HYM simulations of GAEs in DIII-D

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Sub-cyclotron frequency Alfven Eigenmodes were observed in low toroidal field experiments in DIII-D

- High-frequency AEs were observed in DIII-D in low toroidal field discharges (NSTX similarity experiments) mostly when $v_b \gtrsim v_A$.
- These modes are counter-propagating, and driven unstable by Doppler shifted cyclotron resonance with beam ions.
- Mode polarization was not measured directly, but large $dB_\varphi$ was observed near the edge.
- They were identified as compressional Alfven eigenmodes (CAE) based on high frequency ($f \sim 0.6 f_{ci}$) and comparison with previous theoretical instability conditions.
- For comparison:
  
  $f_{CAE} \sim 0.3 - 0.5 \ f_{ci}$, $f_{GAE} \sim 0.1 - 0.3 \ f_{ci}$ (NSTX);
  $f_{GAE} \sim 0.4 \ f_{ci}$ (NSTX-U)
Some observed features could not be explained by CAE theory

- Frequency correlate strongly with $B_{\text{tor}}$, but weakly with $n_e$.
- $f/f_{\text{ci}}$ for the most unstable modes remains $\approx$ const as $B_{\text{tor}}$ is varied.
- Three levels of frequency splitting were observed, not consistent with CAE dispersion:
  - $\Delta f/f \sim 0.8\text{MHz}/2.5\text{MHz}=0.32$,
  - $\Delta f/f \sim 110\text{kHz}/2.5\text{MHz}=0.04$,
  - $\Delta f/f \sim 20\text{kHz}/2.5\text{MHz}=0.008$,

Normalized frequency of the strongest mode vs $n_e$ did not show $\omega \sim n_e^{-1/2}$ scaling (red line) [Heidbrink, NF2006]
Early CAE/GAE theory predicted instability for larger values of $k_{\perp}\rho_b$

- Experimental estimates got $k_{\perp}\rho_b \leq 1$ (based on CAE dispersion i.e. from observed $\omega/v_A$) and local $B$ value.
- Re-scaling for $B$ on axis gives even lower values: $k_{\perp}\rho_b \leq 0.6 \, k_{\perp}v_A/\omega$.
- Theory predicted the CAE instability for: $1 < k_{\perp}\rho_b < 2$
- For Global Alfven eigenmode (GAE) instability: $2 < k_{\perp}\rho_b < 4$
- New GAE/CAE theory [Belova,2019, Lestz, 2020] predicts stronger instability for small $k_{\perp}/k_{\parallel}$ ($k_{\perp}\rho_b << 1$), and GAEs more unstable than CAEs.

[Heidbrink, NF2006]
New analytic theory of counter-GAE instability has been developed

Contour plots of growth rate and plots of $\gamma$ vs $k_{\perp}\rho_i$ for very narrow and for wide pitch parameter distributions; blue-green contours correspond to negative values, and orange-red to positive;
(a) $\Delta \lambda=0.03$, $\gamma/\omega_{ci}$ values are between -0.7+2.0;
(b) $\Delta \lambda=0.3$, $\gamma/\omega_{ci}$ values are between -0.22+0.036.

$\gamma \approx \pi \frac{n_b}{n_i} \frac{\omega_{ci}^2}{2\omega} v_{||res} A \int_0^{\lambda_m} \frac{d\lambda}{(1-\lambda)^2} \frac{\partial f}{\partial \lambda} \frac{\xi^2}{1}$

$v_{||res} = (\omega_{ci} - \omega)|k||$, $\lambda_m = 1 - v_{||res}^2/v_0^2$

$v_0$ – injection velocity,
$f = A \exp[-(\lambda-\lambda_0)^2/\Delta \lambda^2] / (v^3 + v^3)$.

- Resonant beam ions drive the instability provided: $\partial f/\partial \lambda > 0$, i.e. when $\lambda < \lambda_0$, and stabilizing otherwise.

- Most unstable modes have $k_{\perp}\rho_b < 1$, and are in the range [1,2]:

$$(1 + v_0/v_A)^{-1} < \omega/\omega_{ci} \leq (1 + v_0/v_A \sqrt{1-\lambda_0})^{-1}$$

- follows from cyclotron resonance condition and instability condition: $\lambda_m < \lambda_0$

- Higher frequency modes with $\omega/\omega_{ci} > (1 + v_0/v_A \sqrt{1-\lambda_0})^{-1}$ have smaller growth rates and unstable if: $2 < k_{\perp}\rho_b < 4$ (Bessel regime [3]).

Dependence on frequency and $k_\perp \rho_i$ for counter-GAEs

\[ \gamma \approx \pi \frac{n_b}{n_i} \frac{\omega_{ci}^2}{2\omega} \nu|_{res}^3 A \int_0^{\lambda_m} \frac{d\lambda}{(1-\lambda)^2} \frac{\partial f}{\partial \lambda} j^2 |_{\perp \rho_i} \left| \right. \left. \right|_{\nu=\nu_{res}} \]

\[ \lambda = \frac{v_\perp^2}{v^2} \ \ , \ \ \lambda_m = 1 - \frac{v|_{res}^2}{v_0^2} \]

\[ \nu|_{res} = (\omega_{ci} - \omega)/|k| \]

\[ \xi^2 = (k_\perp \nu_\perp/\omega_{ci})^2 = \lambda/(1-\lambda) (k_\perp \nu|_{res}/\omega_{ci})^2 \]

\[ f = A \exp[-(\lambda-\lambda_0)^2/\Delta\lambda^2] / (v^3 + v_*^3) \]

The sign of the integrand is determined by sign of \( \partial f/\partial \lambda = 2(\lambda_0-\lambda)/\Delta\lambda^2 f \)

→ particles with small $v_\perp$ (\( \lambda<\lambda_0 \)) are always destabilizing,

→ particles with large $v_\perp$ (\( \lambda>\lambda_0 \)) are always stabilizing.

1. For $\omega < \omega_{ci}$, $v|_{res} \sim v_0$ and $\lambda_m < 1$
   - \( \gamma \) is always positive if $\lambda_m \leq \lambda_0$ – sufficient condition for instability
   - gives an approximate range of unstable frequencies:
     \[ \omega/\omega_{ci} \leq (1 + v_0/v_A \sqrt{[1-\lambda_0]})^{-1} \]
   - most unstable modes have $k_\perp \rho_i << 1$ when $\left( J_1/\xi \right)^2 = 1/4$

2. High-frequency limit $\omega \approx \omega_{ci}$, $v|_{res} << v_0$ and $\lambda_m \approx 1$
   - for small $k_\perp \rho_i$ → $\left( J_1/\xi \right)^2 = 1/4$ and \( \gamma \) is always negative
   - for $k_\perp \rho_i \geq 2$, Bessel factor reduces stabilizing effect of large $v_\perp$ (\( \lambda>\lambda_0 \)) particles.

This clarifies the role of fast ion FLR effects: small $k_\perp \rho_i$ are destabilizing (\( \lambda<\lambda_0 \)), while large $k_\perp \rho_i$ reduces the stabilizing effect of particles with $\lambda>\lambda_0$. 
Counter-CAE are predicted to be less unstable than GAEs

Contour plots of growth rate and plots of $\gamma$ vs $k_\perp p_i$ for wide pitch parameter distributions; blue-green contours correspond to negative values, and orange-red to positive.

(a) GAE: $\Delta \lambda_0=0.3$, $\gamma/\omega_{ci}$ values are between -0.22÷0.036;
(b) CAE: $\Delta \lambda_0=0.3$, same range of $\gamma/\omega_{ci}$ values as in (a).

\[
\gamma \approx \pi n_b \frac{\omega^2_{ci}}{2} v_{res}^3 n_i \int_0^{\lambda_m} \frac{d\lambda}{(1-\lambda)^2} \frac{\partial f}{\partial \lambda} \frac{J_0^2 - J_2^2}{4},
\]

\[
v_{res}=(\omega_{ci} - \omega)/|k_i|, \lambda_m = 1 - v_{res}^2/v_0^2.
\]

- Same (sufficient) instability condition: $\lambda_m<\lambda_0$ as GAEs. Same $\gamma$ in the limit $k_\perp=0$.
- Most unstable modes have $k_\perp p_b<1$, and are in the range:
  \[
  (1 + \alpha v_0/v_A)^{-1} < \omega/\omega_{ci} < (1 + \alpha v_0/v_A \sqrt{[1-\lambda_0]^{-1}}
  \]
  where $\alpha=|k_\parallel/k|$; follows from cyclotron resonance condition and $\lambda_m<\lambda_0$.
- Counter-CAEs have much smaller growth rates than GAEs for $k_\perp/k_\parallel \gtrsim 1$.
- Two-fluid effects / coupling to shear AW reduce growth rate of CAEs [2].

Theory predicts scaling of most unstable GAE with $B_{\text{tor}}, n_e, \lambda_0$

Predicted range of most unstable counter-GAEs:

$$\frac{1}{(1 + v_0/v_A)} < \frac{\omega}{\omega_{ci}} \leq \frac{1}{(1 + v_0/v_A \sqrt{1 - \lambda_0})},$$

- $\omega \sim \omega_{ci}$ → Nearly linear scaling with $B_{\text{tor}}$.
- Weaker than $1/\sqrt{n_e}$ scaling with density.
- Larger $v_0/v_A$ results in smaller values of $\omega/\omega_{ci}$:
  - $\omega/\omega_{ci} \approx 0.6$ for $v_0/v_A \sim 1$ and $\lambda_0 \sim 0.6$ (DIII-D?),
  - $\omega/\omega_{ci} \approx 0.4$ for $v_0/v_A \sim 2$ (NSTX-U),
  - $\omega/\omega_{ci} \lesssim 0.2$ for $v_0/v_A \gtrsim 4$ (NSTX),

- Scaling with $\lambda_0$: larger $\lambda_0$ → larger $\omega/\omega_{ci}$
  - $\omega/\omega_{ci} \approx 0.6$ for $\lambda_0 = 0.5$,
  - $\omega/\omega_{ci} \approx 0.7$ for $\lambda_0 = 0.8$ and $v_0/v_A \sim 1$

  – consistent with DIII-D ‘left’ and ‘right’ ($\omega/\omega_{ci} = 0.56$ and $\omega/\omega_{ci} = 0.69$ [Heidbrink,NF06]) beam sources.

DIII-D: (a) Spectra for $B_{\text{tor}} = 0.6$ T with 80 keV left beams; (b) Frequency of the strongest mode vs the line-average $n_e$ for all the discharges in the database [Heidbrink,NF06].
# HYM – HYbrid and MHD code

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Self-consistent MHD + fast ions coupling scheme

Background plasma - fluid:

\[ \rho \frac{dV}{dt} = -\nabla p + (j - j_b) \times B - n_b (E - \eta j) \]

\[ E = -V \times B + \eta j \]

\[ B = B_0 + \nabla \times A \]

\[ \frac{\partial A}{\partial t} = -E \]

\[ j = \nabla \times B \]

\[ \frac{\partial p^{1/\gamma}}{\partial t} = -\nabla \cdot (V p^{1/\gamma}) \]

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (V \rho) \]

Fast ions – delta-F scheme:

\[ \frac{dx}{dt} = v \]

\[ \frac{dv}{dt} = E - \eta j + v \times B \]

\[ w = \delta F / F \quad \text{– particle weight} \]

\[ \frac{dw}{dt} = -(1 - w) \frac{d(\ln F_0)}{dt} \]

\[ F_0 = F_0(\varepsilon, \mu, p_\phi) \]

\( \rho, V \) and \( p \) are thermal plasma density, velocity and pressure, \( n_b \) and \( j_b \) are beam ion density and current, and \( n_b \ll n_e \) – is assumed.
Fast ions – delta-f scheme: $F_0 = F_0(\varepsilon, \mu, p_\varphi)$

Equilibrium distribution function $F_0 = F_1(v) F_2(\lambda) F_3(p_\varphi, v)$

- $F_1(v) = \frac{1}{v^3 + v_*^3}$, for $v < v_0$
- $F_2(\lambda) = \exp(- (\lambda - \lambda_0)^2 / \Delta \lambda^2)$
- $F_3(p_\varphi, v) = \frac{(p_\varphi - p_0)^\beta}{(R_0 v - \psi_0 - p_0)^\beta}$, for $p_\varphi > p_0$

where $v_0 = 2-5v_A$, $v_* = v_0/2$, $\lambda = \mu B_0/\varepsilon$ – pitch angle parameter, $\lambda_0 = 0.5-0.7$, and $\mu = \mu_0 + \mu_1$ includes first-order corrections [Littlejohn’81]:

$$\mu = \frac{(v_\perp - v_d)^2}{2B} - \frac{\mu_0 v_\parallel}{2B} [\hat{b} \cdot \nabla \times \hat{b} - 2(\hat{a} \cdot \nabla \hat{b}) \cdot \hat{c}]$$

$v_d$ is magnetic gradient and curvature drift velocity, $\hat{c} = v_\perp/v_\perp$, $\hat{a} = \hat{b} \times \hat{c}$.

Parameters are chosen to match TRANSP beam profiles.
HYM simulations for DIII-D

Two basic cases are considered:

1. **NSTX-similarity experiments on DIII-D from [W. Heidbrink et al, NF 2006]**
   
   \[ B_{\text{tor}} = 0.6 \text{T}, \quad R_0 = 1.63 \text{m}, \quad a = 0.56 \text{m}, \quad I = 0.6 \text{MA}, \quad q_0 = 1.2, \quad q_{\text{max}} = 4.5, \quad \beta_{\text{tot}} \sim 9\% \]
   
   Beam parameters: \( E = 80 \text{keV}, \quad V_0/V_A = 1.5, \quad n_b/n_e \sim 4\%, \quad \beta_{\text{beam}} \sim 3\% \)
   
   Observed mode parameters: \( f = 2.5 \text{MHz}, \quad f_{\text{ci}} = 4.5 \text{MHz} \)

2. **More recent dedicated GAE/CAE experiments [N. Crocker, IAEA 2020]**

   \[ B_{\text{tor}} = 1.24 \text{T}, \quad R_0 = 1.72 \text{m}, \quad I = 0.62 \text{MA}, \quad \beta_{\text{tot}} \sim 2\% \]
   
   Beam parameters: \( E = 78 \text{keV}, \quad V_0/V_A = 0.8, \quad n_b/n_e \sim 3\%, \quad \beta_{\text{beam}} \sim 0.5\% \)
   
   Observed mode parameters: \( f = 5.5 \text{MHz}, \quad f_{\text{ci}} = 9.5 \text{MHz} \)
Case 1: NSTX-similarity experiments on DIII-D

B_{tor} = 0.6T, R_0 = 1.63m, a = 0.56m, I = 0.6MA, q_0 = 1.2, q_{max} = 4.5, \beta_{tot} \sim 9\%

Beam parameters: E = 80keV, V_0/V_A = 1.5, n_b/n_e \sim 4\%, \beta_{beam} \sim 3\%

Observed mode parameters: f = 2.5MHz, f_{ci} = 4.5MHz

HYM parameters:
V_0/V_A = 1.5, n_b/n_e = 4\%, \lambda_0 = 0.75, \Delta \lambda = 0.2

- Unstable modes have shear Alfven polarization with \delta B_\parallel << \delta B_\perp and are counter-rotating – GAEs
- |n| = 14-18 with m \sim 2-3,
- frequencies \omega/\omega_{ci} = 0.52-0.69, and growth rates \gamma/\omega_{ci} = 0.003 – 0.007.
- Can estimate unstable |n| from \n \approx R_0 k_\parallel and resonant condition to get
  \n = R_0 (\omega_{ci} - \omega)/v_\parallel,
and from \lambda < \lambda_0 \rightarrow v_\parallel > 0.5v, so that n = 13-20.

Toroidal mode numbers were not measured in experiments, but estimated as n = O(10). For #120196 shot n = -16 +/-5 was inferred from correlation with changes in Mirnov signal [W. Heidbrink et al, NF 2006]
Modes have shear polarization (GAEs)

- Simulations show unstable counter-propagating GAEs with $\delta B_\parallel < 0.1 \delta B_\perp$
- Located near the magnetic axis

Radial profiles of $\delta B$ for $n = -16$ counter-GAE
Large number of sideband resonances can be seen for each unstable GAE.

HYM fast-ion energy vs pitch distribution from n=-16 GAE simulations; resonant line is shown for $v_\parallel=0.8v_A$; colour dots show resonant particles.

Location of resonant particles in phase space: $\lambda=\mu B_0/\varepsilon$ vs $p_\phi$. Particle color corresponds to different energies: from $E=0$ (purple) to $E=80$ keV (red).
Case 2: Higher B\textsubscript{tor} experiments on DIII-D

B\textsubscript{tor} = 1.24T, R\textsubscript{0} = 1.72m, I = 0.62MA, q\textsubscript{0} = 0.94, q\textsubscript{max} = 6, \(\beta\textsubscript{tot} \approx 2\%

Beam parameters: E = 75-80keV, V\textsubscript{0}/V\textsubscript{A} = 0.8, n\textsubscript{b}/n\textsubscript{e} = 3\%, \(\beta\textsubscript{beam} \approx 0.5\%

Observed mode parameters: f = 5.5 MHz, f\textsubscript{ci} = 9.5 MHz

HYM parameters:

\(V\textsubscript{0}/V\textsubscript{A} = 0.8-0.9\), \(n\textsubscript{b}/n\textsubscript{e} = 3-6\%\), \(\lambda\textsubscript{0} = 0.65-0.75\), \(\Delta\lambda = 0.2\)

- Unstable modes have shear Alfven polarization with \(\delta B\| < < \delta B\perp\) and are counter-rotating – GAEs
- \(|n| = 22-24\) with \(m \approx 3-4\)
- Frequencies \(\omega/\omega\textsubscript{ci} = 0.6-0.75\), and growth rates \(\gamma/\omega\textsubscript{ci} = 0.001 - 0.003\).
- Estimate for \(\lambda < \lambda\textsubscript{0} \rightarrow v\| > 0.5v\) gives:
  \(|n| = R\textsubscript{0}(\omega\textsubscript{ci} - \omega)/v\| \approx 23-33\),

Measured toroidal mode numbers were \(n \approx -7\), possibly aliased by -20 [S.Tang, 2020].

Growth rates and frequencies of unstable counter-GAEs from HYM simulations for \(v\textsubscript{0}/v\textsubscript{A} = 0.8\), \(n\textsubscript{b} = 6\%\), \(\lambda\textsubscript{0} = 0.75\).
HYM simulations demonstrate that unstable modes in DIII-D have SAW polarization (GAEs)

- Simulations show unstable counter-propagating GAEs with $\delta B_\parallel <0.1\delta B_\perp$,
- High toroidal mode numbers $|n|>20$
- Frequency $\omega/\omega_{ci} \sim 0.6-0.7$,
- $k_\perp \rho_b \sim 0.5$, 
- Located near the magnetic axis
HYM simulations for DIII-D agree with analytic predictions

- Frequency of most unstable modes: $\omega/\omega_{ci} \sim 0.6-0.7$,
- GAE linear growth rate is largest for small values of $k_\perp (k_\perp \rho_i < 1)$
- NBI injection velocity $V_\phi/V_A = 0.9$ and density $n_b/n_e = 0.05$.
- $\lambda_0 = 0.7$, $\Delta \lambda = 0.2$

From local dispersion relation:
(a) contour plot of GAE growth rate and (b) plot of $\gamma(k_\perp \rho_i)$ for DIII-D parameters; blue-green contours correspond to negative values, and orange-red to positive; $\gamma/\omega_{ci}$ values are between $-0.08:0.011$. 
Large number of sideband resonances can be seen for each unstable GAE.
Two groups of resonant particles: driving ($\lambda < \lambda_0$) or damping ($\lambda > \lambda_0$)

Energy exchange rate between the beam ions and the mode $\Delta K = \int (\delta j_b \cdot \delta E) d^3x = \sum (v_m \cdot \delta E) w_m$.

Scatter plot of resonant particles from linear phase of $n=-22$ GAE simulations. Time-averaged values of $\langle v \cdot \delta E \rangle w$ [a.u.] of resonant particles. Particle color corresponds to different energies: from $E=35$keV (green) to $E=80$keV (red).
Summary and Future Work

- HYM simulations show that sub-cyclotron frequency modes in DIII-D are counter-GAE, with very small compressional component.
- Simulations reproduce experimentally observed frequencies and toroidal mode numbers.
- A simple analytical theory based on local dispersion relation is very successful in predicting the counter-GAE instabilities.
- New analytic theory explains range of most unstable modes, and GAE frequency scaling across different devices.

Future work:

- Need to include 2-fluid (Hall) effects in thermal plasma description to account for finite frequency effects $\sim O(\omega/\omega_{ci})$. At present, HYM overestimates unstable frequencies, and underestimates toroidal mode numbers.
- Need to implement more efficient coupling between field and particle grids for simulations of large $|n|$ and large aspect ratio devices.
Case 2: Higher $B_{\text{tor}}$ experiments on DIII-D

$B_{\text{tor}} = 1.24\, \text{T}, R_0 = 1.72\, \text{m}, I = 0.62\, \text{MA}, q_0 = 0.94, q_{\text{max}} = 6, \beta_{\text{tot}} \sim 2\%$

Beam parameters: $E = 75-80\, \text{keV}, \frac{V_0}{V_A} = 0.8, \frac{n_b}{n_e} = 3\%, \beta_{\text{beam}} \sim 0.5\%$

Observed mode parameters: $f = 5.5\, \text{MHz}, f\text{ci} = 9.5\, \text{MHz}$