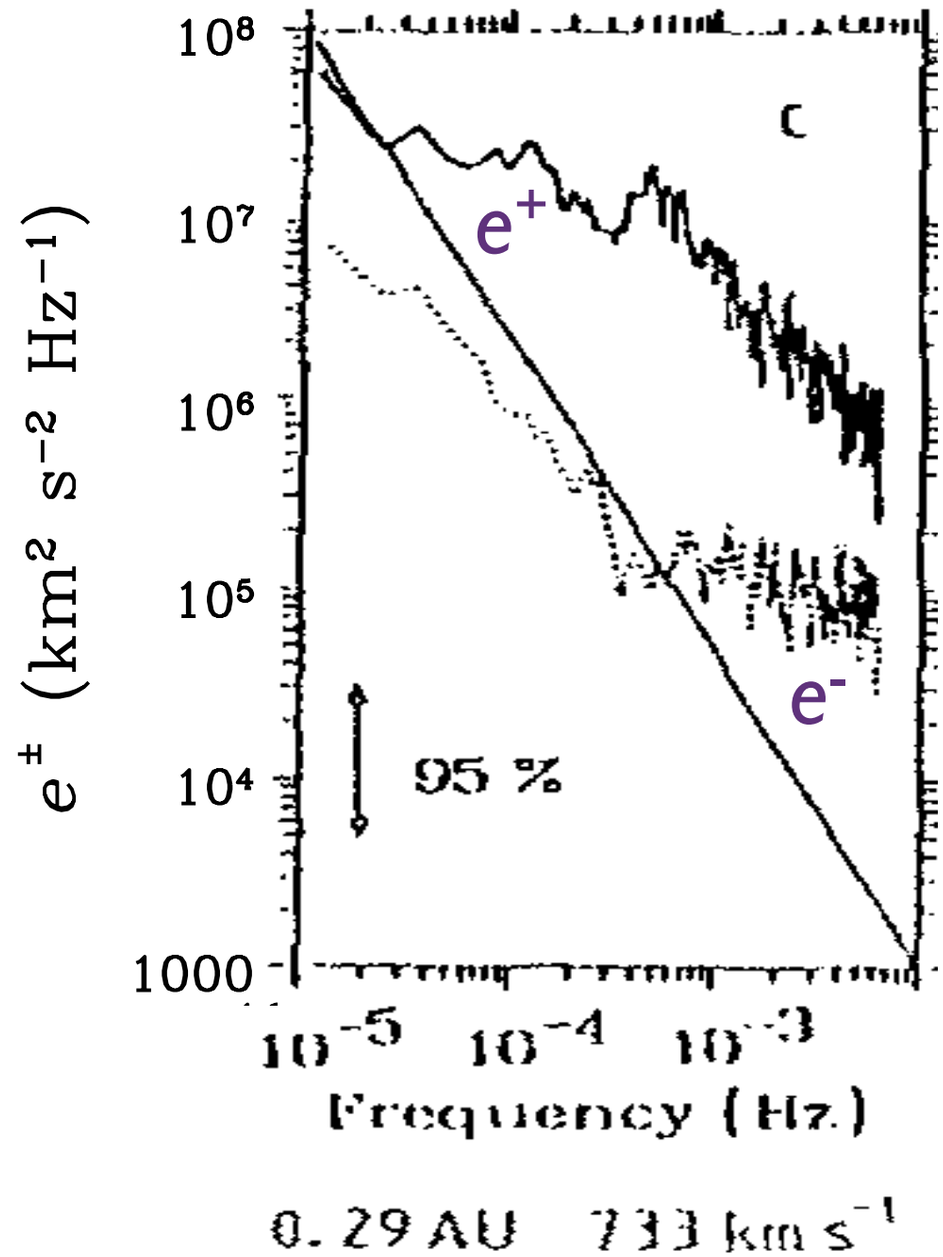
dotted lines in upper left show evolutionary tracks of spectral peaks in an approximate analytic solution

(Chandran 2018)

Comparison Between Numerical Solution and Helios Measurements



Helios Measurements



Alfven speed = 150 km/s. Initial dominant frequency (maximum of f x E_f) is 0.01 Hz. Alfven travel time to 0.29 AU is 12 hours.

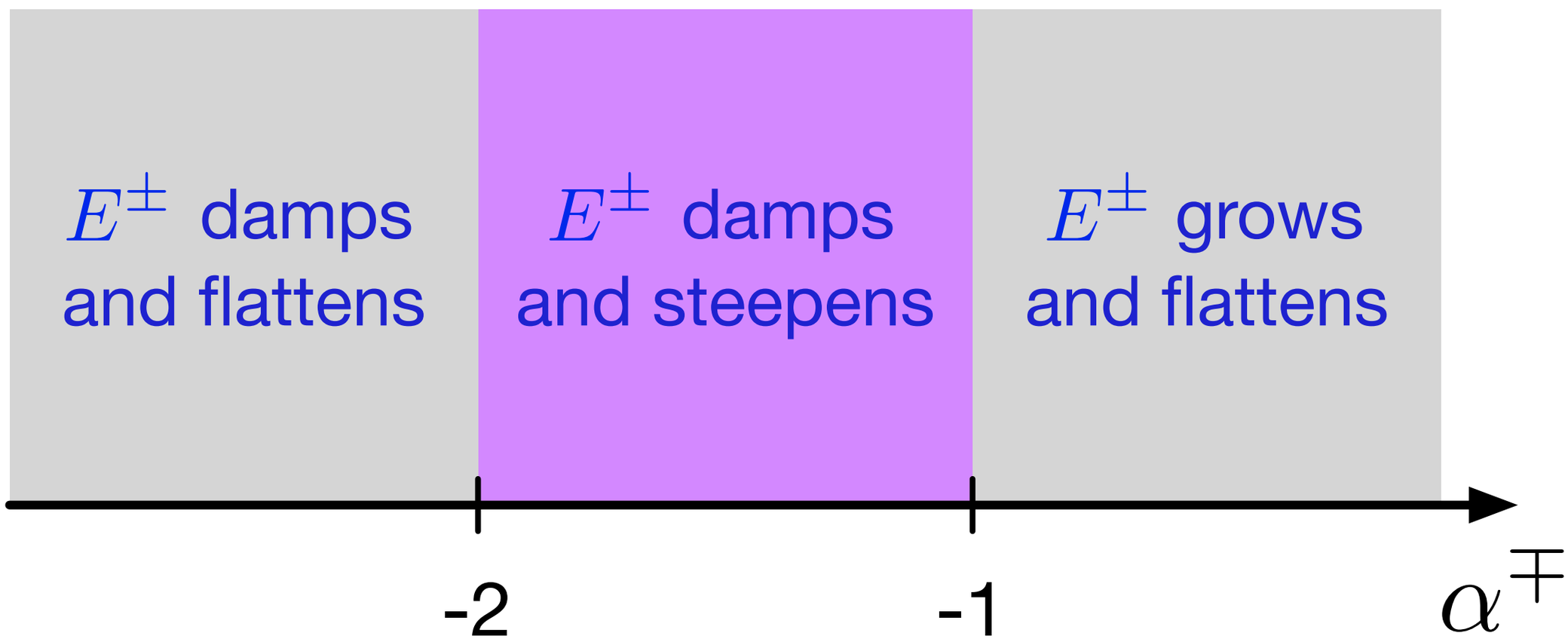
Tu & Marsch (1995)

(Chandran 2018)

Nonlinear Evolution of the Parametric Instability

$$\frac{\partial E^{\pm}}{\partial t} = \frac{\pi}{v_{\mathrm{A}}} k_z^2 E^{\pm} \frac{\partial}{\partial k_z} \left(k_z E^{\mp} \right)$$

$$E^{\mp} \propto k_z^{\alpha^{\mp}} \longrightarrow \frac{\partial}{\partial t} \ln E^{\pm} \propto (1 + \alpha^{\mp}) k_z^{2+\alpha^{\mp}}$$



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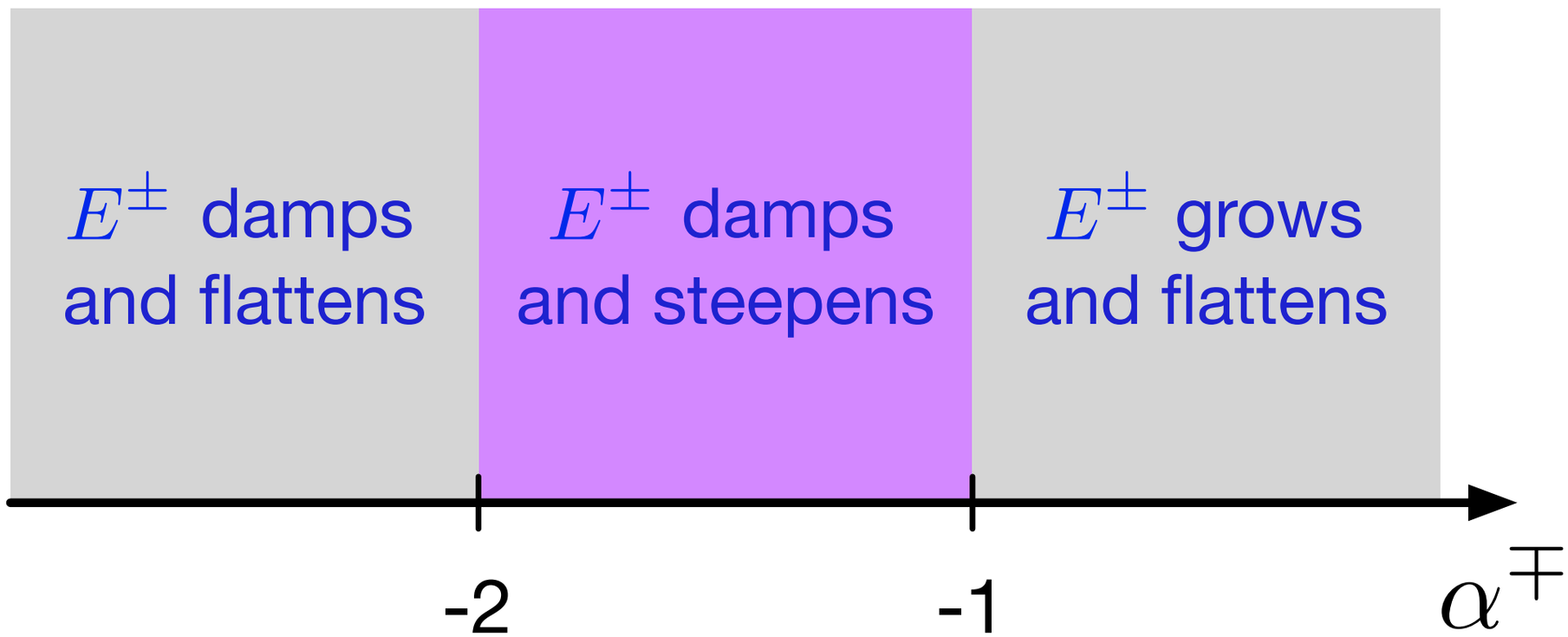

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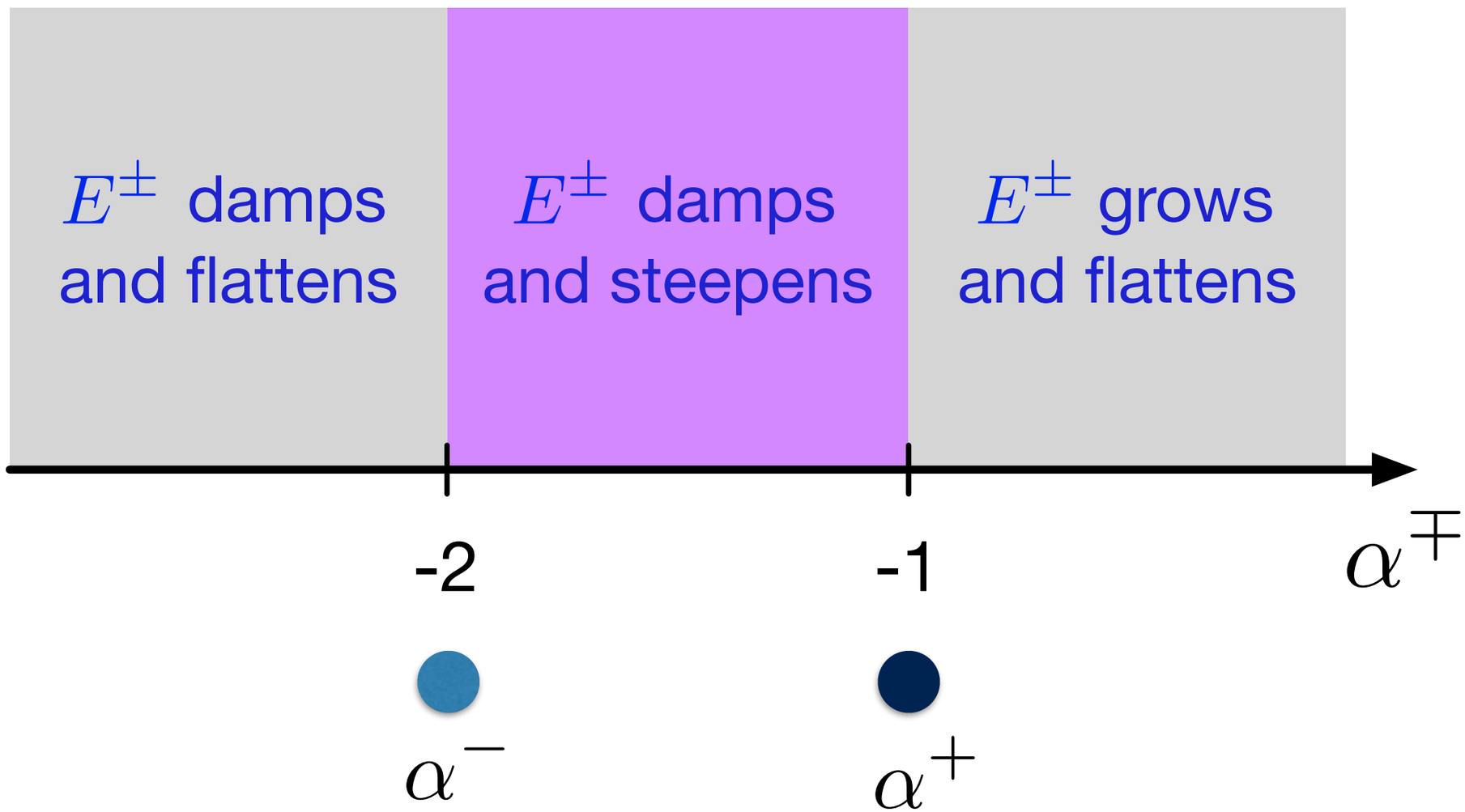

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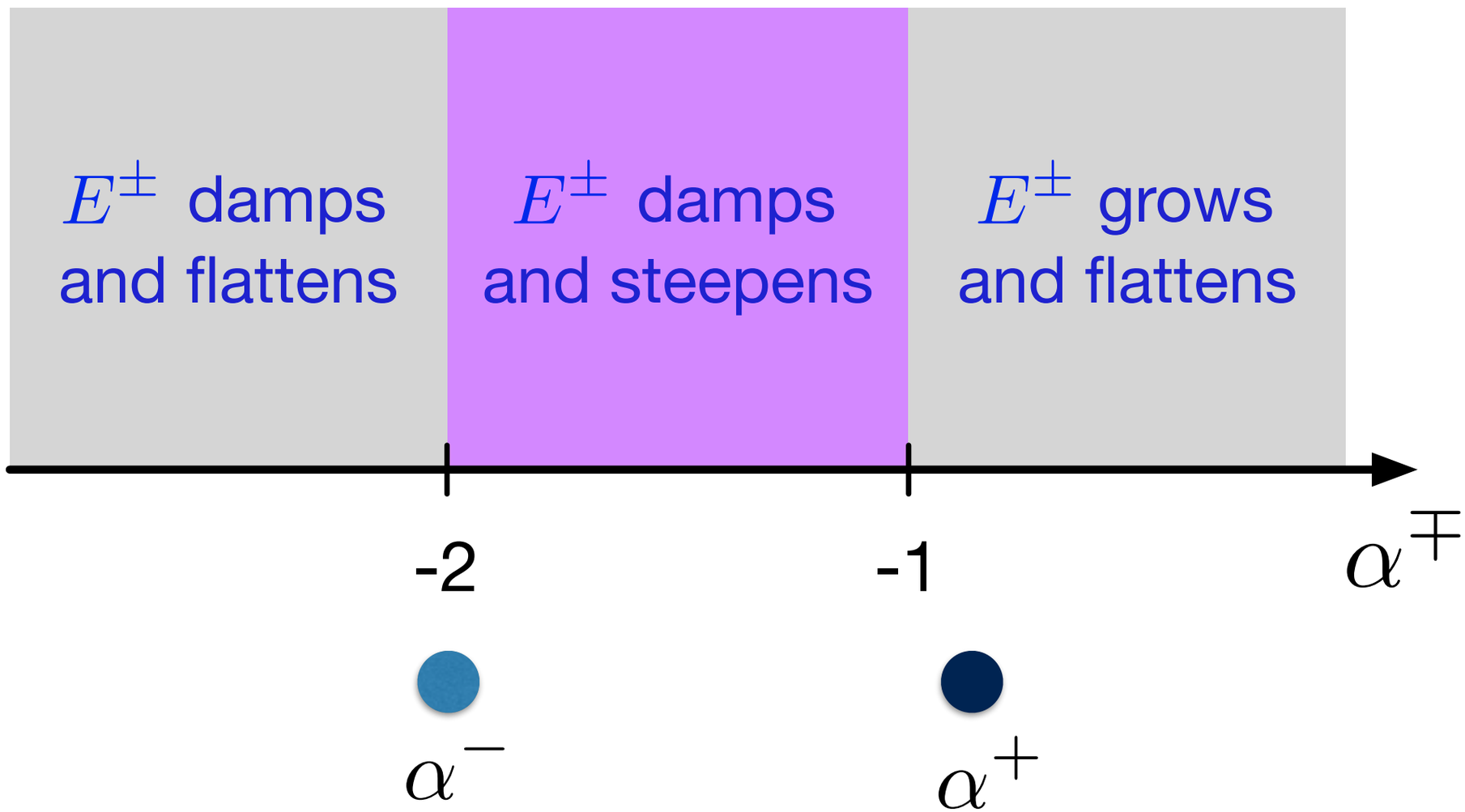

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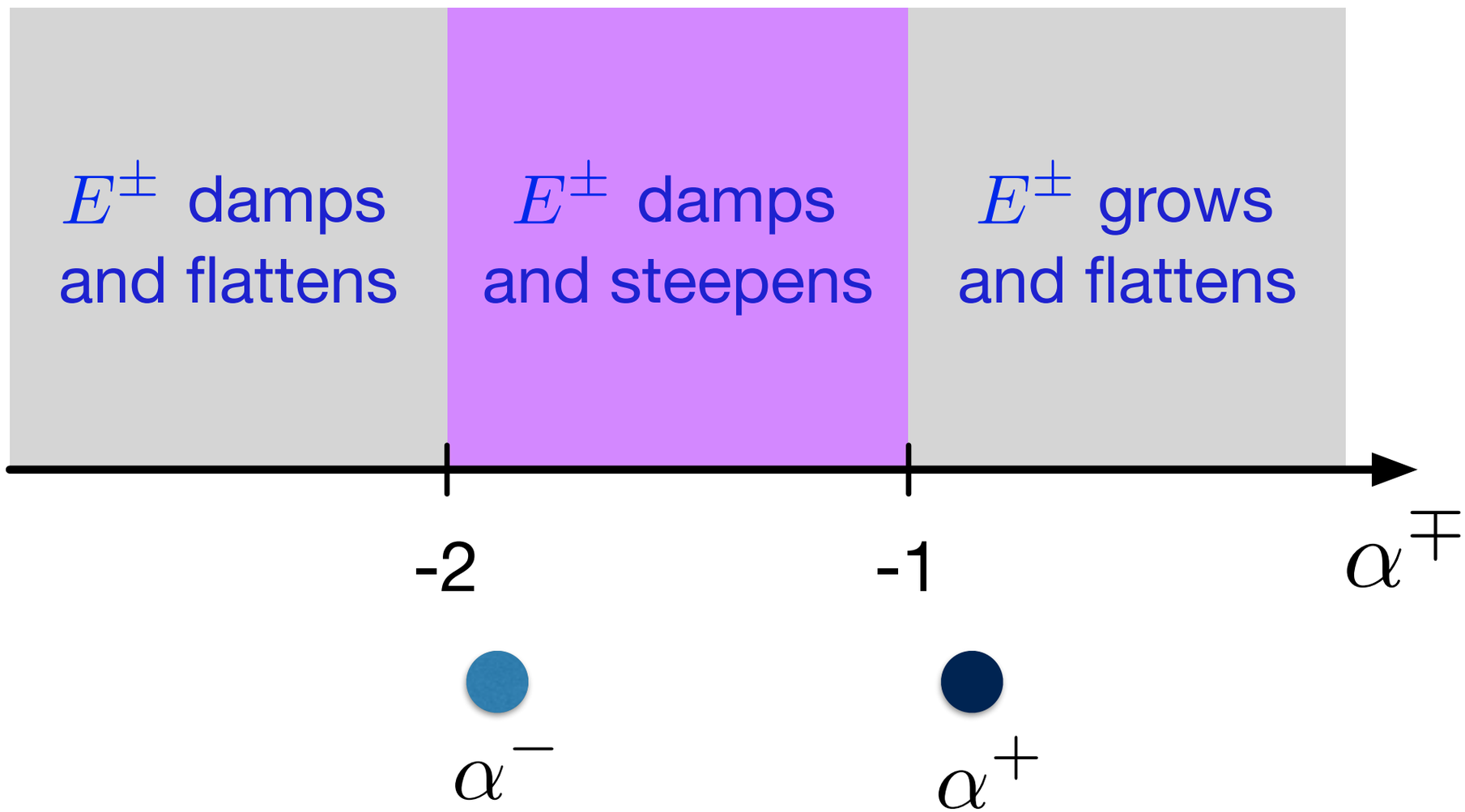

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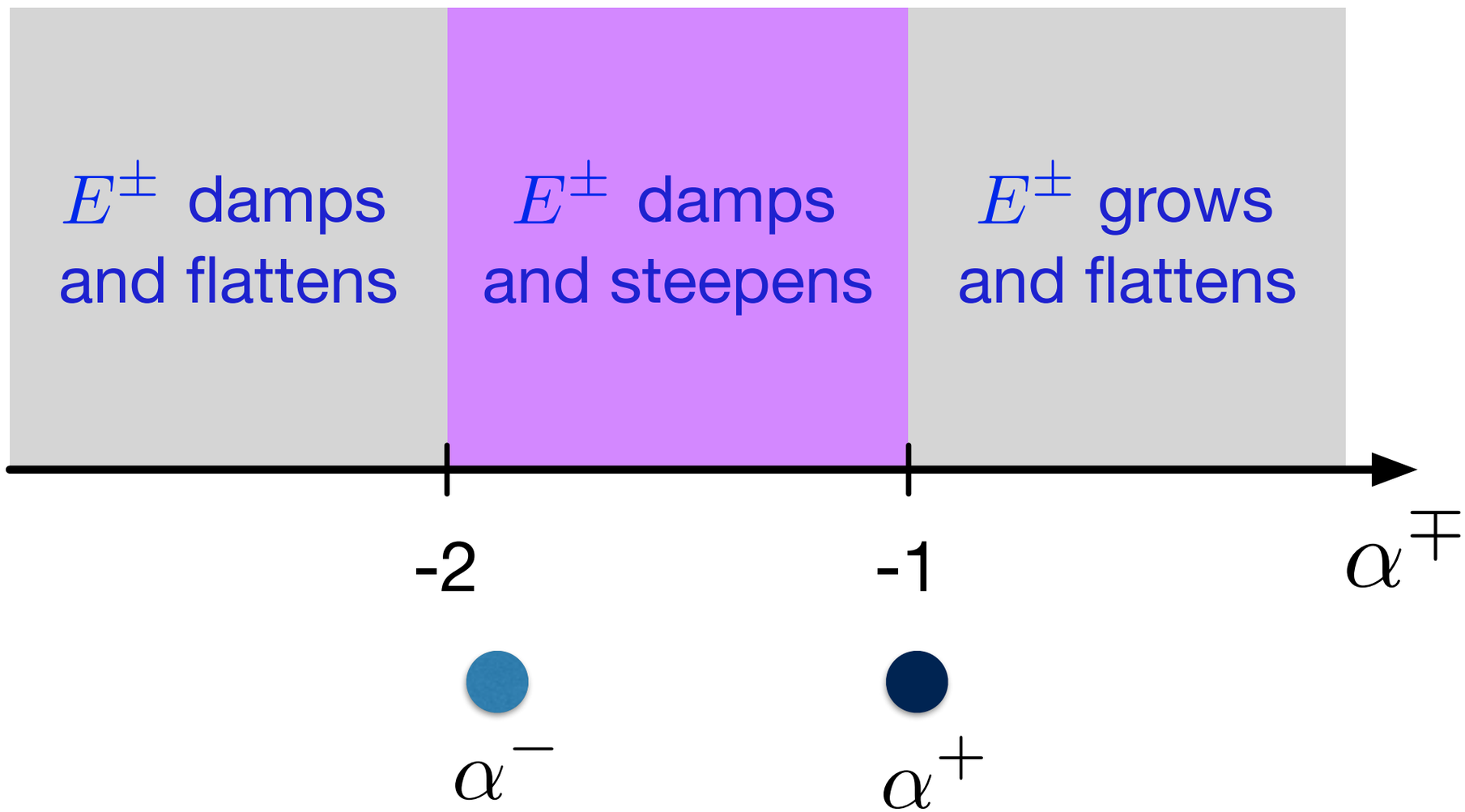

Many Open Questions…

- errors introduced by weak turbulence theory
- finite beta effects
- better treatment of slow-wave damping
- interplay between parametric instability and other types of nonlinear interactions, as well as linear non-WKB reflection.
- effects of solar-wind expansion and radial evolution
- saturation into arc-polarized waves? (Tenerani & Velli 2017)
- connection to velocity spikes? (Horbury talk)

Predictions

- In low-latitude fast wind streams, a 1/f magnetic spectrum at sub-hour timescales (f>0.0003 Hz) emerges dynamically between 10 Rs and 60 Rs via parametric instability and inverse cascade. (The 1/f scaling is a marginally stable state for the parametric instability.)

- The 1/f range is much broader at 60 Rs than at 10 Rs.

- The 1/f range spreads out in both directions from the initial energy-dominating frequency (at which fe_f is maximized). As PSP gets closer to the Sun, the 1/f range that it sees in fast wind will narrow from both the high and low-frequency ends, eventually disappearing at small enough r.