Landau Collision Operator:
Conservative Discontinuous Galerkin Discretization

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June 21, 2018

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Properties of the Landau Collision Operator
The Landau Collision Operator

- The Landau (or Fokker-Planck-Landau) collision kernel is given by

\[
L(f)(v, t) = \frac{\partial}{\partial v} \cdot \int_{\mathbb{R}^3} Q(v - v') \left( f(v', t) \frac{\partial f(v, t)}{\partial v} - f(v, t) \frac{\partial f(v', t)}{\partial v'} \right) dv',
\]

with a particle distribution function

\[ f(v, t) : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \]

and the inversely scaled projection matrix

\[
Q(v) = \frac{1}{|v|^3} \left( |v|^2 \mathbb{1} - v \otimes v \right).
\]

- It describes binary collisions of (single species) charged particles with long-range Coulomb interactions.

- Hence, the time evolution (spatially homogeneous Landau equation)

\[
\frac{\partial f(v, t)}{\partial t} = L(f)(v, t)
\]

describes the collisional relaxation of a plasma.
Properties of the Landau Equation

- Mass, momentum and energy are conserved
  \[
  \frac{d}{dt} \begin{pmatrix}
    m \\
    p \\
    E
  \end{pmatrix} \sim \frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \begin{pmatrix}
    1 \\
    v \\
    |v|^2
  \end{pmatrix} dv = \int_{\mathbb{R}^3} L(f)(v, t) \begin{pmatrix}
    1 \\
    v \\
    |v|^2
  \end{pmatrix} dv = \begin{pmatrix}
    0 \\
    0 \\
    0
  \end{pmatrix}
  \]

- Dissipation of Entropy is non-negative
  \[
  \frac{d}{dt} S = - \frac{d}{dt} \int_{\mathbb{R}^3} f(v, t) \ln(f(v, t)) dv = - \int_{\mathbb{R}^3} L(f)(v, t) \ln f dv \geq 0
  \]

- Distribution function satisfying the equilibrium condition \( L(f)(v, t) = 0 \) is a Maxwellian.

- The positivity of \( f \) is preserved.
Analytic Conservation
Multiplying the Landau equation with a time-independent test function $g(v)$ and integrating over the whole space gives a weak formulation. Assuming $f$ is compactly supported on a finite domain in velocity space a partition with elements $\Omega_k$ and edges $e_{ij}$ is introduced

$$\Omega = \bigcup_k \Omega_k, \quad e_{ij} = \Omega_i \cap \Omega_j = \partial \Omega_i \cap \partial \Omega_j, i \neq j.$$ 

Integrating by parts then yields

$$\sum_k \int_{\Omega_k} g(v) \frac{\partial f(v, t)}{\partial t} \, dv = - \sum_k \int_\Omega \int_{\Omega_k} \frac{\partial g(v)}{\partial v} \cdot Q(v - v') \Gamma(f)(v, v', t) \, dv' \, dv$$

$$+ \sum_k \int_{\partial \Omega_k} \int_\Omega g(v) Q(v - v') \Gamma(f)(v, v', t) \, dv' \cdot n_k \, d\sigma.$$ 

with symmetric matrix $Q(v - v') = Q(v' - v)$ and antisymmetric vector

$$\Gamma(f)(v, v', t) = -\Gamma(f)(v', v, t) = f(v', t) \frac{\partial f(v, t)}{\partial v} - f(v, t) \frac{\partial f(v', t)}{\partial v'}.$$
Analytic Conservation: Symmetrization of Volume Term

Looking at the volume term, also split the inner integral and divide into a same element part and a mixed element part

\[
\text{volume part} = - \sum_k \int_{\Omega_k} \int_{\Omega_k} \frac{\partial g(v)}{\partial v} \cdot Q(v - v') \Gamma(f)(v, v', t) dv' dv \\
- \sum_k \sum_{l \neq k} \int_{\Omega_k} \int_{\Omega_l} \frac{\partial g(v)}{\partial v} \cdot Q(v - v') \Gamma(f)(v, v', t) dv' dv
\]

Symmetrize first term by using the symmetry of \( Q \), antisymmetry of \( \Gamma \) and relabeling of primed and unprimed \( v \) since integration domains are the same.

\[
- \frac{1}{2} \sum_k \int_{\Omega_k} \int_{\Omega_k} \left( \frac{\partial g(v')}{\partial v'} - \frac{\partial g(v)}{\partial v} \right) \cdot Q(v - v') \Gamma(f)(v, v', t) dv' dv
\]

Symmetrize second term since for all \((k, l)\) there exits an \((l, k)\) for which using the symmetry of \( Q \), antisymmetry of \( \Gamma \), relabeling and switching integrals to \((k, l)\)

\[
- \sum_k \sum_{l > k} \int_{\Omega_k} \int_{\Omega_l} \left( \frac{\partial g(v')}{\partial v'} - \frac{\partial g(v)}{\partial v} \right) \cdot Q(v - v') \Gamma(f)(v, v', t) dv' dv
\]
For boundary part change sum to sum over edges, split into inner and outer edges and use that on \( e_{ij} \mathbf{n}_i = -\mathbf{n}_j \).

All terms combined read

\[
\frac{d}{dt} \sum_k \int_{\Omega_k} g(\mathbf{v}) f(\mathbf{v}, t) \, d\mathbf{v}
\]

\[
= -\frac{1}{2} \sum_k \int_{\Omega_k} \int_{\Omega_k} \left( \frac{\partial g(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{\partial g(\mathbf{v})}{\partial \mathbf{v}} \right) \cdot Q(\mathbf{v} - \mathbf{v}') \Gamma(f)(\mathbf{v}, \mathbf{v}', t) \, d\mathbf{v}' \, d\mathbf{v}
\]

\[
- \sum_k \sum_{l > k} \int_{\Omega_k} \int_{\Omega_l} \left( \frac{\partial g(\mathbf{v}')}{\partial \mathbf{v}'} - \frac{\partial g(\mathbf{v})}{\partial \mathbf{v}} \right) \cdot Q(\mathbf{v} - \mathbf{v}') \Gamma(f)(\mathbf{v}, \mathbf{v}', t) \, d\mathbf{v}' \, d\mathbf{v}
\]

\[
+ \sum_{e_{ij} \in \mathcal{E}_{\text{inner}}} \int_{e_{ij}} (g(\mathbf{v})|_{\Omega_i} - g(\mathbf{v})|_{\Omega_j}) \int_{\Omega} Q(\mathbf{v} - \mathbf{v}') \Gamma(\hat{f}(f|_{\Omega_i}, f|_{\Omega_j}))(\mathbf{v}, \mathbf{v}', t) \, d\mathbf{v}' \cdot \mathbf{n}_i
\]

Choosing \( g(\mathbf{v}) \in \{1, \mathbf{v}, |\mathbf{v}|^2\} \) gives conservation of mass-, momentum- and energy since 1 and \( \mathbf{v} \) give trivially zero, \( |\mathbf{v}|^2 \) generates an eigenvector of \( Q \) with zero eigenvalue and all three are continuous across elements. This is also true if \( f(\mathbf{v}, t) \) is discontinuous.
Discontinuous Galerkin Discretization
DG: Properties of the Method

- Combination of finite element and finite volume method
- In contrast to the standard finite element method the approximation space is chosen to consist of only element-wise continuous functions
- High order accuracy and able to handle complicated geometries, while good locality of data makes it easy to parallelize
- Mass matrix block diagonal
- Increased amount of degrees of freedom (dof), can not share dof on element interface
Choose a tensor product mesh with elements $\Omega_n$ and basis functions $\varphi_m^n(v)$ on each element spanning global DG space $\nabla_h$.

Choose basis that is able to represent $1, v, |v|^2$ exactly to maintain conservation. Approximate solution on element $k$ as

$$f_h(v, t) = \sum_{k, i} f^k_i(t) \varphi^k_i(v)$$

Choose test function from same space and insert both in weak form, find $f_h \in \nabla_h$ such that $\forall n, m$

$$\int_{\Omega_n} \varphi^n_m(v) \frac{\partial f_h(v, t)}{\partial t} \, dv = - \int_{\Omega_n} \int_{\Omega} \frac{\partial \varphi^n_m(v)}{\partial v} \cdot Q(v - v') \Gamma[f_h](v, v') \, dv' \, dv$$

$$+ \int_{\partial \Omega_n} \int_{\Omega} \varphi^n_m(v) Q(v - v') \tilde{\Gamma}[\tilde{f}_h, \tilde{f}_h, f_h](v, v') \, dv' \cdot n^n \, ds$$

with

$$\tilde{\Gamma}[\tilde{f}_h, \tilde{f}_h, f_h](v, v') = f_h(v') \frac{\partial \tilde{f}_h(v)}{\partial v} - \tilde{f}_h(v) \frac{\partial f_h(v')}{\partial v'}$$

Note: This is only one possible weak form others exist by integrating by parts differently.
First look at

\[ \Gamma_l[f_h](v, v') = \sum_{k, p, i, j} f^k_i(t) f^p_j(t) \Gamma_l[\varphi^k_i, \varphi^p_j](v, v') \]

which has still the symmetry \( \Gamma_l[\varphi^k_i, \varphi^p_j](v, v') = -\Gamma_l[\varphi^k_i, \varphi^p_j](v', v) \).

The whole volume term can be written as

\[
- \int_{\Omega_n} \int_{\Omega} \frac{\partial \varphi^n_m(v)}{\partial v} \cdot Q(v - v') \Gamma[f_h](v, v') \, dv' \, dv
\]

\[ = - \sum_{k, p, i, j} f^k_i(t) f^p_j(t) D^{nkp}_{mij} \]

with the constant tensor

\[
D^{nkp}_{mij} = \int_{\Omega_n} \int_{\Omega} \sum_{q, l} \frac{\partial \varphi^q_m(v)}{\partial v_q} Q_{ql}(v - v') \left( \varphi^k_i(v') \frac{\partial \varphi^p_j(v)}{\partial v_l} - \varphi^k_i(v) \frac{\partial \varphi^p_j(v')}{\partial v'_l} \right) \, dv' \, dv
\]
Problem: What is the value of $f$ at the interface of two elements? What is the value of $\partial_v f$?

For the convective term introduce the numerical flux $\hat{f}_h$. There is no unique definition, here choose centered flux, i.e.

$$\hat{f}_h(v, t) \equiv \{f_h(v, t)\} = \frac{1}{2}(f_h^+(v, t) + f_h^-(v, t)),$$

where $f^-$ and $f^+$ are the limits of $f$ approaching the boundary from the current element and the next element, respectively, i.e. for $v \in \partial \Omega_k$

$$f^\pm(v) = \lim_{\epsilon \to \infty} (v \pm \epsilon n_k).$$

For the diffusive part a first derivative of numerical flux is obtained by a recovery method.
DG: Recovery Method

Idea: Project the solution on $\Omega_n \cup \Omega_{n+}$ which is discontinuous at the interface onto a new space that is continuous in this domain.

Denote recovery solution on $\Omega_n \cup \Omega_{n+}$ by

$$\tilde{f}^{n\cup n+}(v) = \sum_i \tilde{f}^{n\cup n+}_i \psi_i^{n\cup n+}(v)$$

Global DG solution is $f_h(v) = \sum_{n,m} f^n_m \varphi^n_m(v)$

Recovery basis can be of max degree $2p - 1$ for $p$ degree of DG basis.

The $L_2$ projection reads

$$\int_{\Omega_n \cup \Omega_{n+}} \left( \tilde{f}^{n\cup n+}_h(v) - f_h(v) \right) \psi_j^{n\cup n+}(v) \, dv = 0 \quad \forall j$$

$$\Leftrightarrow \sum_i \tilde{f}^{n\cup n+}_i \int_{\Omega_n \cup \Omega_{n+}} \psi_i^{n\cup n+}(v) \psi_j^{n\cup n+}(v) \, dv$$

$$- \sum_l f^n_l \int_{\Omega_n} \varphi^n_l(v) \psi_j^{n\cup n+}(v) \, dv - \sum_l f^{n+}_l \int_{\Omega_{n+}} \varphi^{n+}_l(v) \psi_j^{n\cup n+}(v) \, dv = 0 \quad \forall j$$
The coefficients can thus be written as

$$\tilde{f}_j^{n \cup n^+} = \sum_l f_l^n P_{jl}^n + \sum_l f_l^{n^+} P_{jl}^{n^+}$$

with the constant tensors

$$\tilde{M}_{ji} = \int_{\Omega_n \cup \Omega_{n^+}} \psi_i^{n \cup n^+}(v) \psi_j^{n \cup n^+}(v) \, dv,$$

$$P_{jl}^n = \sum_i \tilde{M}^{-1}_{ji} \int_{\Omega_n} \varphi_i^n(v) \psi_i^{n \cup n^+}(v) \, dv,$$

$$P_{jl}^{n^+} = \sum_i \tilde{M}^{-1}_{ji} \int_{\Omega_{n^+}} \varphi_i^{n^+}(v) \psi_i^{n \cup n^+}(v) \, dv.$$

The derivative at the interface $\Omega_n \cap \Omega_{n^+}$ is now definable as

$$\frac{\partial}{\partial v} \tilde{f}_h^{n \cup n^+}(v) = \sum_i \tilde{f}_i^{n \cup n^+} \frac{\partial}{\partial v} \psi_i^{n \cup n^+}(v).$$

Note: The recovery coefficients are obtained by a linear combination of the solution coefficients, the corresponding matrix can be precomputed.
Inserting the central numeric flux and the recovered distribution function in the boundary term yields

\[
\begin{aligned}
&\int_{\partial \Omega} \int_{\Omega} \varphi_m^n(v) Q(v - v') \tilde{\Gamma}[\tilde{f}_h, \tilde{f}_h, f_h](v, v') \, dv' \cdot n^n \, d\sigma_n \\
&= \sum_{k,p,i,j} \left( f_i^k(t) f_j^p(t) G_{mij}^{nkp-} + f_i^k(t) f_j^p(t) G_{mij}^{nkp+} \\
&\quad - f_i^{k+}(t) f_j^p(t) B_{mij}^{nk+p} - f_i^{k-}(t) f_j^p(t) B_{mij}^{nk-p} \right)
\end{aligned}
\]

with

\[
G_{mij}^{nkp\pm} = \int_{\partial \Omega} \int_{\Omega} \sum_{q,l} \varphi_m^n(v) Q_{ql}(v - v') \varphi_i^k(v') \sum_s P_{sj}^{p\pm} \frac{\partial}{\partial v_l} \psi_{s}^{p&p'}(v) n_q^n \, dv' \, d\sigma_n
\]

\[
B_{mij}^{nk+p} = \frac{1}{2} \int_{\partial \Omega} \int_{\Omega} \sum_{q,l} \varphi_m^n(v) Q_{ql}(v - v') \varphi_i^{k\pm}(v) \frac{\partial}{\partial v_l} \varphi_j^p(v') n_q^n \, dv' \, d\sigma_n
\]
Combining the previous results to obtain the final semi-discrete form

\[
\sum_{k,i} M_{mi}^{nk} \frac{\partial f_i^k(t)}{\partial t} = - \sum_{k,p,i,j} f_i^k(t) f_j^p(t) D_{mij}^{nkp} + \sum_{k,p,i,j} \left( f_i^k(t) f_j^p-(t) G_{mij}^{nkp-} + f_i^k(t) f_j^p+(t) G_{mij}^{nkp+} - f_i^{k+}(t) f_j^p(t) B_{mij}^{nk+p} - f_i^{k-}(t) f_j^p(t) B_{mij}^{nk-p} \right)
\]

\[
= \sum_{k,p,i,j} f_i^k(t) f_j^p(t) A_{mij}^{nkp}, \quad \forall n, m
\]

Note that the tensors are sparse with regards to two of the element indices \( n, k, p \), since basis functions have only support on their respective element.
Conservation of Fully Discrete System
Conservation for Explicit Time Stepping

The problem can be stated as an initial value problem

\[ \partial_t f_s^r(t) = G_s^r[f](t), \quad f_s^r(t_0) = (f_0)_s \]

with \( G_s^r[f] = \sum_{n,m,k,p,i,j} f^k_i(t) f^p_j(t) (M^{-1})_{sm}^n \alpha_{np} \).

A general form for explicit Runge-Kutta methods is

\[ f_s^r(t_{n+1}) = f_s^r(t_n) + \Delta t \sum_{i=1}^{I} w_i k_i, \quad k_i = G_s^r[f(t_n) + \sum_{j=1}^{i-1} \alpha_{ij} k_j] \]

Because of linearity of \( G \) recursively simplifies to one case

\[ f_s^r(t_{n+1}) - f_s^r(t_n) = \Delta t G_s^r[f(t_n)] \]

Multiply with \( M \) and contract with dofs for 1, \( \nu, |\nu|^2 \)

\[ \text{lhs} = \sum_{a,b,r,s} \begin{pmatrix} 1^a_b \\ v^a_b \\ e^a_b \end{pmatrix} M_{bs}^{ar} \left( f_s^r(t_{n+1}) - f_s^r(t_n) \right) = \begin{pmatrix} m(t_{n+1}) - m(t_n) \\ p(t_{n+1}) - p(t_n) \\ E(t_{n+1}) - E(t_n) \end{pmatrix} \]

\[ = \text{rhs} = \Delta t \sum_{a,b,k,p,i,j} (1^a_b, v^a_b, e^a_b) \top \left( f^k_i(t_n) f^p_j(t_n) A_{bij}^{akp} \right) = 0 \]
Numerical Test Problem
Two dimensional relaxation problem
Initial condition given by Bi-Gaussian with $\sigma = 0.25$, $\mathbf{v}_{\text{in}} = (0.4, 0)^{\top}$

$$f(\mathbf{v}, t = 0) = \frac{1}{\sigma \sqrt{2\pi}} \left( e^{-|\mathbf{v} - \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} + e^{-|\mathbf{v} + \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} \right).$$
Two dimensional relaxation problem
Initial condition given by Bi-Gaussian with $\sigma = 0.25$, $\mathbf{v}_{\text{in}} = (0.4, 0)^\top$

$$f(\mathbf{v}, t = 0) = \frac{1}{\sigma \sqrt{2\pi}} \left( e^{-|\mathbf{v} - \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} + e^{-|\mathbf{v} + \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} \right).$$
Two-dimensional relaxation problem

Initial condition given by Bi-Gaussian with $\sigma = 0.25$, $\mathbf{v}_{\text{in}} = (0.4, 0)^T$

$$f(\mathbf{v}, t = 0) = \frac{1}{\sigma \sqrt{2\pi}} \left( e^{-|\mathbf{v} - \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} + e^{-|\mathbf{v} + \mathbf{v}_{\text{in}}|^2/(2\sigma^2)} \right).$$
2D Test Problem

Initial condition given by anisotropic distribution with discontinuity, i.e. Gaussian with cutout, e.g. due to loss cone in a magnetic mirror.
Initial condition given by anisotropic distribution with discontinuity, i.e. Gaussian with cutout, e.g. due to loss cone in a magnetic mirror.
• With \( m \) the order of the basis, \( n \) the number of elements per dimension and \( d \) the number of dimensions the storage complexity of the system tensor is \( O((mn)^{3d}) \).

\[ \Rightarrow \text{For 2d, 5 elements per dimension, quadratic basis, double precision: } A \text{ has 686 MB.} \]

• Further investigations making use of tensor decompositions/approximations might be interesting. E.g. for rank \( r \), dimensions \( d \), mode length \( n \):

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• Method has many degrees of freedom which are worth investigating, e.g. choice of: flux, projection for recovery, basis functions and order, time stepping scheme, tensor format, ...
Summary

We . . .

- Introduced the nonlinear Landau collision operator for binary Coulomb interactions
- Showed that even for a discontinuous space mass, momentum and energy are conserved if the basis can represent $1, v, |v|^2$ globally exactly.
- Discretized the space homogeneous Landau equation using a discontinuous Galerkin ansatz and a central numerical flux as well as a recovery method.
- Showed that for an explicit time stepping scheme the conservation properties are also true for the fully discrete system.
- Gave two numerical test cases that confirmed conservation up to machine precision and the capability to handle discontinuities in the solution.