Transport-Driven Toroidal Rotation with General Viscosity Profiles

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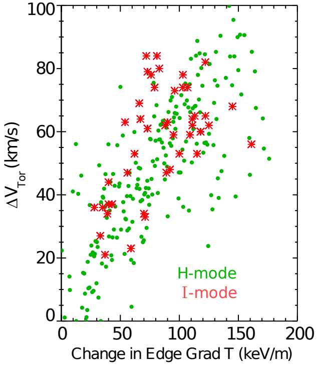
Overview and Context

- Rotation protects against resistive-wall modes, which can cause disruptions. NBI won't drive strong rotation in ITER or an FPP.
    - Luckily, even without applied torque, the edge usually rotates co-current.
- The modulated-transport model explains this rotation with an interaction of ion drift orbits with rapid spatial variation of turbulent viscosity [1].
    - So far, it's been tested on TCV [2], DIII-D [3, 4], and ASDEX-Upgrade [5], in a wide variety of conditions.
    - However, the model assumed that turbulent viscosity decayed exponentially in the radial direction—want to relax this assumption.
- In this new work, we let the turbulent viscosity depend on space in an axisymmetric but otherwise arbitrary way.
    - To do this, we assume normalized viscosity is weak, roughly: pedestal-top ion transit time much shorter than transport across the pedestal
    - The result is more flexible and the calculation is technically much easier.
    - We test the simplified calculation and bound its error using a rigorous (but much more challenging) semi-differential operator based calculation.

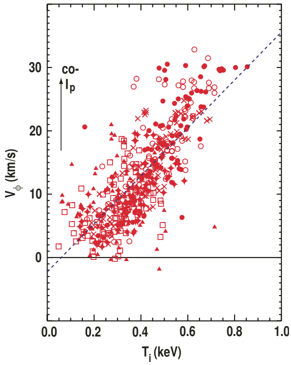
Outline

- Experimental background and basic model
- Simple boundary-layer calculation
- Application to rotation: simple formulas for experimental use
- Semi-differential operators
- Sketch of technical solution using semi-differential operators

Experimentally, H-mode plasmas rotate spontaneously, without external torque.



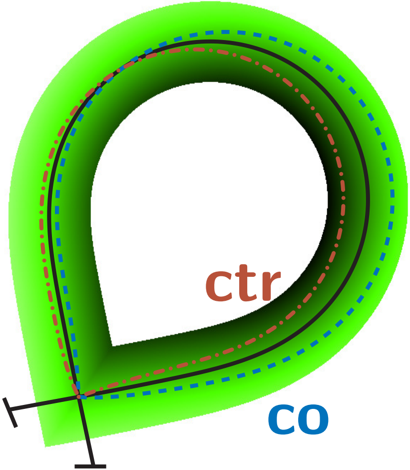
Rice et al PRL 2011, Fig. 5b



deGrassie et al NF 2009, Fig. 7

- Co-current, especially in the edge.
- $v_\phi / v_{ti} \sim O(10^{\text{ths}})$ at the pedestal top.
- Spin-up at L–H transition.
- Roughly proportional to W/I_p.
- Edge rotation proportional to T or ∇T?

Differential transport can be caused by drift orbits' interaction with the diffusivity's spatial variation.



We begin with a deceptively simple transport model,

$$\partial_t f_i + b_\phi v \partial_y f_i - b_\phi \delta v^2 (\sin y) \partial_x f_i - \partial_x [D(x,y) \partial_x f_i] = 0$$

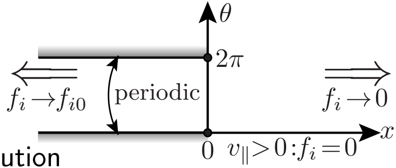
Gyrokinetic equation⇒average over turbulence⇒$\frac{\rho_i}{L_x}, \frac{L_x}{a}, \frac{1}{q}, \frac{a}{R_0}, \frac{v_E/a}{v_{ti}/qR_0} \ll 1$

- Turbulent $D \Longrightarrow$purely diffusive turbulence, "null hypothesis"
  - arbitrary x, y dependence, except $D > 0$
  - assumed small $D \ll 1$: central ordering of this work
    - dimensionally, $(D^{\text{dim}}/L_x^2) \ll (v_{ti}|_{\text{pt}}/qR_0)$
- No acceleration of ions' parallel velocity v: allows v-by-v solve
  - Collisionless: good for superthermal ions
  - No $\mu \nabla B$ force: "deeply-passing" approximation
- Axisymmetric, radially-thin simple-circular geometry
- $\mathrm{E} \times \mathrm{B}$ flows below poloidal sound speed

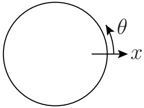
Normalizations: v:$v_{ti}|_{\text{pt}}$, y:a, x:L_x, t:$qR_0/v_{ti}|_{\text{pt}}$, f_i:$n_i|_{\text{pt}}/v_{ti}|_{\text{pt}}$, D:$L_x^2 v_{ti}|_{\text{pt}}/aB_0$

which captures the radially-global nature of the problem.

$$\partial_t f_i + b_\phi v \partial_y f_i - b_\phi \delta v^2 (\sin y) \partial_x f_i - \partial_x [D(x,y) \partial_x f_i] = 0$$



- Solve for $f_i \sim F_0$, "equilibrium" distribution
  - necessary to resolve global problem structiure
  - integrated over μ
  - resulting f_i not symmetric in parallel velocity v or poloidal angle y
  - Pedestal-SOL formulation in boundary conditions:
  - Spatial variation of turbulent diffusivity
  - $\delta \doteq q\rho_i|_{\mathrm{pt}}/L_x$ a free parameter (may ≈ 1 in experiment)



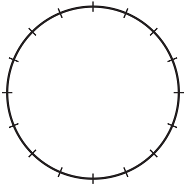
  - Invariant to rigid toroidal rotation v_{rig}
  - Trivial conservation of a simplified toroidal momentum:

$$\int \mathrm{d}v \left(v + v_{\mathrm{rig}}\right) f_i$$

An initial variable transform simplifies the equations.

Transform to drift surface label

$$\bar{x} \doteq x - \delta v \left(\cos y - \cos y_0\right)$$

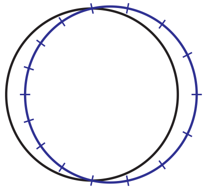


to get a simple form for the equation

$$b_\phi v \partial_y f_i = \partial_{\bar{x}}[D\big(x(\bar{x}, y), y\big)\partial_{\bar{x}} f_i],$$

boundary conditions

$$f_i(\bar{x} \leq 0, y_0) = f_i(\bar{x} \leq 0, y_0 + 2\pi),$$
$$f_i(\bar{x} > 0, y_0, b_\phi v > 0) = 0,$$
$$f_i(\bar{x} > 0, y_0 + 2\pi, b_\phi v < 0) = 0,$$
$$f_i(\bar{x} \to \infty, y) \to 0,$$
$$f_i(\bar{x} = \bar{x}_\ell, y) = f_{i0}(v),$$



and v-dependent surface-integrated flux (constant across $\bar{x}_\ell \leq \bar{x} \leq 0$):

$$\Gamma(v) \doteq \oint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{\Gamma} = -\oint \mathrm{d}y \left(D\partial_{\bar{x}} f_i\right)(\bar{x}, y),$$

In the bulk, away from the LCDS, f_i is constant along drift orbits.

In the bulk, meaning $\bar{x} < 0$, $|\bar{x}| \sim O(1)$, decompose

$$\bar{f}_i(\bar{x}) \doteq \frac{1}{2\pi} \oint \mathrm{d}y \, f_i, \qquad \tilde{f}_i(\bar{x}, y) \doteq f_i - \bar{f}_i,$$

then our simple equation becomes

$$b_\phi v \partial_y \tilde{f}_i = \partial_{\bar{x}}[D(\bar{x}, y)\partial_{\bar{x}}(\bar{f}_i + \tilde{f}_i)]$$

- Since $\oint \mathrm{d}y \, \tilde{f}_i = 0$, must have $\partial_y \tilde{f}_i \sim \tilde{f}_i$.
- On the bulk, $\partial_{\bar{x}} \sim O(1)$, so $\partial_{\bar{x}}(D\partial_{\bar{x}}\tilde{f}_i) \ll \partial_y \tilde{f}_i$.
- Neglect $\partial_{\bar{x}}(D\partial_{\bar{x}}\tilde{f}_i)$, then integrate $\oint \mathrm{d}y$ for the solvability constraint

$$0 \approx \oint \mathrm{d}y \, \partial_{\bar{x}}[D\big(x(\bar{x}, y), y\big)\partial_{\bar{x}}\bar{f}_i] = \partial_{\bar{x}}(\bar{D}\partial_{\bar{x}}\bar{f}_i), \text{ and } \Gamma \approx -\bar{D}\partial_{\bar{x}}\bar{f}_i, \text{ where}$$

$$\bar{D}(\bar{x}) \doteq \oint \mathrm{d}y \, D(\bar{x}, y), \text{ thus}$$

$$f_{i0} - \bar{f}_i(\bar{x} = 0) = -\int_{\bar{x}_\ell}^{0} \mathrm{d}\bar{x} \, \partial_{\bar{x}}\bar{f}_i \approx \Gamma \int_{\bar{x}_\ell}^{0} \mathrm{d}\bar{x} \, \bar{D}^{-1}.$$

In the layer, use local LCDS values to simplify.

Outside the "last closed drift orbit" (LCDS), meaning $\bar{x} > 0$, $f_i = \int_{y_0}^{y} \mathrm{d}y \, \partial_y f_i$ so $\tilde{f}_i \sim \bar{f}_i$, inconsistent with the bulk orderings.

So, the equation $b_\phi v \partial_y f_i = \partial_{\bar{x}}(D \partial_{\bar{x}} f_i)$ implies steep gradients $\partial_{\bar{x}} \sim O(D^{-1/2})$, in the near-LCDS layer $|\bar{x}| \sim O(D^{1/2})$.

Since $|\bar{x}| \ll 1$ in the layer, take $D(\bar{x}, y) \approx D(0, y)$, then use a y transform

$$\bar{y}(y) \doteq \frac{1}{\bar{D}_0} \int_{y_0}^{y} \mathrm{d}y' \, D(0, y'),$$

$$\bar{D}_0 \doteq \bar{D}(\bar{x} = 0),$$

switch $\bar{y} \to 1 - \bar{y}$ for $v < 0$, and define

$$u(\bar{x}) \doteq (|v|/\bar{D}_0)^{1/2} \bar{x},$$

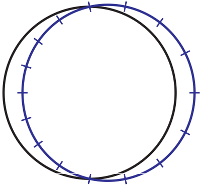
then

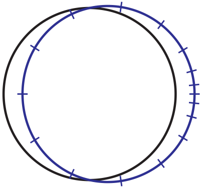
$$\partial_{\bar{y}}|_u f_i = \partial_u|_{\bar{y}}^2 f_i$$

with left-hand matching condition:

$$f_i(u \to -\infty, \bar{y}) \approx c_0 + c_1 u.$$

Baldwin *et al* found $c_0/c_1 = \zeta(1/2)/\sqrt{\pi} \approx -0.824$ [6].





Match the solutions to determine flux Γ and rotation v_{rig}.

Match $\bar{f}_i$ and its radial derivative for bulk $\bar{x} \to 0_-$ and layer $u \to -\infty$:

$$\bar{f}_i(\bar{x} = 0) = c_0,$$

$$\Gamma \approx -(\bar{D}\partial_{\bar{x}}\bar{f}_i)(\bar{x} = 0) = -(|v|\bar{D}_0)^{1/2} c_1$$

thus $\bar{f}_i(\bar{x} = 0) = (-c_0/c_1)\Gamma/(|v|\bar{D}_0)^{1/2}$ and

$$\Gamma(v) \approx \frac{f_{i0}}{\int_{\bar{x}_\ell}^0 \mathrm{d}\bar{x}\, \bar{D}^{-1} + (-c_0/c_1)/(|v|\bar{D}_0)^{1/2}}$$

Γ implies a viscous momentum flux $v_{\mathrm{rig}}\Gamma^p$, intrinsic momentum flux Π, and ion heat flux $(\Gamma^p + Q_\parallel)$ for moments

$$\Gamma^p \doteq \int \mathrm{d}v\, \Gamma, \quad \Pi \doteq \int \mathrm{d}v\, v\Gamma, \quad Q_\parallel \doteq \int \mathrm{d}v\, \frac{1}{2} v^2 \Gamma.$$

With applied torque τ^N, steady-state momentum conservation demands

$$\tau^N = v_{\mathrm{rig}}\Gamma^p + \Pi.$$

In simple limits, we get simple dimensional formulas.

Assume $D^{1/2}, \delta \ll 1$, $D(x,y) = D_x(x)D_y(y)$, canonical Maxwellian f_{i0}.
Model predicts the dimensional intrinsic and viscous momentum fluxes:

$$\Pi^{\text{int}} \approx 1.39\left(\bar{R}_X - \frac{1}{2} d_{\text{c}}^{\text{eff}}\right) \frac{(\mu_i/2) q R_0(\text{m}) Q_i(\text{MW})}{Z_i B_0(\text{T}) L_\phi^{\text{eff}}(\text{cm})} \text{N}\cdot\text{m},$$

$$\Pi^{\text{visc}} \approx 0.0139 \frac{(\mu_i/2) R_0(\text{m}) Q_i(\text{MW})}{T_i|_{\text{pt}}(\text{keV})} v_\varphi(\text{km/s}) \text{N}\cdot\text{m}.$$

Dimensional momentum balance with applied torque τ,

$$\tau = \Pi^{\text{int}} + \Pi^{\text{visc}},$$

is easily solved for core-edge-boundary rotation

$$v_\varphi \approx v_{\text{int}} + 71.9 \frac{T_i|_{\text{pt}}(\text{keV}) \tau(\text{N}\cdot\text{m})}{(\mu_i/2) R_0(\text{m}) Q_i(\text{MW})} \text{km/s},$$

$$v_{\text{int}} \approx 100\left(d_{\text{c}}^{\text{eff}}/2 - \bar{R}_X\right) \frac{q T_i|_{\text{pt}}(\text{keV})}{Z_i B_0(\text{T}) L_\phi^{\text{eff}}(\text{cm})} \text{km/s}.$$

As always, care is needed for theory-experiment comparison.

In the formulas, μ_i is ion mass (in amu), $R_0 \doteq (R_{\mathrm{in}} + R_{\mathrm{out}})/2$, and

$$\bar{R}_X \doteq [2R_X - (R_{\mathrm{out}} + R_{\mathrm{in}})]/(R_{\mathrm{out}} - R_{\mathrm{in}}),$$

with R_X, R_{in}, and R_{out} the major radii of the X-point and inner- and outer-most points of the LCFS.

- Usually $d_{\mathrm{c}}^{\mathrm{eff}} \approx d_{\mathrm{c}}$ for $d_{\mathrm{c}}(x) \doteq \frac{2}{D_{\mathrm{z}}} \oint \mathrm{d}y\, D \cos y$, $D_{\mathrm{z}}(x) \doteq \oint \mathrm{d}y\, D$.
- The effective decay length is defined for D with any x dependence:
$$L_\phi^{\mathrm{eff}} \doteq \int_{x_{\ell,\mathsf{dim}}}^{0} \mathrm{d}x_{\mathsf{dim}}\, [D_{\mathrm{z}}(0)/D_{\mathrm{z}}(x_{\mathsf{dim}})],$$
- The safety factor is really measuring orbit width, one best uses either
$$q \approx q_{\mathrm{eff},B} \doteq \frac{B_0(R_{\mathrm{out}} - R_{\mathrm{in}})}{2B_{\mathrm{p}}R}, \;\text{ or }\; q \approx q_{\mathrm{eff},I} \doteq 5\frac{R_{\mathrm{out}} - R_{\mathrm{in}}}{2R_0}\,\frac{B_0(\mathrm{T})\,\ell_{\mathrm{p}}(\mathrm{m})/2\pi}{I_p(\mathrm{MA})},$$
- Torque τ refers to true torque: Include NTV torque and actual deposited NBI torque [4]. Exclude "intrinsic torque."
- Best radial point often just inside the pedestal top, or L-mode analog.
- Kludge for ion trapping: multiply Π^{int} and v_{int} by f_{pass} [4].

We can define a definite semi-integral and semi-derivative.

For arbitrary function $h(0 \le \bar{y} \le 1)$, define:

$$(\partial_{\bar{y}0}^{-1/2} h)(\bar{y}) \doteq \int_0^{\bar{y}} \mathrm{d}\upsilon \, \frac{h(\upsilon)}{\sqrt{\pi(\bar{y}-\upsilon)}} = \int_0^{\bar{y}} \mathrm{d}\tau \, \frac{h(\bar{y}-\tau)}{\sqrt{\pi\tau}},$$

$$(\partial_{\bar{y}0}^{+1/2} h)(\bar{y}) \doteq \partial_{\bar{y}} \partial_{\bar{y}0}^{-1/2} h,$$

The convenient integral $$\frac{1}{\pi}\int_{\upsilon}^{\bar{y}} \mathrm{d}\bar{y}' \, \frac{1}{\sqrt{(\bar{y}-\bar{y}')(\bar{y}'-\upsilon)}} = 1.$$

then implies that $$\partial_{\bar{y}0}^{-1/2} \partial_{\bar{y}0}^{-1/2} h = \int_0^{\bar{y}} \mathrm{d}\upsilon \, h.$$

One may similarly show that $$\partial_{\bar{y}0}^{+1/2} \partial_{\bar{y}0}^{+1/2} h(\bar{y}) \doteq \partial_{\bar{y}} h.$$

These operations, and more general cases, are discussed in great detail by Oldham and Spanier, "The Fractional Calculus," Academic Press Inc.

Semi-integrals and semi-derivatives for periodic functions:

Expand arbitrary function $h(0 \leq \bar{y} \leq 1)$ in Fourier series:

$$h(\bar{y}) = \sum_m \hat{h}_m e^{2\pi i m \bar{y}}, \;\; \tilde{h} \doteq \sum_{m \neq 0} \hat{h}_m e^{2\pi i m \bar{y}}$$

The derivative and zero-mean ("periodic") integral of $\tilde{h}$ are

$$\partial_{\bar{y}} \tilde{h} = \sum_m 2\pi i m \, \hat{h}_m e^{2\pi i m \bar{y}}, \quad \int_{\mathrm{p}} \mathrm{d}y \, \tilde{h} \doteq \sum_{m \neq 0} \hat{h}_m e^{2\pi i m \bar{y}} / 2\pi i m$$

Define a "periodic semi-derivative" and "periodic semi-integral" as

$$\partial_{\bar{y}\mathrm{p}}^{+1/2} \tilde{h} = \sum_m \sqrt{2\pi i m} \, \hat{h}_m e^{2\pi i m \bar{y}}, \quad \partial_{\bar{y}\mathrm{p}}^{-1/2} \tilde{h} \doteq \sum_{m \neq 0} \hat{h}_m e^{2\pi i m \bar{y}} / \sqrt{2\pi i m}$$

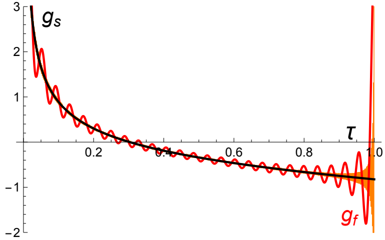
We can also evaluate these in real space:

$$\partial_{\bar{y}\mathrm{p}}^{-1/2} \tilde{h} \doteq \int_0^1 \mathrm{d}\tau \, g(\tau) h_{\mathrm{ext}}(\bar{y} - \tau),$$

$$\partial_{\bar{y}\mathrm{p}}^{+1/2} \tilde{h} \doteq \partial_{\bar{y}} \partial_{\bar{y}\mathrm{p}}^{-1/2} \tilde{h},$$

$$g(\tau) = g_{\mathrm{f}}(\tau) \doteq \sum_{m \neq 0} e^{2\pi i m \tau} / \sqrt{2\pi i m}, \text{ or}$$

$$g(\tau) = g_{\mathrm{s}}(\tau) \doteq 1/\sqrt{\pi \tau} + \sum_{n=0}^{\infty} g_{\mathrm{s},n} \tau^n.$$



An alternative reformulation facilitates more detailed mathematical analysis.

Recast the exact kinetic equation with the layer $\bar{y}$ and a new radial variable

$$\bar{u}(\bar{x}) \doteq (|v|\bar{D}_0)^{1/2} \int_0^{\bar{x}} \mathrm{d}\bar{x}' \,/\, \bar{D}(\bar{x}'), \text{ obtaining}$$

$$\partial_{\bar{y}} f_i - \partial_{\bar{u}}^2 f_i = \partial_{\bar{u}}(\tilde{d}\partial_{\bar{u}} f_i) - \bar{d}\partial_{\bar{y}} f_i, \text{ with}$$

$$\bar{d}(\bar{u}) \doteq [\bar{D}(\bar{u}) - \bar{D}_0]/\bar{D}_0 = \bar{D}(\bar{u})/\bar{D}_0 - 1,$$

$$\tilde{d}(\bar{u},\bar{y}) \doteq [D(\bar{u},\bar{y})/\bar{D}(\bar{u})]/[D(0,\bar{y})/\bar{D}_0] - 1,$$

where $\bar{d}(0) = \tilde{d}(0,\bar{y}) = 0$, $\partial_u \bar{d}, \partial_u|_{\bar{y}}\tilde{d} \sim O(D^{1/2})$, and $\int_0^1 \mathrm{d}\bar{y}\, \tilde{d} = 0$.
Most boundary conditions unchanged:

$f_i(\bar{u} \leq 0, \bar{y} = 0) = f_i(\bar{u} \leq 0, \bar{y} = 1)$, $f_i(\bar{u} > 0, \bar{y} = 0) = 0$, $f_i(\bar{u} \to \infty, y) \to 0$.

But at the core-edge boundary, require only $(\partial_{\bar{y}} f_i)(\bar{u}_\ell, \bar{y}) = 0$. Set overall magnitude by setting $\Gamma = \Gamma_0$ for the exact flux

$$\Gamma = -(|v|\bar{D}_0)^{1/2} \int_0^1 \mathrm{d}\bar{y}\,(1 + \tilde{d})\partial_{\bar{u}} f_i.$$

Take asymptotic small-D limit of the reformulated problem.

To find f_a, a $D \ll 1$ approximation to f_i, we neglect $\bar{d}, \tilde{d} \sim O(D^{1/2}|\bar{u}|)$, obtaining

$$\partial_{\bar{y}} f_\mathrm{a} - \partial_{\bar{u}}^2 f_\mathrm{a} = 0.$$

Boundary conditions are unchanged, except now $(\partial_{\bar{y}} f_\mathrm{a})(\bar{u} \to -\infty, \bar{y}) = 0$, and $\Gamma = \Gamma_0$ applies to

$$-\Gamma/(|v|\bar{D}_0)^{1/2} = \int_0^1 \mathrm{d}\bar{y}\, \partial_{\bar{u}} f_\mathrm{a} = \partial_{\bar{u}} \bar{f}_\mathrm{a},$$

$$\text{where} \quad \bar{f}_\mathrm{a} \doteq \int_0^1 \mathrm{d}\bar{y}\, f_\mathrm{a}, \quad \tilde{f}_\mathrm{a} \doteq f_\mathrm{a} - \bar{f}_\mathrm{a}.$$

To calculate $f_{i0} \approx \bar{f}_\mathrm{a}(\bar{u}_\ell)$ as a function of Γ_0:

$$\bar{f}_\mathrm{a}(\bar{u}_\ell) = -\int_{\bar{u}_\ell}^0 \mathrm{d}\bar{u}\, \partial_{\bar{u}} \bar{f}_\mathrm{a} + \bar{f}_\mathrm{a}(0) = \frac{\Gamma|\bar{u}_\ell|}{(|v|\bar{D}_0)^{1/2}} + \bar{f}_\mathrm{a}(0),$$

we need "only" find $\bar{f}_\mathrm{a}(0, \bar{y})$. But this requires to solve separately for the edge $f_\mathrm{a}(\bar{u} \leq 0)$, SOL $f_\mathrm{a}(\bar{u} > 0)$, and then enforce continuity at $\bar{u} = 0$.

A basic Green's function form facilitates our solution.

If we knew $f_{\mathrm{a}}(\bar{u} \leq 0, \bar{y} = 1)$, our solution would be

$$f_{\mathrm{a}}(\bar{u}, \bar{y}) = \int_{-\infty}^{0} \mathrm{d}\xi \, f_{\mathrm{a}}(\xi, \bar{y} = 1) \, G(\bar{u} - \xi, \bar{y}),$$

with standard diffusion Green's function (with general arguments $\breve{u}, \breve{y}$)

$$G(\breve{u}, \breve{y}) \doteq \exp(-\breve{u}^2/4\breve{y})/\sqrt{4\pi\breve{y}}.$$

Since $G(\breve{u}, \breve{y} > 0)$ is smooth, so is $f_{\mathrm{a}}(\bar{u}, \bar{y} > 0)$, thus also $f_{\mathrm{a}}(\bar{u}, \bar{y} = 0)$, except right at $\bar{u} = \bar{y} = 0$.

Since $f_{\mathrm{a}}(\bar{u}, 1)$ is smooth, we may Taylor expand it about $\bar{u} = 0$ and substitute in GF formula to get

$$f_{\mathrm{a}}(\bar{u} = 0, \bar{y}) = \sum_{n=0}^{\infty} b_n \bar{y}^{n/2},$$

from which we may evaluate

$$f_{\mathrm{a}}(\bar{u} = 0, \bar{y} = 0) = \frac{1}{2} f_{\mathrm{a}}(\bar{u} = 0, \bar{y} = 1).$$

The edge solution is captured by a semidifferential relation.

For $\bar{u} \leq 0$, expand

$$f_{\rm a}(\bar{u} \leq 0, \bar{y}) = \bar{f}_{\rm a}(\bar{u}) + \sum_{m \neq 0} \hat{f}_m(\bar{u}) e^{2\pi i m \bar{y}}$$

in our differential equation

$$\partial_{\bar{y}} f_{\rm a} - \partial_{\bar{u}}^2 f_{\rm a} = 0,$$

then trivially solve for

$$\bar{f}_{\rm a}(\bar{u} \leq 0) = c_0 + c_1 \bar{u},$$
$$\tilde{f}_{\rm a}(\bar{u} \leq 0, \bar{y}) = \sum_{m \neq 0} \hat{f}_{m{\rm c}} e^{\sqrt{2\pi i m}\,\bar{u}} e^{2\pi i m \bar{y}},$$

with unknown constants c_0, c_1, $\hat{f}_{m{\rm c}}$.
By $\sqrt{2\pi i m}$, we always intend the branch with positive real part, $\propto 1 \pm i$.

Recalling our definitions, this implies the semi-differential relationship

$$\partial_{\bar{u}} \tilde{f}_{\rm a}(\bar{u} \leq 0, \bar{y}) = \partial_{\bar{y}{\rm p}}^{+1/2} \tilde{f}_{\rm a}(\bar{u} \leq 0, \bar{y}).$$

The SOL solution follows a different semidifferential relation.

For $\bar{u} > 0$, use GF form for f_{a}, along with its $\bar{u}$ partial:

$$(\partial_{\bar{u}} f_{\mathrm{a}})(\bar{u}, \bar{y}) = \frac{-1}{2\bar{y}} \int_{-\infty}^{0} \mathrm{d}\xi \, f_{\mathrm{a}}(\xi, 1)\,(\bar{u} - \xi)\, G(\bar{u} - \xi, \bar{y})$$

If we recall the definite semi-integral,

$$(\partial_{\bar{y}0}^{-1/2} h)(\bar{y}) \doteq \int_{0}^{\bar{y}} \mathrm{d}\upsilon \, \frac{h(\upsilon)}{\sqrt{\pi(\bar{y} - \upsilon)}} = \int_{0}^{\bar{y}} \mathrm{d}\tau \, \frac{h(\bar{y} - \tau)}{\sqrt{\pi\tau}},$$

then we can carry out the integral to get

$$\partial_{\bar{y}0}^{-1/2} \partial_{\bar{u}} f_{\mathrm{a}} = -\int_{-\infty}^{0} \mathrm{d}\xi \, f_{\mathrm{a}}(\xi, 1)\, G(\bar{u} - \xi, \bar{y})\, \mathrm{sign}(\bar{u} - \xi),$$

$$(\partial_{\bar{y}0}^{-1/2} \partial_{\bar{u}} f_{\mathrm{a}})(\bar{u} \geq 0, \bar{y}) = -f_{\mathrm{a}}(\bar{u} \geq 0, \bar{y}).$$

Enforce continuity of f_a and $\partial_{\bar{u}} f_\mathrm{a}$ at $\bar{u}=0$ to find self-consistent solution.

Use $f_\mathrm{a}(0_-,\bar{y}>0) = f_\mathrm{a}(0_+,\bar{y})$ in $(\partial_{\bar{y}0}^{-1/2}\partial_{\bar{u}} f_\mathrm{a})(0_-,\bar{y}>0) = (\partial_{\bar{y}0}^{-1/2}\partial_{\bar{u}} f_\mathrm{a})(0_+,\bar{y})$:

$$\partial_{\bar{y}0}^{-1/2}\partial_{\bar{y}\mathrm{p}}^{+1/2}\tilde{f}_\mathrm{a} + 2c_1\sqrt{\bar{y}/\pi} = -f_\mathrm{a},$$

with c_1 already known. Rearrange, using $\partial_{\bar{y}0}^{-1/2}\partial_{\bar{y}0}^{+1/2}\tilde{f}_\mathrm{a} = \tilde{f}_\mathrm{a}$:

$$2f_\mathrm{a} - c_0 + 2c_1\sqrt{\bar{y}/\pi} = \partial_{\bar{y}0}^{-1/2}(\partial_{\bar{y}0}^{+1/2} - \partial_{\bar{y}\mathrm{p}}^{+1/2})\tilde{f}_\mathrm{a}.$$

One may derive an integral form for the operator on the RHS:

$$\partial_{\bar{y}0}^{-1/2}(\partial_{\bar{y}0}^{+1/2} - \partial_{\bar{y}\mathrm{p}}^{+1/2})\tilde{h} = \int_0^1 \mathrm{d}\tau\, g_\Delta(\bar{y},\tau)h(1-\tau),$$

$$g_\Delta(\bar{y},\tau) = \frac{\sqrt{\bar{y}}/\pi}{\sqrt{\tau}(\bar{y}+\tau)} + \frac{\sqrt{\bar{y}}}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(n+\frac{3}{2},\tau+1)\bar{y}^n - 1.$$

Define $f_{\mathrm{a,sh}}(\bar{y}) \doteq f_\mathrm{a}(0,\bar{y}) - c_0/2 = \tilde{f}_\mathrm{a} + c_0/2$, then we have

$$2f_{\mathrm{a,sh}} + 2c_1\sqrt{\bar{y}/\pi} = \int_0^1 \mathrm{d}\tau\, g_\Delta(\bar{y},\tau)f_{\mathrm{a,sh}}(1-\tau).$$

Iteration on the semi-differential operator is convergent.

Consider the problem

$$h(0<\bar{y}\leq 1)-h^{(0)}(\bar{y})=\frac{1}{2}\int_0^1 \mathrm{d}\tau\, g_\Delta(\bar{y},\tau)h(1-\tau),$$

with specified function $h^{(0)}$ satisfying $|h^{(0)}|(\bar{y})\leq b_0$.
Define

$$h^{(q+1)}(\bar{y})=\frac{1}{2}\int_0^1 \mathrm{d}\tau\, g_\Delta(\bar{y},\tau)h^{(q)}(1-\tau),$$

then one may bound

$$|h^{(q\geq 1)}|(\bar{y})\leq b_q(1-\sqrt{\bar{y}}/4),\ \text{for } b_{q\geq 1}\doteq b_0 r^{q-1},$$

for positive constant $r<0.82$. This implies that h may be expressed as a convergent sum

$$h=\sum_{q=0}^{\infty}h^{(q)}, \text{which is bounded by } \sum_{q=0}^{\infty}b_q=\frac{2-r}{1-r}b_0<6.46b_0.$$

Iterative approximations for $f_\mathrm{a}(0,\bar{y})$ converge rapidly.

Use the iterative scheme with $h = f_{\mathrm{a,sh}}$ and $h^{(0)} = -c_1\sqrt{\bar{y}/\pi}$.
Exact results: $c_0/c_1 = \zeta(1/2)/\sqrt{\pi} \approx -0.823917$, $f_\mathrm{a}(0_+) = f_\mathrm{a}(1)/2$.

Define: $c_0 \approx c_0^{(Q)} \doteq \sum_{q=0}^{Q} 2\int_0^1 \mathrm{d}\bar{y}\, f_{\mathrm{a,sh}}^{(q)}$

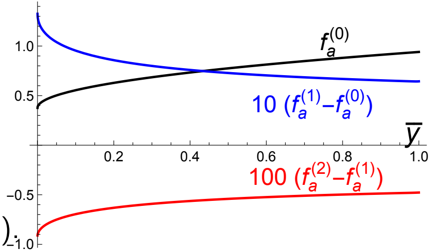
$f_\mathrm{a}^{(Q)} \doteq \sum_{q=0}^{Q} f_{\mathrm{a,sh}}^{(q)} + c_0^{(Q)}/2.$

Convergence is rapid:

$f_\mathrm{a}^{(0)}$: $c_0^{(0)} \approx -0.752 c_1$,
$f_\mathrm{a}^{(0)}(\bar{y} \to 0_+)/f_\mathrm{a}^{(0)}(\bar{y} = 1) = 2/5$.

$f_\mathrm{a}^{(1)}$: $c_0^{(1)} \approx -0.829249 c_1$ (<1% off),
$f_\mathrm{a}^{(1)}(\bar{y} \to 0_+)/f_\mathrm{a}^{(1)}(\bar{y} = 1) \approx 0.506$.

$f_\mathrm{a}^{(2)}$: $c_0^{(2)} \approx -0.82356 c_1$ (<0.05% off),
$f_\mathrm{a}^{(2)}(0_+)/f_\mathrm{a}^{(2)}(1) \approx 0.49964$ (<0.1% off).



The asymptotic solution reproduces the boundary-layer solution, and can be shown to be $O(D^{1/2})$ accurate.

Substitute
$$\bar{f}_a(0) = c_0 = (c_0/c_1)c_1 = -\Gamma(c_0/c_1)/(|v|\bar{D}_0)^{1/2}$$
$$|\bar{u}_\ell| = (|v|\bar{D}_0)^{1/2} \int_{\bar{x}_\ell}^{0} \mathrm{d}\bar{x}\, \bar{D}^{-1}(\bar{x})$$
into $\bar{f}_a(\bar{u}_\ell)$ equation to get
$$\bar{f}_a(\bar{u}_\ell) = \Gamma \left[\int_{\bar{x}_\ell}^{0} \mathrm{d}\bar{x}\, \bar{D}^{-1} + (-c_0/c_1)/(|v|\bar{D}_0)^{1/2} \right],$$
equivalent to boundary layer results for $f_{i0} = \bar{f}_a(\bar{u}_\ell)$.

We were then able to demonstrate $D^{1/2}$ convergence as follows:

- Define $f_\Delta \doteq f_i - f_a$, which solves (exact eqn) - (asymptotic eqn).
- Solve for its leading-order portion f_δ, using similar methods as here.
- Eval $|f_i(\bar{u}_\ell) - \bar{f}_a(\bar{u}_\ell)|$ with $f_\Delta \to f_\delta$, it's bounded by $O(D^1)$ constant.
- Show that one may bound the ratio of (error after f_δ solution) over (error after f_a solution) by a constant of $O(D^{1/2})$.

Summary

- Drift orbits beat with spatial variation of turbulent D causing unequal orbit-averaged diffusivity for co- and counter- ions.
- Assuming $(D^{\rm dim}/L_x^2) \ll (v_{ti}|_{\rm pt}/qR_0)$, we may approximately solve for the resulting rotation, even for a $D(x,y)$ with arbitrary spatial dependence.
- A simple boundary-layer method produces a quick, intuitive answer.
  - This method may be applied to more general problems, e.g. self-consistent rotation with short-charge-exchange-length neutrals
- Simple limits of this answer give relatively convenient dimensional formulas for rotation at the core-edge boundary.
  - As always, care is needed when using idealized theory to model experiment.
- A more detailed calculation allows us to verify the boundary layer answer and its accuracy to $O(D^{1/2})$.
  - This approach also produces concrete approximate forms for the strong SOL and near-LCFS flows.

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