Nonlinear Evolution of Instabilities due to Drag and Large Effective Scattering

Jeff Lestz

University of California, Irvine

Vinícius Duarte, Nikolai Gorelenkov, Roscoe White

Princeton Plasma Physics Lab

PPPL EP Meeting
Zoom, NJ
July 8, 2020

Motivation and Main Results

- **Motivation**: studying simple models can provide insight into rich nonlinear behavior that can guide our understanding of more complex systems
- **Goal**: understand the influence of drag on the nonlinear evolution of an isolated eigenmode in the presence of large scattering (common tokamak regime)
  - Role of drag in chirping has been explored extensively, but less for steady solutions
- **Main results**: new analytic solutions are found for the electrostatic bump on tail problem near marginal stability in the large *effective* scattering limit with drag
  - Drag increases the saturation amplitude and shifts the oscillation frequency
  - A quasilinear equation for δF naturally emerges from nonlinear theory
  - Drag fundamentally modifies the resonance lines – shifting and splitting

Outline

- Introduction: the Berk-Breizman cubic equation
- The time-local cubic equation and its analytic solution
- Spontaneous emergence of a quasilinear system
- Implications for the resonance condition and future applications

Outline

- Introduction: the Berk-Breizman cubic equation
- The time-local cubic equation and its analytic solution
- Spontaneous emergence of a quasilinear system
- Implications for the resonance condition and future applications

Assumptions to Derive the Berk-Breizman Cubic Equation

- The electrostatic Vlasov equation with scattering and drag is written

$$\frac{\partial F}{\partial t} + v\frac{\partial F}{\partial x} + \frac{E(x,t)}{k}\frac{\partial F}{\partial v} = \underbrace{\frac{\nu^3}{k^2}\frac{\partial^2 \delta F}{\partial v^2}}_{\text{diffusive}} + \underbrace{\frac{\alpha^2}{k}\frac{\partial \delta F}{\partial v}}_{\text{convective}}$$

- Monochromatic wave $E(x,t) = \mathrm{Re}\left[\hat{E}(t)e^{i(kx-\omega t)}\right]$, where $\omega_b^2(t) = ek\hat{E}(t)/m$ for particles deeply trapped within the resonant phase space island.
- Bump on tail: analyze near a region where $F_0(v)$ has constant slope $\propto \gamma_L$
- Assume marginal stability: $\gamma \equiv \gamma_L - \gamma_d \ll \gamma_L$
- **Goal**:[1] solve for $\hat{E}(t)$ by perturbatively expanding in $\omega_b^2/\nu^2 \ll 1$

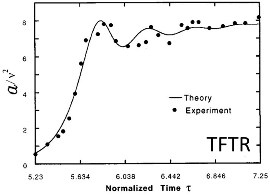
[1]H.L. Berk *et al.* Phys. Rev. Lett. **76**, 1256 (1996)

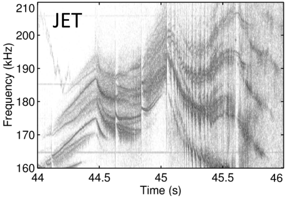
Cubic Equation Contains Rich Nonlinear Behavior

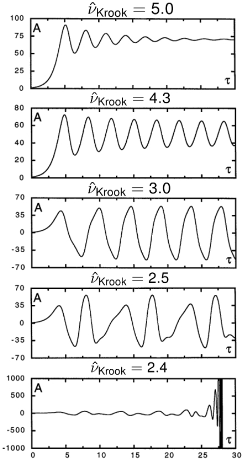
- Cubic equation with scattering ($\hat{\nu} \equiv \nu/\gamma$) and drag ($\hat{\alpha} \equiv \alpha/\gamma$) describes the evolution of the complex amplitude $A(\tau) \propto \hat{E}(\gamma t)$

$$\frac{dA(\tau)}{d\tau} = A(\tau) - \frac{1}{2}\int_0^{\tau/2} dz \int_0^{\tau-2z} dx$$
$$z^2 e^{-\hat{\nu}^3 z^2(2z/3+x)+i\hat{\alpha}^2 z(z+x)} A(\tau - z)A(\tau - z - x)A^*(\tau - 2z - x)$$

- Paradigm to interpret nonlinear experimental phenomena[2,3]



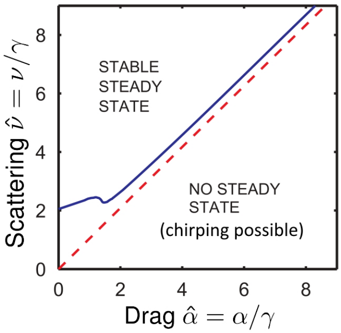




[2] K.L. Wong *et al.* Phys. Plasmas **4**, 393 (1997)
[3] R.F. Heeter *et al.* Phys. Rev. Lett. **85**, 3177 (2000)

Steady vs. Non-Steady Solutions

- Cubic equation includes both steady state and dynamical (non-steady) solutions
- Scattering is stabilizing while drag is destabilizing
  - Diffusion tends to smooth the distribution
  - Convection carries flattened gradients out of resonant region, replacing with new particles
- Non-steady solutions occur[4] when $\alpha/\nu > 0.96$
  - In reality, must integrate over 6D phase space, leading to the more complicated chirping criteria[5]
- **Not previously investigated**: how are the steady state solutions modified by drag?



[4] M.K. Lilley *et al.* Phys. Rev. Lett. **102**, 195003 (2009)
[5] V.N. Duarte *et al.* Nucl. Fusion **57**, 054001 (2017)

Outline

- Introduction: the Berk-Breizman cubic equation
- The time-local cubic equation and its analytic solution
- Spontaneous emergence of a quasilinear system
- Implications for the resonance condition and future applications

Derivation of the Time-Local Cubic Equation

- Berk-Breizman cubic equation:

$$\frac{dA(\tau)}{d\tau} = A(\tau) - \frac{1}{2}\int_0^{\tau/2} dz \int_0^{\tau-2z} dx \; z^2 e^{-\hat{\nu}^3 z^2 (2z/3+x) + i\hat{\alpha}^2 z(z+x)} A(\tau - z) A(\tau - z - x) A^*(\tau - 2z - x)$$

- When *effective* collisions are large relative to the growth rate, $\hat{\nu} = \nu/\gamma \gg 1$, the amplitudes pass through the integral, leading to the time-local cubic equation:
  - Physically, large collisions erase phase correlations, making time delays irrelevant
    - In tokamaks, $\hat{\nu} \gtrsim 10$ is typical due to small angle collisions, turbulence, *etc.*

$$\frac{dA(\tau)}{d\tau} = A(\tau) - b(\hat{\nu}, \hat{\alpha}) A(\tau) |A(\tau)|^2 \quad \text{where}$$

$$b(\hat{\nu}, \hat{\alpha}) = \frac{1}{2\hat{\nu}^4} \int_0^{\infty} \frac{e^{-2u^3/3 + iu^2\hat{\alpha}^2/\hat{\nu}^2}}{1 - i\hat{\alpha}^2/(\hat{\nu}^2 u)} du$$

Analytic Solution of Time-Local Cubic Equation

- Solve for the amplitude and phase evolution separately: $A(\tau) = |A(\tau)|\, e^{i\phi(\tau)}$

Amplitude

$$|A(\tau)|' = |A(\tau)| - \mathrm{Re}\,[b]\,|A(\tau)|^3$$

$$|A(\tau)| = \frac{|A_0|\, e^{\tau}}{\sqrt{1 - \mathrm{Re}\,[b]\,|A_0|^2\,(1 - e^{2\tau})}}$$

$$A_{\mathrm{sat}} \equiv \lim_{\tau\to\infty} |A(\tau)| = 1/\sqrt{\mathrm{Re}\,[b(\hat{\nu},\hat{\alpha})]}$$

Phase

$$\phi'(\tau) = -\mathrm{Im}\,[b]\,|A(\tau)|^2$$

$$\phi(\tau) = \phi_0 - \frac{\mathrm{Im}\,[b]}{2\mathrm{Re}\,[b]} \log\left[1 - \mathrm{Re}\,[b]\,|A_0|^2\,(1 - e^{2\tau})\right]$$

$$\frac{\delta\omega_{\mathrm{sat}}}{\gamma} \equiv -\lim_{\tau\to\infty} \phi'(\tau) = \mathrm{Im}\,[b(\hat{\nu},\hat{\alpha})]\,/\mathrm{Re}\,[b(\hat{\nu},\hat{\alpha})]$$

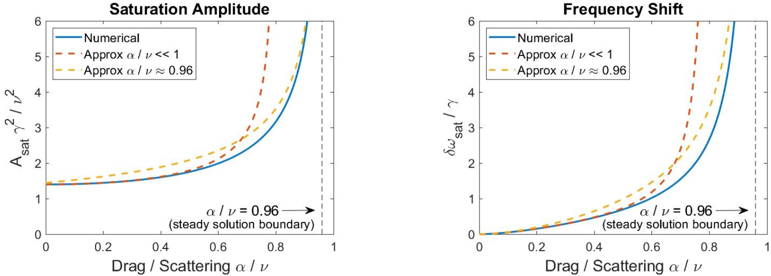
- $\mathrm{Re}\,[b] > 0$ corresponds to steady state solutions $(\alpha/\nu < 0.96)$
- **Any** amount of drag $\hat{\alpha} > 0$ leads to a finite frequency shift $\delta\omega_{\mathrm{sat}}$
  - $E(x,t) = \hat{E}(t)e^{i(kx-\omega t)} \propto |A(\tau)|\, e^{i\phi(\tau)} e^{-i(kx-\omega t)} \xrightarrow{\tau\to\infty} A_{\mathrm{sat}} e^{-i(kx-(\omega+\delta\omega_{\mathrm{sat}})t)}$

Saturation Amplitude and Frequency Shift Depend on Ratio of Drag to Scattering

- Larger α/ν leads to larger saturation amplitude

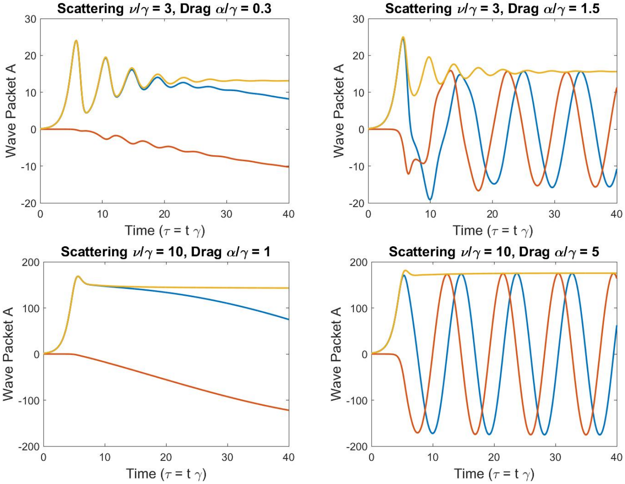
$$
\begin{aligned}
\alpha \ll \nu \qquad & A_{\text{sat}} \propto \hat{\nu}^2 / \sqrt{1 - \pi\alpha^2/2\nu^2} \\
\alpha/\nu \approx 0.96 \quad & A_{\text{sat}} \propto \hat{\nu}^2 / \sqrt{0.96 - \alpha/\nu}
\end{aligned}
$$

- Larger α/ν leads to a larger shift in frequency due to wave packet modulation
  - Approximate trends are more complicated, but note $\delta\omega_{\text{sat}} = h(\alpha/\nu)\gamma$



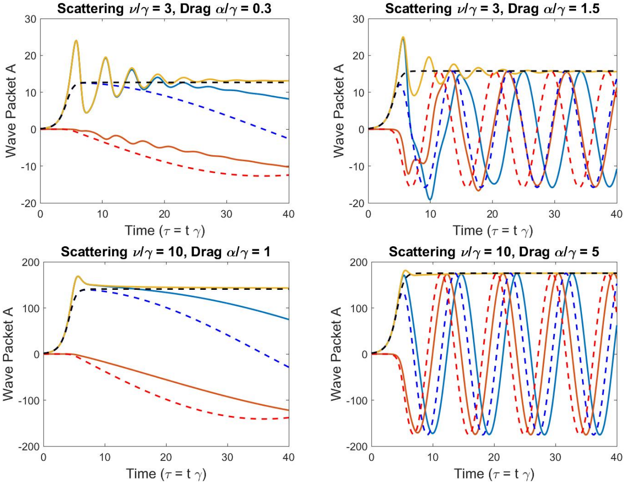
Analytic Solution Compares Well With Full Cubic Equation

- Solid curves: numerically integrated cubic equation
  - Blue: Re $[A]$
  - Red: Im $[A]$
  - Gold: $|A|$



Analytic Solution Compares Well With Full Cubic Equation

- Solid curves: numerically integrated cubic equation
  - Blue: Re $[A]$
  - Red: Im $[A]$
  - Gold: $|A|$
- Dashed curves: analytic solution to time-local cubic equation
  - Blue: Re $[A]$
  - Red: Im $[A]$
  - Black: $|A|$
- Convergence of phase lag is not yet understood



Outline

- Introduction: the Berk-Breizman cubic equation
- The time-local cubic equation and its analytic solution
- Spontaneous emergence of a quasilinear system
- Implications for the resonance condition and future applications

Perturbed Distribution Satisfies a Quasilinear Diffusion Equation When $\nu/\gamma \gg 1$

- Review: perturbative expansion of the Vlasov system (in $\omega_b^2/\nu^2 \ll 1$) yields the Berk-Breizman cubic equation, describing the nonlinear evolution of $A(t) \propto \hat{E}(t)$
- Time evolution equation for $\langle \delta F(v,t) \rangle_x \equiv \int \delta F(v,x,t) dx$ is found the same way
- When $\hat{\nu} \gg 1$, the evolution of $\langle \delta F(v,t) \rangle_x$ is identical to a quasilinear system

$$\frac{\partial \delta F}{\partial t} - \frac{\partial}{\partial v}\left[\underbrace{\frac{\pi \gamma^4 \left(1 - \gamma_d/\gamma_L\right)}{2k^3} |A(t)|^2 \boldsymbol{\mathcal{R}(v)}}_{\text{quasilinear diffusion coefficient}} \frac{\partial \delta F}{\partial v} \right] = \frac{\nu^3}{k^2} \frac{\partial^2 \delta F}{\partial v^2} + \frac{\alpha^2}{k} \frac{\partial \delta F}{\partial v}$$

- Quasilinear theory usually requires overlapping resonances to destroy coherence
  - Remarkably, near marginal stability with sufficiently large collisions, kinetic theory is equivalent to quasilinear theory even for a single, isolated resonance[6]

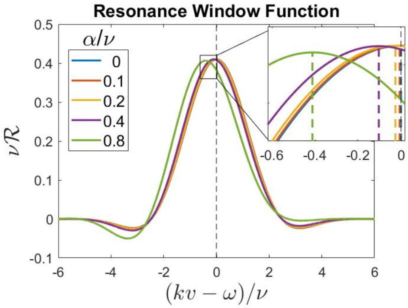
[6] V.N. Duarte *et al.* Phys. Plasmas **26**, 120701 (2019)

Drag Leads to a Shift of Resonance Lines

- $\mathcal{R}(v)$ is the <u>resonance window function</u>, which weights the quasilinear diffusion coefficient
- The window function is calculated self-consistently from first principles
  - Needed for realistic quasilinear modeling

$$\mathcal{R}(v) = \frac{k}{\pi \nu} \int_0^\infty \cos\left(\frac{kv-\omega}{\nu} s + \frac{\alpha^2}{\nu^2}\frac{s^2}{2}\right) e^{-s^3/3} ds$$

- In the absence of collisions, $\mathcal{R}(v) = \delta(\omega - kv)$
- Scattering broadens the resonance $\propto \mathcal{O}(\nu)$
- Drag breaks symmetry, shifting the peak
  - Peaks at $\omega - kv \approx \dfrac{3^{1/3}\Gamma[4/3]}{2}\dfrac{\alpha^2}{\nu} \equiv \Delta\Omega_{\mathrm{win}}$

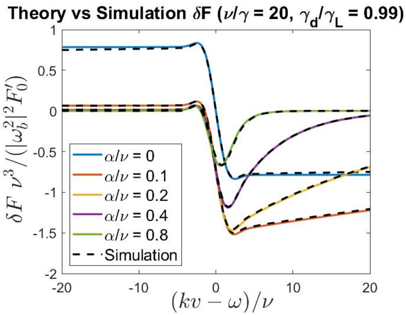


Perturbed Distribution is Sensitive to Drag

- Saturated δF can also be calculated

$$\langle \delta F(v) \rangle_{x,\mathrm{sat}} \propto -\int_{-\infty}^{kv-\omega} \mathcal{R}(v') e^{-\frac{\alpha^2}{\nu^2}\frac{k(v-v')}{\nu}} dv'$$

- Perturbed distribution exhibits sensitive dependence on α/ν due to exponential factor
  - In contrast, the window function modification is relatively less substantial
- $\langle \delta F(v) \rangle_{x,\mathrm{sat}}$ agrees with 1D Vlasov code BOT[7]
  - Caveat: simulations were run *very* close to marginal stability ($\gamma_d/\gamma_L = 0.99$)



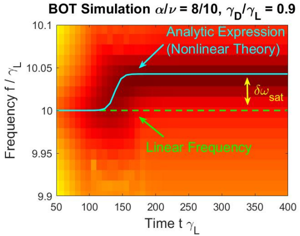
[7] M.K. Lilley *et al.* Phys. Plasmas **17**, 092305 (2010)

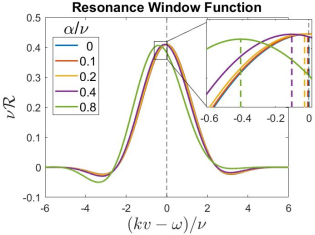
Outline

- Introduction: the Berk-Breizman cubic equation
- The time-local cubic equation and its analytic solution
- Spontaneous emergence of a quasilinear system
- Implications for the resonance condition and future applications

Frequency Shifts Modify the Resonance Condition

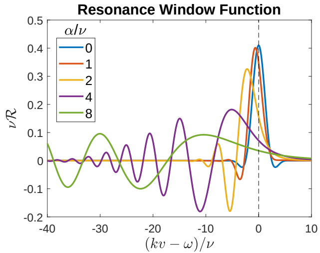
- Two distinct frequency shifts have been derived
  1. Drag modulates the real part of mode amplitude
     - $\delta\omega_{\text{sat}}/\gamma = \text{Im}\left[b(\hat{\nu},\hat{\alpha})\right]/\text{Re}\left[b(\hat{\nu},\hat{\alpha})\right]$
  2. Drag shifts the peak in the window function
     - $\Delta\Omega_{\text{win}}/\nu = 3^{1/3}\Gamma[4/3]\alpha^2/2\nu^2$
- **Interpretation**: the "most resonant" velocity changes as the system evolves: $\omega_{NL}(t) - kv_{\text{res}}(t) = \Delta\Omega_{\text{win}}$
  - Linear stage: $\omega_L - kv_{\text{res,L}} = \Delta\Omega_{\text{win}}$
  - NL stage: $\underbrace{\omega_L + \delta\omega_{\text{sat}}}_{\omega_{NL}} - k\big(\underbrace{v_{\text{res,L}} + \delta v_{\text{res,NL}}}_{\text{new } v_{\text{res}}}\big) = \Delta\Omega_{\text{win}}$

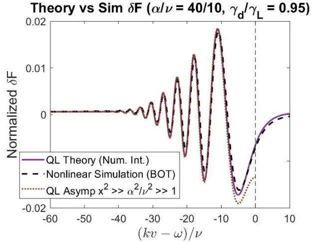




Very Large Drag Induces Resonance Splitting

- To this point: all results have assumed $\alpha/\nu < 0.96$, which ensures a steady solution
- What about the early phase of the non-steady solutions with large drag $\hat{\alpha} > \hat{\nu} \gg 1$?
  - The formalism remains valid until $\omega_b^2/\nu^2 \ll 1$ is violated
- For $\hat{\alpha} \gg \hat{\nu}$, the window function splits with many peaks
- $\langle \delta F(v,t) \rangle_x$ has stationary, growing holes and clumps
  - This is not the saturated $\langle \delta F(v) \rangle_{x,\mathrm{sat}}$ from before, as $\omega_b^2/\nu^2 \sim 1$ occurs prior to saturation in this regime
- **Open question**: how are these features connected to the system's nonlinear fate?





Fusion Applications

- Reduced quasilinear models for wave-particle interactions are further justified
  - Equivalent to full nonlinear theory in the typical $\hat{\nu} \gg 1$ regime, even with drag
  - The previously *ad-hoc* window function has now been rigorously derived
- Resonance-broadened-quasilinear model (RBQ)[8] for realistic yet reduced simulations of AE-induced fast ion transport in present and future burning devices
  - Motivated in part by DIII-D critical gradient experiments[9]
  - Similar methods could be applied to study RF heating
- How can the α/ν knob be turned experimentally?
  - Change the level of microturbulence, which contributes to $\hat{\nu}$
    - DIII-D negative triangularity experiments led to chirping[10]
  - Possibly other dependencies, TBD
    - NBI injection angle, magnetic shear, temperature, others?

---

[8] N.N. Gorelenkov *et al.* Phys. Plasmas **26**, 072507 (2019)
[9] C.S. Collins *et al.* Phys. Rev. Lett. **116**, 095001 (2016)
[10] M.A. Van Zeeland *et al.* Nucl. Fusion **59**, 086028 (2019)

Summary and Outlook

Problem

- The nonlinear evolution of instabilities was studied in the presence of drag and large *effective* scattering (relative to growth rate) near marginal stability

Main Results

- The time-local cubic equation was derived, leading to new analytic solutions
- Drag increases the saturation amplitude and introduces a frequency shift
- δF satisfies a quasilinear system, demonstrating NL theory $\xrightarrow{\hat{\nu} \gg 1}$ QL theory
- The resonance lines can be shifted – and even split – due to drag

Future Work

- Explore the consequences of resonance splitting for non-steady solutions
- Understand dependence of α/ν on plasma properties for experimental verification