# Theoretical expressions for TAE EP driven growth rates

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We start with zero orbit radial width approximation for energetic particle trajectories. General equation for the growth rate due to Alfvén instability [1] in this case is

$$\frac{\gamma}{\omega} \simeq -\frac{v_A^2 m_b^2 \pi^2}{2\omega R^2 B^2 E_{b0}} \int d^3v \left[-\frac{E_{bo}\partial}{\partial E} - \frac{\omega_*}{\omega}\right] f_b \left(\frac{v_\perp^2}{2} + v_\parallel^2\right)^2 \delta\left(\omega - \left(k_\parallel \pm \frac{1}{qR}\right) v_\parallel\right), \quad (1)$$

where $\omega_*$ is the diamagnetic frequency of energetic ions. Note, that in used approximation of strongly passing beam ions $\chi$ is not changing along the particle drift trajectory.

## I. MAXWELLIAN DISTRIBUTION FUNCTION

Distribution function in this case is

$$f_M = \frac{n_f}{\left(\sqrt{\pi} v_T\right)^3} e^{-v^2/v_T^2}, \quad (2)$$

where $v_T = \sqrt{2T/m}$, $m$ is mass, and $n_f$ is the fast ion density.

$$\frac{\gamma}{\omega} \simeq \sum_{\pm} \frac{-\sqrt{\pi}\beta_f}{2^2 R^2} \frac{v_A^2}{\omega^2} \chi_l \left(\frac{1}{2} + \chi_l^2 + \chi_l^4\right) \left(1 - \frac{\omega_*}{\omega}\right) e^{-\chi_l^2},$$

where $\chi_l = (v_A/v_0) / |1 + l v_A/qR\omega|$, $l = \pm 1$. Two signs of $l$ correspond to two fundamental resonances with passing particles, $v_\parallel = v_A$ and $v_\parallel = v_A/3$.

## II. SLOWING DOWN DISTRIBUTION FUNCTION

Take isotropic EP distribution in the form

$$f_b \simeq \frac{3\beta(r) B^2}{2^5 \pi^2 E_{b0}} \left(1 - \frac{3}{5}\bar{v}_{cr}^2\right)^{-1} H(v_0 - v) \left[\frac{H(v - v_{cr})}{v^3} + \frac{H(v_{cr} - v)}{v_{cr}^3}\right], \quad (3)$$

where $H$ is the step function, $v_{cr}$ is the critical velocity below which the drag on thermal ions becomes dominant, and $\bar{v}_{cr} = v_{cr}/v_0$. This expression captures the effect of slowing down on electrons (first term in square brackets) and thermal ions (second term in square brackets), but is sufficiently simple and accurate. Substituting Eq.(3) into Eq.(1) we find

$$\frac{\gamma}{\omega} \simeq \sum\nolimits_{\pm} \frac{-3\pi\beta}{2^5 R^2} \frac{v_A^2}{\omega^2} \left\{ \frac{3}{2}\chi_l^2 \left(2 + \frac{\chi_{lcr}^2}{3} + \frac{\chi_l \chi_{lcr}}{3} + \frac{\chi_l^2}{3} + \frac{1}{\chi_l \chi_{lcr}}\right) (\chi_{lcr} - \chi_l) + \frac{\chi_l}{2}\left(\chi_l^4 + 2\chi_l^2 + 1\right) \right. \quad (4)$$

$$- \quad \frac{\omega_*}{\omega}\chi_l^4\left(\chi_{lcr}-\chi_l\right)\left(1+\frac{2}{\chi_l\chi_{lcr}}+\frac{1}{3\chi_l\chi_{lcr}^3}+\frac{1}{3\chi_l^3\chi_{lcr}}+\frac{1}{3\chi_l^2\chi_{lcr}^2}\right) \qquad (5)$$

$$- \quad \frac{\omega_*}{\omega}\frac{\chi_l^7}{2\bar{v}_{cr}^3}\left[\frac{1}{3\chi_{lcr}^6}+\frac{1}{\chi_{lcr}^4}+\frac{1}{\chi_{lcr}^2}-\frac{7}{3}\right]\Bigg\}, \qquad (6)$$

where it is assumed that $v_{cr} < v_0$ and $\chi_{lcr} = \min\left[\left(v_A/v_{cr}\right)/\left|1+lv_A/qR\omega\right|, 1\right]$.

[1] Y. M. Li, S. M. Mahajan, and D. W. Ross, Phys. Fluids **30**, 1466 (1987).