Landau Collision Operator:

Conservative Discontinuous Galerkin Discretization

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Properties of the Landau Collision Operator

The Landau Collision Operator

- The Landau (or Fokker-Planck-Landau) collision kernel is given by

$$L(f)(\boldsymbol{v},t) = \frac{\partial}{\partial \boldsymbol{v}} \cdot \int\limits_{\mathbb{R}^3} Q(\boldsymbol{v}-\boldsymbol{v}') \left(f(\boldsymbol{v}',t) \frac{\partial f(\boldsymbol{v},t)}{\partial \boldsymbol{v}} - f(\boldsymbol{v},t) \frac{\partial f(\boldsymbol{v}',t)}{\partial \boldsymbol{v}'} \right) d\boldsymbol{v}' ,$$

  with a particle distribution function

$$f(\boldsymbol{v},t) : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$$

  and the inversely scaled projection matrix

$$Q(\boldsymbol{v}) = \frac{1}{|\boldsymbol{v}|^3} \left(|\boldsymbol{v}|^2 \, \mathbb{1} - \boldsymbol{v} \otimes \boldsymbol{v} \right) .$$

- It describes binary collisions of (single species) charged particles with long-range Coulomb interactions.
- Hence, the time evolution (spatially homogeneous Landau equation)

$$\frac{\partial f(\boldsymbol{v},t)}{\partial t} = L(f)(\boldsymbol{v},t)$$

  describes the collisional relaxation of a plasma.

Properties of the Landau Equation

- Mass, momentum and energy are conserved

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} m \\ \boldsymbol{p} \\ E \end{pmatrix} \sim \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} f(\boldsymbol{v},t)\begin{pmatrix} 1 \\ \boldsymbol{v} \\ |\boldsymbol{v}|^2 \end{pmatrix}\mathrm{d}\boldsymbol{v} = \int\limits_{\mathbb{R}^3} L(f)(\boldsymbol{v},t)\begin{pmatrix} 1 \\ \boldsymbol{v} \\ |\boldsymbol{v}|^2 \end{pmatrix}\mathrm{d}\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- Dissipation of Entropy is non-negative

$$\frac{\mathrm{d}}{\mathrm{d}t}S = -\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{R}^3} f(\boldsymbol{v},t)\ln\left(f(\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v} = -\int\limits_{\mathbb{R}^3} L(f)(\boldsymbol{v},t)\ln f\,\mathrm{d}\boldsymbol{v} \geq 0$$

- Distribution function satisfying the equilibrium condition $L(f)(\boldsymbol{v},t) = 0$ is a Maxwellian.
- The positivity of f is preserved.

Analytic Conservation

Analytic Conservation: Weak Form Landau Equation

Multiplying the Landau equation with a time-independent test function $g(\boldsymbol{v})$ and integrating over the whole space gives a weak formulation. Assuming f is compactly supported on a finite domain in velocity space a partition with elements Ω_k and edges e_{ij} is introduced

$$\Omega = \bigcup_k \Omega_k \,, \quad e_{ij} = \Omega_i \cap \Omega_j = \partial\Omega_i \cap \partial\Omega_j \,, i \neq j \,.$$

Integrating by parts then yields

$$\sum_k \int_{\Omega_k} g(\boldsymbol{v}) \frac{\partial f(\boldsymbol{v},t)}{\partial t} \,\mathrm{d}\boldsymbol{v} = -\sum_k \int_{\Omega_k} \int_{\Omega} \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}} \cdot Q(\boldsymbol{v}-\boldsymbol{v}') \,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t) \,\mathrm{d}\boldsymbol{v}' \,\mathrm{d}\boldsymbol{v}$$
$$+ \sum_k \int_{\partial\Omega_k} \int_{\Omega} g(\boldsymbol{v}) \, Q(\boldsymbol{v}-\boldsymbol{v}') \,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t) \,\mathrm{d}\boldsymbol{v}' \cdot \boldsymbol{n}_k \,\mathrm{d}\sigma.$$

with symmetric matrix $Q(\boldsymbol{v}-\boldsymbol{v}') = Q(\boldsymbol{v}'-\boldsymbol{v})$ and antisymmetric vector

$$\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t) = -\boldsymbol{\Gamma}(f)(\boldsymbol{v}',\boldsymbol{v},t) = f(\boldsymbol{v}',t)\frac{\partial f(\boldsymbol{v},t)}{\partial \boldsymbol{v}} - f(\boldsymbol{v},t)\frac{\partial f(\boldsymbol{v}',t)}{\partial \boldsymbol{v}'} \,.$$

Analytic Conservation: Symmetrization of Volume Term

Looking at the volume term, also split the inner integral and divide into a same element part and a mixed element part

$$\text{volume part} = -\sum_k \int\limits_{\Omega_k} \int\limits_{\Omega_k} \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}} \cdot Q(\boldsymbol{v}-\boldsymbol{v}')\,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t)\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v}$$

$$-\sum_k \sum_{l\neq k} \int\limits_{\Omega_k} \int\limits_{\Omega_l} \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}} \cdot Q(\boldsymbol{v}-\boldsymbol{v}')\,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t)\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v}$$

Symmetrize first term by using the symmetry of Q, antisymmetry of $\boldsymbol{\Gamma}$ and relabeling of primed and unprimed v since integration domains are the same.

$$-\frac{1}{2}\sum_k \int_{\Omega_k}\int_{\Omega_k} \left(\frac{\partial g(\boldsymbol{v}')}{\partial \boldsymbol{v}'} - \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}}\right) \cdot Q(\boldsymbol{v}-\boldsymbol{v}')\,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t)\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v}$$

Symmetrize second term since for all (k,l) there exits an (l,k) for which using the symmetry of Q, antisymmetry of $\boldsymbol{\Gamma}$, relabeling and switching integrals to (k,l)

$$-\sum_k \sum_{l>k} \int_{\Omega_k}\int_{\Omega_l} \left(\frac{\partial g(\boldsymbol{v}')}{\partial \boldsymbol{v}'} - \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}}\right) \cdot Q(\boldsymbol{v}-\boldsymbol{v}')\,\boldsymbol{\Gamma}(f)(\boldsymbol{v},\boldsymbol{v}',t)\,\mathrm{d}\boldsymbol{v}'\,\mathrm{d}\boldsymbol{v}$$

Analytic Conservation: All Terms

For boundary part change sum to sum over edges, split into inner and outer edges and use that on e_{ij} $\boldsymbol{n}_i = -\boldsymbol{n}_j$.
All terms combined read

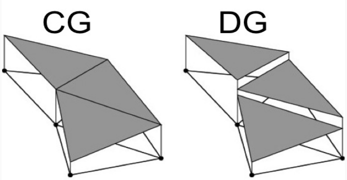
$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \sum_k \int\limits_{\Omega_k} g(\boldsymbol{v}) f(\boldsymbol{v}, t) \,\mathrm{d}\boldsymbol{v} \\
&= -\frac{1}{2} \sum_k \int_{\Omega_k} \int_{\Omega_k} \left(\frac{\partial g(\boldsymbol{v}')}{\partial \boldsymbol{v}'} - \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}} \right) \cdot Q(\boldsymbol{v} - \boldsymbol{v}') \,\boldsymbol{\Gamma}(f)(\boldsymbol{v}, \boldsymbol{v}', t) \,\mathrm{d}\boldsymbol{v}' \,\mathrm{d}\boldsymbol{v} \\
&\quad - \sum_k \sum_{l>k} \int_{\Omega_k} \int_{\Omega_l} \left(\frac{\partial g(\boldsymbol{v}')}{\partial \boldsymbol{v}'} - \frac{\partial g(\boldsymbol{v})}{\partial \boldsymbol{v}} \right) \cdot Q(\boldsymbol{v} - \boldsymbol{v}') \,\boldsymbol{\Gamma}(f)(\boldsymbol{v}, \boldsymbol{v}', t) \,\mathrm{d}\boldsymbol{v}' \,\mathrm{d}\boldsymbol{v} \\
&\quad + \sum_{e_{ij} \in \mathcal{E}_{\text{inner}}} \int_{e_{ij}} \left(g(\boldsymbol{v})|_{\Omega_i} - g(\boldsymbol{v})|_{\Omega_j} \right) \int_{\Omega} Q(\boldsymbol{v} - \boldsymbol{v}') \,\boldsymbol{\Gamma}(\hat{f}(f|_{\Omega_i}, f|_{\Omega_j}))(\boldsymbol{v}, \boldsymbol{v}', t) \,\mathrm{d}\boldsymbol{v}' \cdot \boldsymbol{n}
\end{aligned}
$$

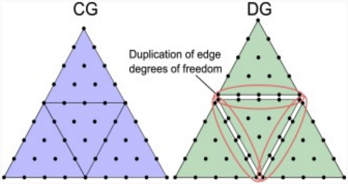
Choosing $g(\boldsymbol{v}) \in \{1, \boldsymbol{v}, |\boldsymbol{v}|^2\}$ gives conservation of mass-, momentum- and energy since 1 and $\boldsymbol{v}$ give trivially zero, $|\boldsymbol{v}|^2$ generates an eigenvector of Q with zero eigenvalue and all three are continuous across elements. This is also true if $f(\boldsymbol{v}, t)$ is discontinuous.

Discontinuous Galerkin Discretization

DG: Properties of the Method

- Combination of finite element and finite volume method
- In contrast to the standard finite element method the approximation space is chosen to consist of only element-wise continuous functions
- High order accuracy and able to handle complicated geometries, while good locality of data makes it easy to parallelize
- Mass matrix block diagonal
- Increased amount of degrees of freedom (dof), can not share dof on element interface





DG: Weak Form

Choose a tensor product mesh with elements $\Omega_{\boldsymbol{n}}$ and basis functions $\varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})$ on each element spanning global DG space $\mathbb{V}_h$.
Choose basis that is able to represent $1, \boldsymbol{v}, |\boldsymbol{v}|^2$ exactly to maintain conservation. Approximate solution on element $\boldsymbol{k}$ as

$$f_h(\boldsymbol{v}, t) = \sum_{\boldsymbol{k},\boldsymbol{i}} f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)\, \varphi_{\boldsymbol{i}}^{\boldsymbol{k}}(\boldsymbol{v})$$

Choose test function from same space and insert both in weak form, find $f_h \in \mathbb{V}_h$ such that $\forall \boldsymbol{n}, \boldsymbol{m}$

$$\begin{aligned}\int_{\Omega_{\boldsymbol{n}}} \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v}) \frac{\partial f_h(\boldsymbol{v}, t)}{\partial t}\, \mathrm{d}v = &- \int_{\Omega_{\boldsymbol{n}}} \int_{\Omega} \frac{\partial \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})}{\partial \boldsymbol{v}} \cdot Q(\boldsymbol{v} - \boldsymbol{v}')\, \boldsymbol{\Gamma}[f_h](\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}\boldsymbol{v}'\, \mathrm{d}\boldsymbol{v} \\ &+ \int_{\partial\Omega_{\boldsymbol{n}}} \int_{\Omega} \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})\, Q(\boldsymbol{v} - \boldsymbol{v}')\, \tilde{\boldsymbol{\Gamma}}[\widetilde{f_h}, \widehat{f_h}, f_h](\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}\boldsymbol{v}' \cdot \boldsymbol{n}^{\boldsymbol{n}}\, \mathrm{d}\sigma\end{aligned}$$

with

$$\tilde{\boldsymbol{\Gamma}}[\widetilde{f_h}, \widehat{f_h}, f_h](\boldsymbol{v}, \boldsymbol{v}') = f_h(\boldsymbol{v}') \frac{\partial \widetilde{f_h}(\boldsymbol{v})}{\partial \boldsymbol{v}} - \widehat{f_h}(\boldsymbol{v}) \frac{\partial f_h(\boldsymbol{v}')}{\partial \boldsymbol{v}'}\,.$$

Note: This is only one possible weak form others exist by integrating by parts differently.

DG: Volume Part

First look at

$$\Gamma_l[f_h](\boldsymbol{v}, \boldsymbol{v}') = \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} f_{\boldsymbol{i}}^{\boldsymbol{k}}(t) f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \Gamma_l[\varphi_{\boldsymbol{i}}^{\boldsymbol{k}}, \varphi_{\boldsymbol{j}}^{\boldsymbol{p}}](\boldsymbol{v}, \boldsymbol{v}')$$

which has still the symmetry $\Gamma_l[\varphi_{\boldsymbol{i}}^{\boldsymbol{k}}, \varphi_{\boldsymbol{j}}^{\boldsymbol{p}}](\boldsymbol{v}, \boldsymbol{v'}) = -\Gamma_l[\varphi_{\boldsymbol{i}}^{\boldsymbol{k}}, \varphi_{\boldsymbol{j}}^{\boldsymbol{p}}](\boldsymbol{v}', \boldsymbol{v})$.
The whole volume term can be written as

$$\begin{aligned} &- \int_{\Omega_{\boldsymbol{n}}} \int_{\Omega} \frac{\partial \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})}{\partial \boldsymbol{v}} \cdot Q(\boldsymbol{v} - \boldsymbol{v}')\, \boldsymbol{\Gamma}[f_h](\boldsymbol{v}, \boldsymbol{v}')\, \mathrm{d}\boldsymbol{v}'\, \mathrm{d}\boldsymbol{v} \\ = &- \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)\, f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \mathcal{D}_{\boldsymbol{mij}}^{\boldsymbol{nkp}} \end{aligned}$$

with the constant tensor

$$\mathcal{D}_{\boldsymbol{mij}}^{\boldsymbol{nkp}} \equiv \int_{\Omega_{\boldsymbol{n}}} \int_{\Omega} \sum_{q,l} \frac{\partial \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})}{\partial v_q} Q_{ql}(\boldsymbol{v} - \boldsymbol{v}') \left(\varphi_{\boldsymbol{i}}^{\boldsymbol{k}}(\boldsymbol{v}') \frac{\partial \varphi_{\boldsymbol{j}}^{\boldsymbol{p}}(\boldsymbol{v})}{\partial v_l} - \varphi_{\boldsymbol{i}}^{\boldsymbol{k}}(\boldsymbol{v}) \frac{\partial \varphi_{\boldsymbol{j}}^{\boldsymbol{p}}(\boldsymbol{v}')}{\partial v_l'} \right) \mathrm{d}\boldsymbol{v}'\, \mathrm{d}\boldsymbol{v}$$

DG: Numerical Flux

Problem: What is the value of f at the interface of two elements? What is the value of $\partial_{\boldsymbol{v}} f$?

For the convective term introduce the numerical flux $\widehat{f_h}$. There is no unique definition, here choose centered flux, i.e.

$$\widehat{f_h}(\boldsymbol{v}, t) \equiv \{f_h(\boldsymbol{v}, t)\} = \frac{1}{2}(f_h^+(\boldsymbol{v}, t) + f_h^-(\boldsymbol{v}, t)),$$

where f^- and f^+ are the limits of f approaching the boundary from the current element and the next element, respectively, i.e. for $\boldsymbol{v} \in \partial\Omega_{\boldsymbol{k}}$
$f^{\pm}(\boldsymbol{v}) = \lim_{\epsilon \to \infty} (\boldsymbol{v} \pm \epsilon\, \boldsymbol{n_k})$.

For the diffusive part a first derivative of numerical flux is obtained by a recovery method

Idea: Project the solution on $\Omega_{\boldsymbol{n}} \cup \Omega_{\boldsymbol{n}+}$ which is discontinuous at the interface onto a new space that is continuous in this domain.
Denote recovery solution on $\Omega_{\boldsymbol{n}} \cup \Omega_{\boldsymbol{n}+}$ by
$\tilde{f}_h^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) = \sum_{\boldsymbol{i}} \tilde{f}_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}\, \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})$
Global DG solution is $f_h(\boldsymbol{v}) = \sum_{\boldsymbol{n},\boldsymbol{m}} f_{\boldsymbol{m}}^{\boldsymbol{n}}\, \varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})$
Recovery basis can be of max degree $2p-1$ for p degree of DG basis.
The L_2 projection reads

$$\int_{\Omega_{\boldsymbol{n}}\cup\Omega_{\boldsymbol{n}+}} \left(\tilde{f}_h^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) - f_h(\boldsymbol{v})\right) \psi_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 0 \quad \forall \boldsymbol{j}$$

$$\Leftrightarrow \sum_{\boldsymbol{i}} \tilde{f}_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+} \int_{\Omega_{\boldsymbol{n}}\cup\Omega_{\boldsymbol{n}+}} \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})\, \psi_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})\,\mathrm{d}v$$

$$- \sum_{\boldsymbol{l}} f_{\boldsymbol{l}}^{\boldsymbol{n}} \int_{\Omega_{\boldsymbol{n}}} \varphi_{\boldsymbol{l}}^{\boldsymbol{n}}(\boldsymbol{v})\, \psi_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} - \sum_{\boldsymbol{l}} f_{\boldsymbol{l}}^{\boldsymbol{n}+} \int_{\Omega_{\boldsymbol{n}+}} \varphi_{\boldsymbol{l}}^{\boldsymbol{n}+}(\boldsymbol{v})\, \psi_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v})\,\mathrm{d}\boldsymbol{v} = 0\,\forall \boldsymbol{j}$$

DG: Recovery Method Continued

The coefficients can thus be written as

$$\tilde{f}_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+} = \sum_{\boldsymbol{l}} f_{\boldsymbol{l}}^{\boldsymbol{n}} \, P_{\boldsymbol{jl}}^{\boldsymbol{n}} + \sum_{\boldsymbol{l}} f_{\boldsymbol{l}}^{\boldsymbol{n}+} \, P_{\boldsymbol{jl}}^{\boldsymbol{n}+}$$

with the constant tensors

$$\tilde{M}_{\boldsymbol{ji}} = \int_{\Omega_{\boldsymbol{n}}\cup\Omega_{\boldsymbol{n}+}} \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) \, \psi_{\boldsymbol{j}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \,,$$

$$P_{\boldsymbol{jl}}^{\boldsymbol{n}} = \sum_{\boldsymbol{i}} \tilde{M}_{\boldsymbol{ji}}^{-1} \int_{\Omega_{\boldsymbol{n}}} \varphi_{\boldsymbol{l}}^{\boldsymbol{n}}(\boldsymbol{v}) \, \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v} \,,$$

$$P_{\boldsymbol{jl}}^{\boldsymbol{n}+} = \sum_{\boldsymbol{i}} \tilde{M}_{\boldsymbol{ji}}^{-1} \int_{\Omega_{\boldsymbol{n}+}} \varphi_{\boldsymbol{l}}^{\boldsymbol{n}+}(\boldsymbol{v}) \, \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) \, \mathrm{d}\boldsymbol{v}$$

The derivative at the interface $\Omega_{\boldsymbol{n}} \cap \Omega_{\boldsymbol{n}+}$ is now definable as

$$\frac{\partial}{\partial \boldsymbol{v}} \tilde{f}_h^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) = \sum_{\boldsymbol{i}} \tilde{f}_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+} \, \frac{\partial}{\partial \boldsymbol{v}} \psi_{\boldsymbol{i}}^{\boldsymbol{n}\cup\boldsymbol{n}+}(\boldsymbol{v}) \,.$$

Note: The recovery coefficients are obtained by a linear combination of the solution coefficients, the corresponding matrix can be precomputed.

DG: Boundary Part

Inserting the central numeric flux and the recovered distribution function in the boundary term yields

$$\int_{\partial\Omega_{\boldsymbol{n}}}\int_{\Omega}\varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})\,Q(\boldsymbol{v}-\boldsymbol{v}')\,\tilde{\boldsymbol{\Gamma}}[\tilde{f}_h,\widehat{f_h},f_h](\boldsymbol{v},\boldsymbol{v}')\,\mathrm{d}v'\cdot\boldsymbol{n}^{\boldsymbol{n}}\,\mathrm{d}\sigma_{\boldsymbol{n}}$$
$$=\sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}}\Big(f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)f_{\boldsymbol{j}}^{\boldsymbol{p}_-}(t)\,\mathcal{G}_{\boldsymbol{mij}}^{\boldsymbol{nkp}_-}+f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)f_{\boldsymbol{j}}^{\boldsymbol{p}_+}(t)\,\mathcal{G}_{\boldsymbol{mij}}^{\boldsymbol{nkp}_+}$$
$$-f_{\boldsymbol{i}}^{\boldsymbol{k}_+}(t)f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\,\mathcal{B}_{\boldsymbol{mij}}^{\boldsymbol{nk}_+\boldsymbol{p}}-f_{\boldsymbol{i}}^{\boldsymbol{k}_-}(t)f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\,\mathcal{B}_{\boldsymbol{mij}}^{\boldsymbol{nk}_-\boldsymbol{p}}\Big)$$

with

$$\mathcal{G}_{\boldsymbol{mij}}^{\boldsymbol{nkp}_\pm}=\int_{\partial\Omega_{\boldsymbol{n}}}\int_{\Omega}\sum_{q,l}\varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})\,Q_{ql}(\boldsymbol{v}-\boldsymbol{v}')\varphi_{\boldsymbol{i}}^{\boldsymbol{k}}(\boldsymbol{v}')\sum_{\boldsymbol{s}}P_{\boldsymbol{sj}}^{\boldsymbol{p}\pm}\frac{\partial}{\partial v_l}\psi_{\boldsymbol{s}}^{\boldsymbol{p}\&\boldsymbol{p}_+}(\boldsymbol{v})\,n_q^{\boldsymbol{n}}\,\mathrm{d}v'\,\mathrm{d}\sigma_{\boldsymbol{n}}$$
$$\mathcal{B}_{\boldsymbol{mij}}^{\boldsymbol{nk}_\pm\boldsymbol{p}}=\frac{1}{2}\int_{\partial\Omega_{\boldsymbol{n}}}\int_{\Omega}\sum_{q,l}\varphi_{\boldsymbol{m}}^{\boldsymbol{n}}(\boldsymbol{v})\,Q_{ql}(\boldsymbol{v}-\boldsymbol{v}')\varphi_{\boldsymbol{i}}^{\boldsymbol{k}\pm}(\boldsymbol{v})\frac{\partial}{\partial v_l'}\varphi_{\boldsymbol{j}}^{\boldsymbol{p}}(\boldsymbol{v}')\,n_q^{\boldsymbol{n}}\,\mathrm{d}v'\,\mathrm{d}\sigma_{\boldsymbol{n}}$$

DG: Semi-Discrete Form

Combining the previous results to obtain the final semi-discrete form

$$
\begin{aligned}
\sum_{\boldsymbol{k},\boldsymbol{i}} M_{\boldsymbol{mi}}^{\boldsymbol{nk}} \frac{\partial f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)}{\partial t} = &- \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)\, f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \mathcal{D}_{\boldsymbol{mij}}^{\boldsymbol{nkp}} \\
&+ \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} \Big(f_{\boldsymbol{i}}^{\boldsymbol{k}}(t) f_{\boldsymbol{j}}^{\boldsymbol{p}_-}(t)\, \mathcal{G}_{\boldsymbol{mij}}^{\boldsymbol{nkp}_-} + f_{\boldsymbol{i}}^{\boldsymbol{k}}(t) f_{\boldsymbol{j}}^{\boldsymbol{p}_+}(t)\, \mathcal{G}_{\boldsymbol{mij}}^{\boldsymbol{nkp}_+} \\
&\qquad - f_{\boldsymbol{i}}^{\boldsymbol{k}_+}(t) f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \mathcal{B}_{\boldsymbol{mij}}^{\boldsymbol{nk}_+\boldsymbol{p}} - f_{\boldsymbol{i}}^{\boldsymbol{k}_-}(t) f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \mathcal{B}_{\boldsymbol{mij}}^{\boldsymbol{nk}_-\boldsymbol{p}} \Big) \\
= &\sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} f_{\boldsymbol{i}}^{\boldsymbol{k}}(t)\, f_{\boldsymbol{j}}^{\boldsymbol{p}}(t)\, \mathcal{A}_{\boldsymbol{mij}}^{\boldsymbol{nkp}}, \quad \forall \boldsymbol{n}, \boldsymbol{m}
\end{aligned}
$$

Note that the tensors are sparse with regards to two of the element indices $\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{p}$, since basis functions have only support on their respective element.

Conservation of Fully Discrete System

Conservation for Explicit Time Stepping

The problem can be stated as an initial value problem

$$\partial_t f^{\boldsymbol{r}}_{\boldsymbol{s}}(t) = G^{\boldsymbol{r}}_{\boldsymbol{s}}[f](t)\,, \quad f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_0) = (f_0)^{\boldsymbol{r}}_{\boldsymbol{s}}$$

$$\text{with } G^{\boldsymbol{r}}_{\boldsymbol{s}}[f] = \sum_{\boldsymbol{n},\boldsymbol{m},\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} f^{\boldsymbol{k}}_{\boldsymbol{i}}(t)\, f^{\boldsymbol{p}}_{\boldsymbol{j}}(t)\, (M^{-1})^{\boldsymbol{rn}}_{\boldsymbol{sm}}\, \mathcal{A}^{\boldsymbol{nkp}}_{\boldsymbol{mij}}\,.$$

A general form for explicit Runge-Kutta methods is

$$f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_{n+1}) = f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_n) + \Delta t \sum_{i=1}^{I} w_i k_i\,, \quad k_i = G^{\boldsymbol{r}}_{\boldsymbol{s}}\Big[f(t_n) + \sum_{j=1}^{i-1} \alpha_{ij} k_j\Big]$$

Because of linearity of G recursively simplifies to one case

$$f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_{n+1}) - f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_n) = \Delta t\, G^{\boldsymbol{r}}_{\boldsymbol{s}}[f(t_n)]$$

Multiply with M and contract with dofs for $1, \boldsymbol{v}, |\boldsymbol{v}|^2$

$$\text{lhs} = \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{r},\boldsymbol{s}} \begin{pmatrix} 1^{\boldsymbol{a}}_{\boldsymbol{b}} \\ \boldsymbol{v}^{\boldsymbol{a}}_{\boldsymbol{b}} \\ e^{\boldsymbol{a}}_{\boldsymbol{b}} \end{pmatrix} M^{\boldsymbol{ar}}_{\boldsymbol{bs}} \Big(f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_{n+1}) - f^{\boldsymbol{r}}_{\boldsymbol{s}}(t_n)\Big) = \begin{pmatrix} m(t_{n+1}) - m(t_n) \\ \boldsymbol{p}(t_{n+1}) - \boldsymbol{p}(t_n) \\ E(t_{n+1}) - E(t_n) \end{pmatrix}$$

$$= \text{rhs} = \Delta t \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{k},\boldsymbol{p},\boldsymbol{i},\boldsymbol{j}} (1^{\boldsymbol{a}}_{\boldsymbol{b}}, \boldsymbol{v}^{\boldsymbol{a}}_{\boldsymbol{b}}, e^{\boldsymbol{a}}_{\boldsymbol{b}})^\top f^{\boldsymbol{k}}_{\boldsymbol{i}}(t_n)\, f^{\boldsymbol{p}}_{\boldsymbol{j}}(t_n)\, \mathcal{A}^{\boldsymbol{akp}}_{\boldsymbol{bij}} = 0$$

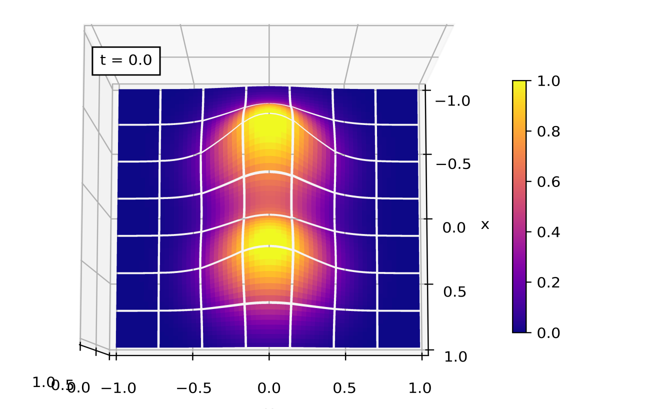
Numerical Test Problem

2D Test Problem

Two dimensional relaxation problem

Initial condition given by Bi-Gaussian with $\sigma = 0.25\,, \boldsymbol{v}_{\mathrm{in}} = (0.4, 0)^\top$

$$f(\boldsymbol{v}, t=0) = \frac{1}{\sigma\sqrt{2\pi}} \left(e^{-|\boldsymbol{v}-\boldsymbol{v}_{\mathrm{in}}|^2/(2\sigma^2)} + e^{-|\boldsymbol{v}+\boldsymbol{v}_{\mathrm{in}}|^2/(2\sigma^2)} \right) .$$



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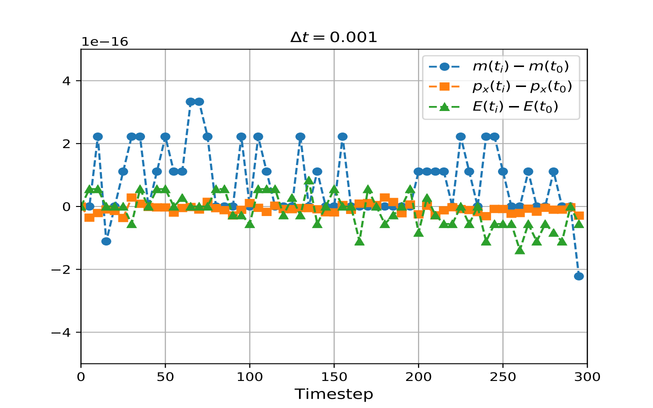
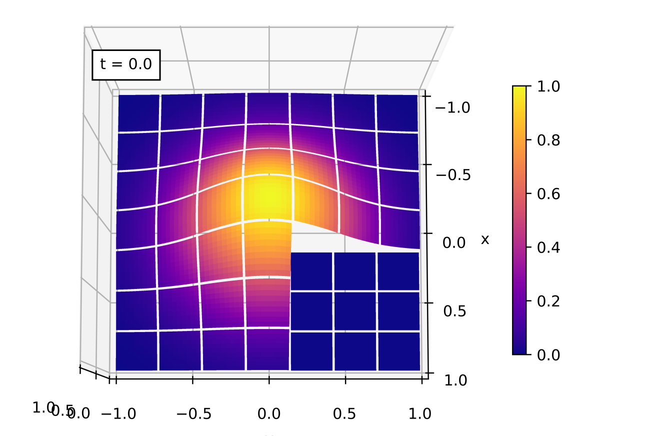


2D Test Problem

Initial condition given by anisotropic distribution with discontinuity, i.e. Gaussian with cutout, e.g. due to loss cone in a magnetic mirror



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Remarks

- With m the order of the basis, n the number of elements per dimension and d the number of dimensions the storage complexity of the system tensor is $\mathcal{O}\big((mn)^{3d}\big)$
  $\Rightarrow$ For 2d, 5 elements per dimension, quadratic basis, double precision: $\mathcal{A}$ has 686 MB.
- Further investigations making use of tensor decompositions/approximations might be interesting.
  E.g. for rank r, dimensions d, mode length n:

	CP	Tucker	Hierarchical Tucker	Tensor Train
complexity	$\mathcal{O}(ndr)$	$\mathcal{O}\Big(r^d + ndr\Big)$	$\mathcal{O}\big(ndr + (d-2)r^3 + r^2\big)$	$\mathcal{O}\big((d-2)nr^2 + 2nr\big)$
closedness	no	yes	yes	yes

- Method has many degrees of freedom which are worth investigating, e.g. choice of: flux, projection for recovery, basis functions and order, time stepping scheme, tensor format, ...

Summary

We . . .

- Introduced the nonlinear Landau collision operator for binary Coulomb interactions
- Showed that even for a discontinuous space mass, momentum and energy are conserved if the basis can represent $1, \boldsymbol{v}, |\boldsymbol{v}|^2$ globally exactly.
- Discretized the space homogeneous Landau equation using a discontinuous Galerkin ansatz and a central numerical flux as well as a recovery method.
- Showed that for an explicit time stepping scheme the conservation properties are also true for the fully discrete system.
- Gave two numerical test cases that confirmed conservation up to machine precision and the capability to handle discontinuities in the solution.