Problems for AST558 seminars on "Introduction to Energetic Particles Physics in Magnetic Fusion" by N.N. Gorelenkov (lecture notes are here http://w3.pppl.gov/~ngorelen/lecture_09vg_p1.pdf and the problems are here http://w3.pppl.gov/~ngorelen/lecture_09prblms.pdf)

Problem 1. Find the non - stationary alpha particle distribution function.
Solve the kinetic equation for alpha particle distribution function f (see p.16 of the first lecture) but with $S_0 = S_0(t)$ and allow for the time dependence of the distribution $f = f(t, v)$:

$$\frac{\partial f}{\partial t} = \frac{1}{\tau_{se} v^2} \frac{\partial}{\partial v} \left(v^3 + v_*^3\right) f + \frac{\delta(v - v_0)}{4\pi} S_0. \tag{1}$$

Assume that the plasma is homogeneus and that $\tau_{se} = const$ and $v_* = const$.

Solution.
For more insight into the method of solving Eq.(1) we rewrite it showing the overset of the downarrow and a variable on which a partial derivative operates, e.g. $\overset{\downarrow}{t}$:

$$\frac{\partial f\left(v, \overset{\downarrow}{t}\right)}{\partial t} = \frac{1}{\tau_{se} v^2} \frac{\partial}{\partial v} \left(v^3 + v_*^3\right) f\left(\overset{\downarrow}{v}, t\right) + \frac{\delta(v - v_0)}{4\pi} S_0. \tag{2}$$

Introduce a new function $F(v,t) = \left(v^3 + v_*^3\right) f(v,t)$. We find an equation for $F(v,t)$;

$$\frac{\tau_{se} v^2}{v^3 + v_*^3} \frac{\partial F\left(v, \overset{\downarrow}{t}\right)}{\partial t} = \frac{\partial}{\partial v} F\left(\overset{\downarrow}{v}, t\right) + \frac{v_0 \tau_{se} \delta(v - v_0)}{4\pi} S_0. \tag{3}$$

Then we will look for the solution using the characteristic method, in which we assume that $t = t(v)$. For that let's write full derivative of F:

$$\frac{dF(v, t(v))}{dv} = \frac{dt}{dv} \frac{\partial F\left(v, \overset{\downarrow}{t}\right)}{\partial t} + \frac{\partial}{\partial v} F\left(\overset{\downarrow}{v}, t\right). \tag{4}$$

By comparing Eq.(4) and Eq.(3) we find an equation for the characteristic $t(v)$:

$$\frac{dt}{dv} = -\frac{\tau_{se} v^2}{v^3 + v_*^3},$$

or

$$t = t_0 - (\tau_{se}/3) \ln\left(v^3 + v_*^3\right) / \left(v_0^3 + v_*^3\right). \tag{5}$$

Again comparing Eq.(4) and Eq.(3) we find an equation for $F(v, t(v))$

$$\frac{dF}{dv} = \frac{v_0 \tau_{se} \delta(v - v_0)}{4\pi} S_0(t(v)).$$

Integrating it from $v_0(1-\epsilon)$ to $v_0(1+\epsilon)$ with $\epsilon \to 0$ we find that

$$F = \frac{v_0 \tau_{se}}{4\pi} S_0 \left(t_0 \left(v, t\right)\right),$$

and

$$f = \frac{v_0 \tau_{se}}{4\pi} \frac{S_0 \left(t_0 \left(v, t\right)\right)}{v^3 + v_*^3}. \tag{6}$$

Note, that another way of solving Eq.(1) is given in Putvinski, Rev. of Plasma Physics, v. 18.

Problem 2. For the solution from 1. above find the criteria for the source evolution when the alpha particle distribution is stable in a homogeneous plasma. That is when $\partial f/\partial v \leq 0$.

Solution.

We take derivative of $f(v,t)$ from a previous problem and find that

$$\frac{\partial f}{\partial v} \leq 0$$

amounts to

$$\frac{\partial S_0}{\partial t} \leq \frac{3 S_0}{\tau_{se}}.$$

Problem 3. In a homogeneous plasma with zero pressure find an equation for the oscillations using two linearized ideal MHD equations:

$$\rho_i \frac{\partial \mathbf{v}}{\partial t} = \frac{1}{4\pi} \left[\left[\nabla \times \tilde{\mathbf{B}} \right] \times \mathbf{B} \right], \tag{7}$$

and

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = \nabla \times \left[\mathbf{v} \times \mathbf{B} \right], \tag{8}$$

where $\mathbf{v}$ and $\tilde{\mathbf{B}}$ are perturbed vectors of the plasma displacement and the magnetic field. Assume that the perturbations are uncompressible and perpendicular to the direction z of the equilibrium magnetic field $\mathbf{B} = B_z \mathbf{e}_z$, $\mathbf{e}_z = \mathbf{B}/B$, $B = |\mathbf{B}|$.

Solution.

From Eq.(8) we find

$$\frac{\partial \tilde{\mathbf{B}}}{\partial t} = B \frac{\partial}{\partial z} \mathbf{v}_\perp.$$

From Eq.(7) we find

$$\rho_i \frac{\partial^2 \mathbf{v}_\perp}{\partial t^2} = \frac{B}{4\pi} \left[\left[\nabla \times \frac{\partial}{\partial z} \mathbf{v}_\perp \right] \times \mathbf{B} \right], \tag{9}$$

and finally

$$\rho_i \frac{\partial^2 \mathbf{v}_\perp}{\partial t^2} = \frac{B^2}{4\pi} \frac{\partial^2}{\partial z^2} \mathbf{v}_\perp. \tag{10}$$

The last shear Alfven wave equation can also be rewritted in the form

$$\frac{\partial^2 \xi_\perp}{\partial t^2} = v_A^2 \frac{\partial^2}{\partial z^2} \xi_\perp, \tag{11}$$

where v_A is the Alfven speed.