Kelvin-Helmholtz Instability is the Result of Parity-Time Symmetry Breaking*

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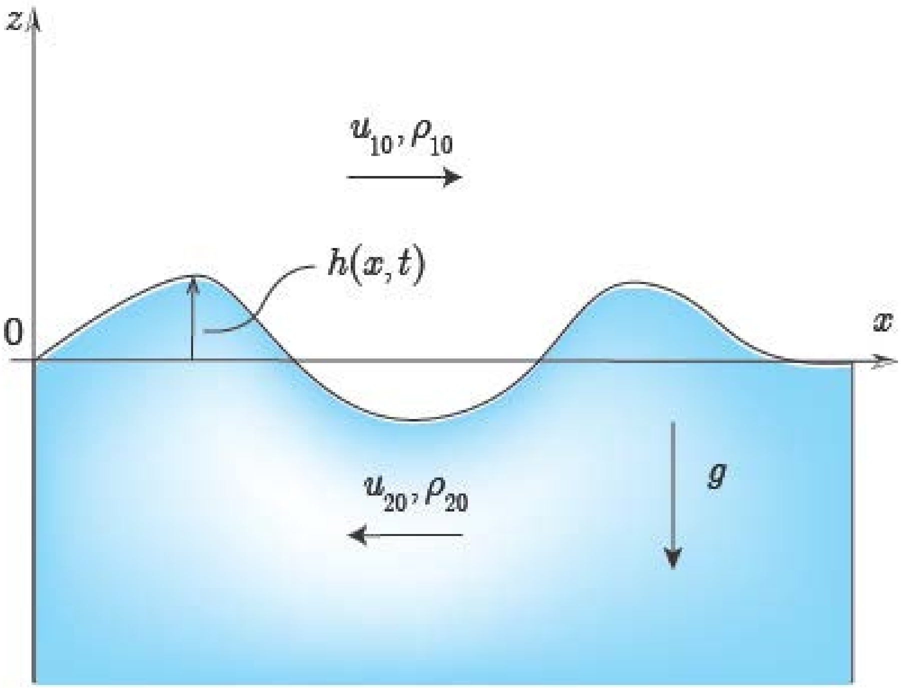
* [Qin, Zhang, Glasser, Xiao arXiv:1810.11460]

Kelvin-Helmholtz-Rayleigh-Taylor Instability



Parity-Time (PT)-symmetry

Standard Quantum Mechanics:

Schrodinger's equation:

$$\dot{\psi} = -iH\psi = A\psi,$$

H : Hermitian $\longmapsto$ real eigenvalues

Bender 1998:

However, H needs not be Hermitian to have real eigenvalues.

H is PT-symmetric if $PTH - HPT = 0$,

where $P^2 = I$, T : complex conjugate operation

$$P\overline{H} - HP = 0 \text{ or } PA + \overline{A}P = 0$$

The physics of PT-symmetry

$P-$reflection: $\phi \equiv P\psi$

$$\dot{\phi} = PAP^{-1}\phi$$

But, in general, $PAP^{-1} \neq A$

Consider an additional $T-$reflection: $t \rightarrow -t$ and $i \rightarrow -i$

$$\dot{\bar{\phi}} = -\overline{PAP^{-1}}\bar{\phi}$$

if $-\overline{PAP^{-1}} = A$, then the system is PT-symmetric.

Spectrum properties of PT-symmetric operators

(I) The spectrum of $H = iA$ is symmetric w.r.t real axis: $\lambda = a \pm ib$.

(II) There are boundaries in the parameter space that separate regions with **unbroken PT-symmetry** where all eigenvalues of H are real from regions with **spontaneously broken PT-symmetry** where H has at least one pair of complex conjugate eigenvalues.

(III) In regions with **unbroken PT-symmetry**, any eigenvector ξ of H is also an eigenvector of the PT operator, i.e., $(PT)\xi = \alpha\xi$, which implies that the PT-symmetry of H is preserved by the eigenvector. In regions with **spontaneously broken PT-symmetry**, the eigenvectors corresponding to the pair of complex conjugate eigenvalues do not preserve the PT-symmetry, i.e., they are not eigenvectors of the PT operator.

Physics of PT-symmetry is an active research field

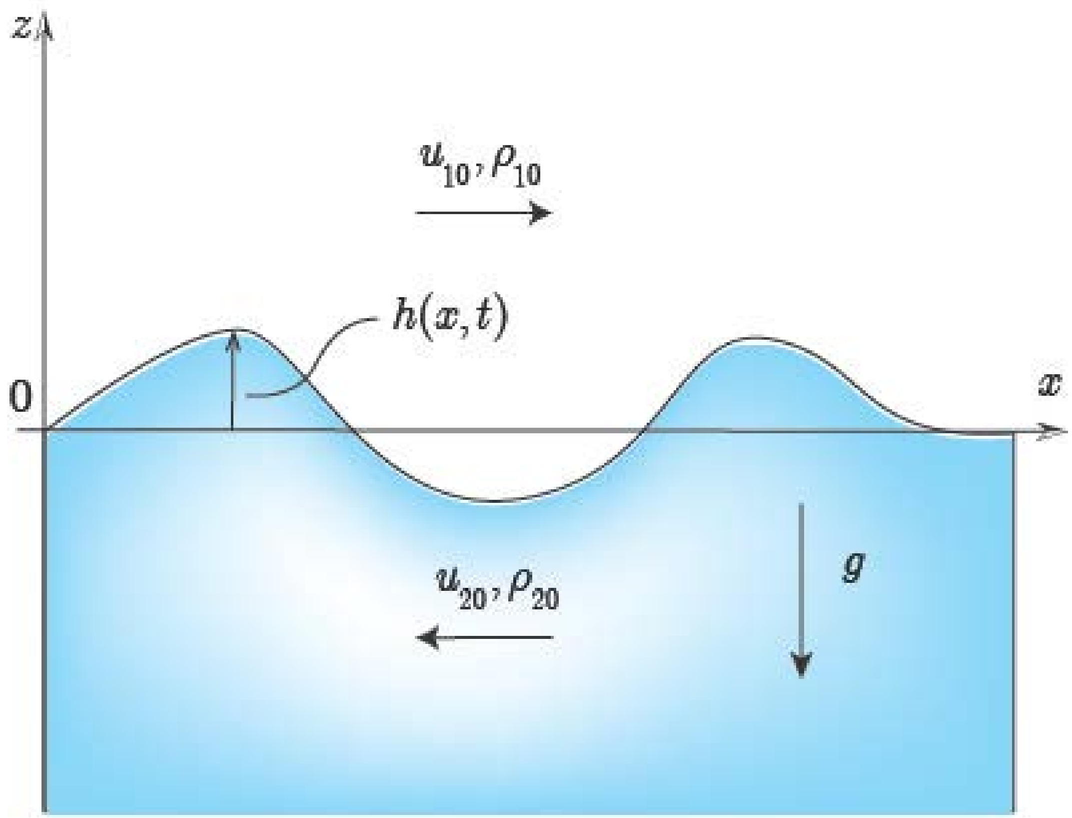
"Space-time reflection symmetry, or PT symmetry, first proposed in quantum mechanics by Bender and Boettcher in 1998 [1], has become an active research area in fundamental physics. More than two thousand papers have been published on the subject and papers have appeared in two dozen categories of the arXiv. Over two dozen international conferences and symposia specifically devoted to PT symmetry have been held and many PhD theses have been written."

--- Bender, Europhysics News 47, 17 (2016)

Main results

- **Conservative systems governed by Newton's law are PT-symmetric.**
- **Classical instabilities of conservative systems occur when and only when PT-symmetry is broken spontaneously.**

Kelvin-Helmholtz-Rayleigh-Taylor Instability



Governing equations – impressible 2D fluid system

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0,$$
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x},$$
$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g,$$
$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+w\frac{\partial \rho}{\partial z}=0.$$

Equilibrium:
$$u_{i0}=const.,w_{i0}=0,\rho_{i0}=const.,$$
$$p_{i0}=p_0-\rho_{i0}gz,$$

Perturbation:
$$u_i=u_{i0}+\delta u_i,w_i=\delta w_i,\rho_i=\rho_{i0},$$
$$p_i=p_0-\rho_{i0}gz+\delta p_i,$$

Mode structure

$$\left(\delta u_i, \delta w_i, \delta p_i\right) \sim \exp\left(ikx\right) \longmapsto$$

$$ik\delta u_i + \frac{\partial \delta w_i}{\partial z} = 0,$$

$$\frac{\partial \delta u_i}{\partial t} + u_{i0} ik\delta u_i = -\frac{ik}{\rho_{i0}}\delta p_i,$$

$$\frac{\partial \delta w_i}{\partial t} + u_{i0} ik\delta w_i = -\frac{1}{\rho_{i0}}\frac{\partial \delta p_i}{\partial z}.$$

$$\downarrow$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial \delta w_1}{\partial t} + iku_{10}\delta w_1\right) = \frac{\partial^2}{\partial z^2}\left(\frac{\partial \delta w_2}{\partial t} + iku_{20}\delta w_2\right)$$

$$\downarrow$$

$$(\delta w_1, \delta p_1) \sim \exp(ikx - |k| z),$$

$$(\delta w_2, \delta p_2) \sim \exp(ikx + |k| z),$$

Derivation

$$ik\delta u_1 - |k|\delta w_1 = 0, \qquad ik\delta u_2 + |k|\delta w_2 = 0,$$

$$\frac{\partial \delta u_1}{\partial t} + iku_{10}\delta u_1 = -\frac{ik}{\rho_{10}}\delta p_1, \qquad \frac{\partial \delta u_2}{\partial t} + iku_{20}\delta u_2 = -\frac{ik}{\rho_{20}}\delta p_2,$$

$$\frac{\partial \delta w_1}{\partial t} + iku_{10}\delta w_1 = \frac{|k|}{\rho_{10}}\delta p_1, \qquad \frac{\partial \delta w_2}{\partial t} + iku_{20}\delta w_2 = -\frac{|k|}{\rho_{20}}\delta p_2.$$

BC at the interface

$$\frac{d}{dt}\left(h(x,t) - z\right) = u_{i0}\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} - \frac{dz}{dt} = 0, (i = 1, 2),$$

$$p_1 |_{z=h} = p_2 |_{z=h},$$

↓

$$u_{i0}ikh + \frac{\partial h}{\partial t} - \delta w_i = 0, (i = 1, 2),$$

$$-\rho_{10}g + \delta p_1 = -\rho_{20}g + \delta p_2,$$

Dispersion Relation

$$\phi_2 \equiv \delta w_2 / |k|$$

$$\phi_1 \equiv -\delta w_1 / |k|$$

$$\phi_1 = -\frac{1}{|k|}\left(\frac{\partial}{\partial t} + iku_{10}\right)h,$$

$$\phi_2 = \frac{1}{|k|}\left(\frac{\partial}{\partial t} + iku_{20}\right)h,$$

$$\rho_{10}\left(\frac{\partial \phi_1}{\partial t} + iku_{10}\phi_1 + gh\right) = -\rho_{20}\left(\frac{\partial \phi_2}{\partial t} + iku_{20}\phi_2 + gh\right).$$

$$(\phi_1, \phi_2, h) \sim \exp(-i\omega t)$$

$$\omega_\pm = k\frac{\rho_{10}u_{10} + \rho_{20}u_{20}}{\rho_{10} + \rho_{20}} \pm \sqrt{|k|\, g\frac{\rho_{20} - \rho_{10}}{\rho_{20} + \rho_{10}} - \frac{k^2\rho_{10}\rho_{20}(u_{10} - u_{20})^2}{(\rho_{10} + \rho_{20})^2}}.$$

Kelvin-Helmholtz-Rayleigh-Taylor Instability

Unstable when $g\left(\rho_{20}^{2}-\rho_{10}^{2}\right)<\left|k\right|\rho_{10}\rho_{20}(u_{10}-u_{20})^{2}$

Has Properties (I) and (II).

Is the system PT-symmetric?

Yes. The system is PT-Symmetric

$$\begin{pmatrix} \dot{\phi}_1 \\ \dot{h} \end{pmatrix} = A \begin{pmatrix} \phi_1 \\ h \end{pmatrix},$$

$$A = \begin{pmatrix} \dfrac{ik\left(-u_{10}\rho_{10} - 2u_{20}\rho_{20} + u_{10}\rho_{20}\right)}{\rho_{10} + \rho_{20}} & \dfrac{-\left|k\right|(u_{10} - u_{20})^2 \rho_{20} + g(\rho_{20} - \rho_{10})}{\rho_{10} + \rho_{20}} \\ -\left|k\right| & -iku_{10} \end{pmatrix}.$$

↓

Symmetric under $\left(\phi_1, h, t, i\right) \rightarrow \left(-\phi_1, h, -t, -i\right)$

↓

$$PA + \overline{A}P = 0 \qquad P = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Instability occurs when and on when the PT-symmetry is broken (spontaneously).

$$v_{\pm} = \begin{pmatrix} \pm i\bar{\phi} \\ 1 \end{pmatrix},$$

$$\bar{\phi} = \frac{k(u_{20} - u_{10})\rho_{20}}{|k|(\rho_{10} + \rho_{20})} + \frac{1}{|k|}\sqrt{|k|\, g\, \frac{\rho_{20} - \rho_{10}}{\rho_{20} + \rho_{10}} - \frac{k^2 \rho_{10}\rho_{20}(u_{10} - u_{20})^2}{(\rho_{10} + \rho_{20})^2}}.$$

$g\left(\rho_{20}^2 - \rho_{10}^2\right) < |k|\, \rho_{10}\rho_{20}(u_{10} - u_{20})^2$ (unstable)

when and only when $v_{\pm}$ is not PT-symmetric.

Where does the PT-symmetry come from?

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0.$$

Invariant under

$(t, u, w, p, \rho) \to (-t, -u, -w, p, g)$

Newton's Law

$+$

Conservativeness

Conclusions

- Parity-time symmetry is an important physics concept.
  - It pertains to the fundamental question of quantum theory: what are observables?

- Parity-time symmetry is also important for classical physics.

- Two-fluid interactions are PT-symmetric

- Instability occurs (Kelvin-Helmholtz-Rayleigh-Taylor instability) when and only when the PT-symmetry is spontaneously broken.

- **Conservative systems governed by Newton's law are PT-symmetric.**

- **Classical instabilities of conservative systems occur when and only when PT-symmetry is broken spontaneously.**